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A DIOPHANTINE EQUATION

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1. Introduction. Partial integer solutions of the diophantine equation

$$X^2 - Y^2 = Z^3 - W^3, \tag{1.1}$$

some involving integer parameters, but all far from complete, have been given. In contrast, the more difficult equation in which the left member is X^2+Y^2 (obviously it is immaterial whether the right member is Z^3-W^3 or Z^3+W^3) was completely solved in integers by Rosenthall [8] by operating in a certain (Dirichlet) biquadratic domain. For (1.1) the rational domain suffices throughout. The solution goes through on the strictly elementary level because Dickson [2, 4, 5] recast his first proof [1] of the theorem of which Lemma 2 below is an instance so as to obviate ideals, and indeed algebraic numbers, entirely. This was possible because multiplication of ideals is equivalent to composition of certain (composable) forms. Each of the three lemmas is a very special case of a corresponding general theorem having numerous applications to diophantine analysis. None has yet been fully exploited.

The structure of the complete integer solution of (1.1) is best seen by leaving it in unreduced form. Each of X, Y is of degree 9, and each of Z, W of degree 6, in the 9 independent integer parameters a, b, c, f, g, h, k, n, p; the integers α, β, γ, ζ, η are defined by

$$\alpha^2 = \beta^2 = \gamma^2 = 1; \quad (\zeta, \eta) = (\gamma, 4\gamma), \quad (2\gamma, 2\gamma), \quad (4\gamma, \gamma). \tag{1.2}$$

The solution falls into two parts, (A), (B).

$$\text{(A)} \qquad \begin{aligned} 8X &= j[\eta x(t^2 + 3v^2) + \zeta w(r^2 + 3s^2)], \\ 8Y &= j[\eta x(t^2 + 3v^2) - \zeta w(r^2 + 3s^2)], \\ 2Z &= j[\alpha(rv + st) + \beta(rt - 3sv)], \\ 2W &= j[\alpha(rv + st) - \beta(rt - 3sv)], \end{aligned}$$

in which j, x, w, r, s, t, v are integers ranging over all solutions $(j, \cdots, v)$ of

$$xw = \beta j(rt - 3sv). \tag{1.3}$$

All solutions of (1.3) are given by

$$\begin{aligned} j &= \beta np, \qquad x = p(ag + bh), \qquad w = ckn, \qquad r = ak, \\ s &= kb, \qquad t = bf + cg, \qquad v = (af - ch)/3, \end{aligned} \tag{1.4}$$

where a, f, c, h are such that

$$af \equiv ch \bmod 3. \tag{1.5}$$

The values (1.4) of $j, \cdots, v$ substituted into (A) give the first part of the complete integer solution of (1.1). We shall omit the details of solving (1.5) explicitly and assigning the forms modulo 8 of the parameters.

$$\text{(B)}\quad \begin{aligned} 4X &= j[\eta x(t^2 + tv + v^2) + \zeta w(r^2 - rs + s^2)],\\ 4Y &= j[\eta x(t^2 + tv + v^2) - \zeta w(r^2 - rs + s^2)],\\ 2Z &= j[\alpha(rv + st) + \beta(2rt - 2sv - st + vr)],\\ 2W &= j[\alpha(rv + st) - \beta(2rt - 2sv - st + vr)], \end{aligned}$$

in which $(j, \cdots, v)$ ranges over all sets of integer solutions of

$$xw = \beta j(2rt - 2sv - st + vr). \tag{1.6}$$

All solutions of (1.6) are given by

$$\begin{aligned} &j = \beta np, \qquad x = -p(ag + bh), \qquad w = -kcn, \qquad r = bf + cg,\\ &s = -af + ch, \qquad t = k(2a + b)/3, \qquad v = -k(a + 2b)/3, \end{aligned} \tag{1.7}$$

where k, a, b are such that

$$k(a - b) \equiv 0 \bmod 3. \tag{1.8}$$

It is readily verified that (A), (B) actually are solutions. Their completeness follows from the lemmas stated next.

2. Three lemmas. In the following lemmas the complete integer solutions of the respective equations are as indicated.

LEMMA 1. [6]

$$RS = TV:$$

$$R = rt, \qquad S = sv, \qquad T = rv, \qquad V = st,$$

r, s, t, v independent integer parameters, t, v, coprime if desired.

LEMMA 2. [4, 5]

$$R^2 + 3S^2 = TV:$$

$$\text{(A')}\quad \begin{aligned} R &= \beta j(rt - 3sv), & S &= j(rv + st),\\ T &= j(r^2 + 3s^2), & V &= j(t^2 + 3v^2); \end{aligned}$$

$$\text{(B')}\quad \begin{aligned} R &= \beta j(2rt - 2sv - st + vr), & S &= j(rv + st),\\ T &= j(2r^2 - 2rs + 2s^2), & V &= j(2t^2 + 2tv + 2v^2), \end{aligned}$$

j, r, s, t, v independent integer parameters.

LEMMA 3. [7]

$$X_1Y_1 + X_2Y_2 + X_3Y_3 = 0:$$

$$\begin{aligned} X_1 &= kx_1, & Y_1 &= x_2y_1 + x_3y_2,\\ X_2 &= kx_2, & Y_2 &= -x_1y_1 + x_3y_3,\\ X_3 &= kx_3, & Y_3 &= -x_1y_2 - x_2y_3, \end{aligned}$$

the small letters independent integer parameters.

3. Verification. By the change of notation $X+Y=X'$, $X-Y=Y'$, (1.1) becomes

$$X'Y' = (Z - W)(Z^2 + ZW + W^2). \tag{3.1}$$

By application of Lemma 1,

$$\begin{aligned} X' &= xz, & Z - W &= xw \\ Y' &= yw, & Z^2 + ZW + W^2 &= yz; \end{aligned} \tag{3.2}$$

whence, by elimination of W,

$$3Z^2 - 3xwZ + x^2w^2 - yz = 0, \tag{3.3}$$

to be solved in integers Z, x, y, z, w. The Z-discriminant of (3.3) is the integer $12yz-3x^2w^2$. This must be the square, $(3u)^2$, of an integer multiple u of 3:

$$(xw)^2 + 3u^2 = 4yz.$$

Referring to (1.2), we write $y'=\eta y$, $z'=\zeta z$. Then

$$(xw)^2 + 3u^2 = y'z'. \tag{3.4}$$

Thus far, with Z, W from (3.2), (3.3), and $2X=X'+Y'$, $2Y=X'-Y'$, we have

$$\begin{aligned} 8X &= \eta xz' + \zeta wy', \quad 8Y = \eta xz' - \zeta wy', \quad 2Z = \alpha u + xw, \\ 2W &= \alpha u - xw, \end{aligned} \tag{3.5}$$

where x, w, u, y', z' are to be determined from (3.4).

Application of Lemma 2 to (3.4) and substitution of the values of x, w, u, y', z' thus determined into (3.5) give the two parts (A), (B) of the solution, where now j, x, w, r, s, t, v are to be found from (1.3) for (A), and from (1.6) for (B).

Application of Lemma 1 to (1.3) gives

$$x = pR, \qquad w = Qn, \qquad \beta j = pn, \qquad rt - 3sv = QR,$$

and by application of Lemma 3 to the last of these written as

$$r\cdot t + s\cdot(-3sv) + Q\cdot(-R) = 0$$

we get

$$\begin{aligned} r &= ka, & t &= bf + cg, \\ s &= kb, & 3v &= af - ch, \\ Q &= kc, & R &= ag + bh. \end{aligned}$$

Collecting results we have (1.4), (1.5). This disposes of (A).

From Lemma 1 applied to (1.6), $x=pR$, $w=Qn$, $\beta j=pn$,

$$(2t + v)\cdot r - (t + 2v)\cdot s - Q\cdot R = 0;$$

whence, by Lemma 3,

$$\begin{aligned} 2t + v &= ka, & r &= bf + cg, \\ t + 2v &= -kb, & s &= -af + ch, \\ Q &= -kc, & R &= -ag - bh. \end{aligned}$$

Collecting results we have (1.7), (1.8), disposing of (B).

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1. L. E. Dickson, A new method in diophantine analysis, Bull. Amer. Math. Soc., vol. 27 (1920–21), pp. 353–365.

2. L. E. Dickson, Integral solutions of $x^2-my^2=zw$, ibid., vol. 29 (1923), pp. 464–467.

3. L. E. Dickson, All integral solutions of $ax^2+bxy+cz^2=w_1w_2 \cdots w_n$, ibid, vol. 32 (1936), pp. 644–648.

4. L. E. Dickson, Introduction to the theory of numbers 1929, Chap. VI. (The discrepancy between Theorem 69, p. 91 and Exercise 1, p. 93, is rectified in [5], p. 186, Exercise 3. Apparently the sign $\pm$ should be prefixed to the value of x in the first part of the solution stated on p. 186.)

5. L. E. Dickson, Modern elementary theory of numbers, 1939, 3d. impression, 1947, Ch. IX.

6. Stated as Problem 1, p. 195, in [5].

7. Alternative form in [5], Section 84. The more symmetrical form of the solution used here is a special case of F. Giudice's solution; see Th. Skolem, Diophantische Gleichungen, 1938.

8. E. Rosenthall, Diophantine equations reducible in biquadratic fields, Duke Math. Jr., vol. 10 (1943), pp. 463–470.

ON THE MONGE POINT OF THE TETRAHEDRON*

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1. Introduction. In recent years some properties, possibly new, have been added to this chapter of the geometry of the tetrahedron [1]. We shall mention some of them after having generalized the definition of the Monge point in order to call attention to various spheres associated with it. Many of the properties of these spheres seem to have remained unpublished.

2. History. In a memoir entitled, *Sur la pyramide triangulaire*, Monge proved that the planes drawn through the midpoints of the edges of a tetrahedron perpendicular to the opposite edges meet at a point M, and that M is the midpoint of the segment joining the centroid and the circumcenter [2]. In this memoir the hyperboloid of center M, determined by the altitudes, is not mentioned. Its fundamental properties seem to have been cited first by Steiner [3]. According to Chasles, Monge and Hachette had found the rectilinear generators of these second degree surfaces analytically. He adds, "For a long time the only demonstration of this property of the hyperboloid was the initial analytical proof. While a student at the École Polytechnique I gave a purely geometric discussion, which became a part of the course at Polytechnique" [4].

* Translated from the French by Col. W. E. Byrne.

3. Generalization. The properties proved by Monge are particular cases of the following:

THEOREM I. *If the planes drawn through the midpoints of the edges of a tetrahedron $T \equiv ABCD$ perpendicular to the corresponding (or opposite) edges of a tetrahedron $T' \equiv A'B'C'D'$ intersect in a point Q', the planes drawn through the midpoints of the edges of T' perpendicular to the corresponding (or opposite) edges of T intersect in a point Q.*

Let A_1, B_1, C_1, A_1', B_1', C_1' and a_1, b_1, c_1, a_1', b_1', c_1' be the midpoints of the edges BC, CA, AB, DA, DB, DC and $B'C'$, $C'A'$, $A'B'$, $D'A'$, $D'B'$, $D'C'$ of T and T'. If the planes drawn through the points A_1, B_1, C_1 perpendicular to the edges $B'C'$, $C'A'$, $A'B'$ of T' intersect in a point Q', these planes are coaxal and the triangles $A_1B_1C_1$ (or ABC) and $A'B'C'$ are said to be skew orthological.* The planes through a_1, b_1, c_1 perpendicular to the sides BC, CA, AB of triangle ABC have in common a line d_d. For the same reason the planes perpendicular to the sides AB, DA, DB of triangle DAB drawn through c_1, a_1', b_1' have a common line d_c. The lines d_d and d_c, both in the plane through c_1 perpendicular to AB, meet in a point Q, common to the five planes considered. The plane drawn through c_1' perpendicular to DC is coaxal with the planes drawn through a_1 perpendicular to BC and through b_1' perpendicular to DB; hence this sixth plane also contains Q, thus proving the theorem.

If we replace the corresponding edges by the opposite edges, the planes in question meet at the points Q_1 and Q_1', symmetrical to Q and Q' with respect to the centroids of T' and T. The points T and T' are orthological with respect to the midpoints of the corresponding (or opposite) edges. The points Q, and Q' (or Q_1, and Q_1') are their respective centers of orthology [6]. If T and T' coincide, Q and Q' coincide with the circumcenter O of T; Q_1 and Q_1' coincide with the point symmetric to O with respect to the centroid G of T. We find thus the construction indicated by Monge of the point M which bears his name and observe at the same time the fundamental property of this point.

4. Notations. Let the edges of $T \equiv ABCD$ be denoted by $BC = a$, $DA = a'$, $CA = b$, $DB = b'$, $AB = c$, $DC = c'$. Also let us designate by (O, R) the circumsphere; G the centroid; G_a, G_b, G_c, G_d the centroids of the faces BCD, CDA, DAB, ABC; G_a', G_b', G_c', G_d' the midpoints of the medians $AG_a = M_a$, $BG_b = M_b$, $CG_c = M_c$, $DG_d = M_d$; O_a, O_b, O_c, O_d the circumcenters of the faces opposite the vertices A, B, C, D; A_1', B_1', C_1', D_1' the points of intersection of the altitudes AA', BB', CC', DD' with the circumsphere.

5. Metric relations. Let O' be the orthogonal projection of O on the altitude AA' and let O_i, G_i'', M_i', $(i = a, b, c, d)$, be the orthogonal projections on the planes BCD, CDA, DAB, ABC of O, G, M. We have (in magnitude and sign)

* If the planes drawn through the vertices of a triangle ABC perpendicular to the corresponding sides of a triangle $A'B'C'$ are coaxal, then the planes drawn through the vertices of $A'B'C'$ perpendicular to the corresponding sides of ABC are coaxal. [5].

$$OO_a = \frac{AA' + A'A_1'}{2} + A_1'A' = \frac{AA' + A_1'A'}{2}, \cdots$$

$$GG_a'' = \frac{OO_a + MM_a'}{2} = \frac{AA'}{4}, \cdots .$$

From the above relations it follows that

$$MM_a' = \frac{AA'}{2} + O_aO = \frac{A'A_1'}{2},$$

and

$$MM_b' = \frac{B'B_1'}{2}, \qquad MM_c' = \frac{C'C_1'}{2}, \qquad MM_d' = \frac{D'D_1'}{2}.$$

Hence we have

THEOREM II. *In a tetrahedron T the necessary and sufficient condition that the Monge point M be interior or exterior to T is that the points A and A_1', B and B_1', C and C_1', D and D_1' be separated, respectively, by the planes of the faces BCD, CDA, DAB, ABC or that one of these pairs of points be on the same side of the plane of the corresponding face.*

COROLLARY. *For the point M to be on the surface of the tetrahedron, in the plane of a face, or on an edge, or at a vertex, it is necessary and sufficient that the foot of the altitude drawn from the opposite vertex, the feet of the two altitudes drawn from the vertices opposite the faces which intersect along the edge in question, the feet of the three altitudes drawn from the three other vertices, respectively, be on the circumsphere.*

We recall the formulas:

$$\overline{MO}^2 = 4R^2 - \sum \frac{a^2 + a'^2}{4}, \tag{1}$$

$$M_a^2 = \frac{a'^2 + b^2 + c^2}{3} - \frac{a^2 + b'^2 + c'^2}{9}, \cdots \tag{2}$$

$$\rho_a^2 = \frac{a^2 + b'^2 + c'^2}{18}, \cdots \tag{3}$$

where ρ_a is the radius of the director circle of the inscribed Steiner ellipse of triangle BCD. By applying the theorem on medians to the triangles AOM, BOM, COM, DOM and AMG_a, BMG_b, CMG_c, DMG_d, then using Stewart's theorem for the four last named triangles to calculate MG_i, we obtain [7],

$$\overline{MA}^2 = R^2 + \frac{a'^2 + b^2 + c^2 - a^2 - b'^2 - c'^2}{4}, \cdots \tag{4}$$

$$\overline{MG}_a^2 = R^2 - \frac{3(a'^2 + b^2 + c^2) + a^2 + b'^2 + c'^2}{36}, \cdots \tag{5}$$

$$\overline{MG}_a'^2 = R^2 - \frac{a^2 + b'^2 + c'^2}{9} = \overline{OG}_a^2, \cdots \tag{6}$$

which will be used later.

6. Spheres associated with the Monge point. (a) The Longchamps sphere of the medial tetrahedron t of T.

THEOREM III. *In a tetrahedron $T \equiv ABCD$, twice the sum of the squares of the edges issued from a vertex diminished by the sum of the squares of the edges of the opposite face equals six times the power (P_i) of that vertex with respect to the Longchamps sphere of the medial tetrahedron $t \equiv G_aG_bG_cG_d$.*

In calling attention to the Longchamps sphere of a tetrahedron T for the first time, we established that its center M_1 coincides with the symmetric of the Monge point M with respect to the circumcenter O of T, and that its radius 3ρ is given by [8],

$$9\rho^2 = 9R^2 - 3\sum \frac{a^2 + a'^2}{4}. \tag{7}$$

The Longchamps sphere of t is the transform of the sphere $(M_1, 3\rho)$ by the homothety $(G, -1/3)$; its center coincides with M.* The theorem may be proved by using (4) and (7) to show that

$$(P_a) = \overline{AM}^2 - \rho^2 = \frac{2(a'^2 + b^2 + c^2) - a^2 - b'^2 - c'^2}{6}, \cdots .$$

COROLLARY. *In a tetrahedron T the power (P_i') of a vertex with respect to the Longchamps sphere $(M_1, 3\rho)$ equals half the sum of the squares of the edges of the opposite face. The sum of the powers of the four vertices equals the sum of the squares of the edges.*

The median theorem applied to triangle AMM_1 gives

$$\overline{AM_1}^2 = 9R^2 - \frac{3(a'^2 + b^2 + c^2) + a^2 + b'^2 + c'^2}{4}$$

and

$$(P_a') = \overline{AM_1}^2 - 9\rho^2 = \frac{a^2 + b'^2 + c'^2}{2}, \cdots$$

so that

$$\sum (P_i') = \sum (a^2 + a'^2).$$

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* The sphere (M, ρ) coincides with the quasi-polar sphere of T of N. A. Court. [9]. Several of its properties given by N. A. Court are identical with those contained in our memoirs of 1932 and 1937 [8].

THEOREM IV. *In a tetrahedron $T \equiv ABCD$ the Monge point M coincides with the radical center of: the spheres described on the medians as diameters, the spheres which admit as great circles the director circles of the inscribed Steiner ellipses of the faces, the Longchamps spheres of the medial tetrahedrons of $T_a \equiv MBCD$, $T_b \equiv MCDA$, $T_c \equiv MDAB$, $T_d \equiv MABC$. These twelve spheres are orthogonal to the Longchamps sphere of the medial tetrahedron t.*

From formulas (1) to (7) we have

$$\overline{MG_a'}^2 - \frac{\overline{AG_a}^2}{4} = R^2 - \sum \frac{a^2 + a'^2}{12} = \frac{\overline{MO}^2 - R^2}{3} = \overline{MG_a}^2 - \rho_a^2.$$

Furthermore, by using the formula for (P_a) we obtain the power (P_M) of M with respect to the Longchamps where (M_a'') of the medial tetrahedron of T_a,

$$(P_M) = \frac{2(\overline{MB}^2 + \overline{MC}^2 + \overline{MD}^2) - (\overline{BC}^2 + \overline{BD}^2 + \overline{CD}^2)}{6}$$

$$= R^2 - \sum \frac{a^2 + a'^2}{12}.$$

Hence the point M is the radical center of the twelve spheres in question and its power with respect to these spheres is

$$R^2 - \sum \frac{a^2 + a'^2}{12} = \frac{\overline{MO}^2 - R^2}{3} = \rho^2;$$

they are all orthogonal to the Longchamps sphere (M, ρ) of t.

COROLLARY. *In a tetrahedron T the spheres $(G_i', M_i/2)$ described on the medians as diameters intersect the spheres (G_i, ρ_i) traced on the director circles of the inscribed Steiner ellipses of the faces in four circles located on the director sphere (G) of the inscribed Steiner ellipsoid of T [10].*

COROLLARY. *The Longchamps sphere (M, ρ) of the medial tetrahedron t is orthogonal to the sphere (G) mentioned above [8].*

THEOREM V. *The thirteen spheres $(G_i', M_i/2)$, (G_i, ρ_i), (M_i'') and (G) are orthogonal to (M, ρ).*

(b) Orthocentric sphere. By analogy with the orthocentric circle of a triangle (Tucker) let us agree to call the sphere (L) of center L described on MG a diameter the orthocentric sphere of T [11].

THEOREM VI. *Eight times the power of a vertex of T with respect to (L) equals twice the sum of the squares of the edges issued from this vertex less the sum of the squares of the edges of the opposite face.*

The powers (P_i) of the vertices A, B, C, D of T with respect to (L) are equal to

$$\frac{4\overline{AL}^2 - \overline{GM}^2}{4}, \quad \frac{4\overline{BL}^2 - \overline{GM}^2}{4}, \quad \frac{4\overline{CL}^2 - \overline{GM}^2}{4}, \quad \frac{4\overline{DL}^2 - \overline{GM}^2}{4}.$$

By application of the median theorem to triangles AGM, BGM, CGM, CGM we have, using formulas (1) to (6),

$$(P_a) = \frac{2(a'^2 + b^2 + c^2) - a^2 - b'^2 - c'^2}{8}, \cdots \tag{8}$$

thus proving the theorem.

COROLLARY. *In a tetrahedron T the sum of the powers of the vertices with respect to (L) equals the sum of the square of the distances from the centroid G to the four vertices.*

By adding relations (8) we find

$$\sum (P_i) = \sum \frac{a^2 + a'^2}{4} = \overline{GA}^2 + \overline{GB}^2 + \overline{GC}^2 + \overline{GD}^2.$$

THEOREM VII. *In a tetrahedron $T \equiv ABCD$ the Monge point M is the radical center of the orthocentric spheres (L_a), (L_b), (L_c), (L_d) of tetrahedrons T_a, T_b, T_c, T_d.*

By virtue of (8) the power $P(M, L_a)$ of M with respect to (L_a) may be written

$$P(M, L_a) = \frac{2(\overline{MB}^2 + \overline{MC}^2 + \overline{MD}^2) - \overline{BC}^2 - \overline{CD}^2 - \overline{DB}^2}{8},$$

or

$$P(M, L_a) = \frac{[3R^2 - \sum (a^2 + a'^2)/4]}{4} = \frac{\overline{MO}^2 - R^2}{4} = \sigma^2. \tag{9}$$

Likewise,

$$P(M, L_b) = P(M, L_c) = P(M, L_d) = \sigma^2. \tag{10}$$

COROLLARY. *The diameter of the sphere (M, σ) orthogonal to the orthocentric spheres of T_i is equal to the radius of the director sphere of the Longchamps sphere of the medial tetrahedron t.*

We have

$$4\sigma^2 = \overline{MO}^2 - R^2 = 3R^2 - \sum \frac{a^2 + a'^2}{4} = 3\rho^2.$$

As a more general proposition, we have

THEOREM VIIa. *The radical center of spheres* (ω_i), $(i=a, b, c, d)$, *such that the powers of the vertices of* T_i *with respect to* (ω_i) *are* k *times twice the sum of the squares of the edges issued from these vertices less the sum of the squares of the edges of the opposite faces, is the Monge point* M *of* T, k *being positive or negative.*

By formula (4) we find for the power of M with respect to (ω_d),

$$k[2(\overline{MA}^2 + \overline{MB}^2 + \overline{MC}^2) - \overline{BC}^2 - \overline{CA}^2 - \overline{AB}^2] = k\left[\frac{12R^2 - \sum (a^2 + a'^2)}{2}\right] = 2k(\overline{MO}^2 - R^2).$$

As we shall see later, for $k=1/6$ the spheres (ω_i) are the Longchamps spheres of the medial tetrahedrons t_i of T_i. Theorems VII and VII a correspond to $k=1/8$. For $k=1/9$, the (ω_i) are the twelve-point spheres of T_i. From this we obtain the following proposition.

COROLLARY. *The Monge point* M *of* T *is the radical center of the twelve-point spheres of the tetrahedrons* T_i.

THEOREM VIII. *In a tetrahedron* T *the Monge point is the radical center of the spheres described on* Ag_a, Bg_b, Cg_c, Dg_d *as diameters, where* g_i *is the centroid of* T_i.

The twelve-point sphere $(\omega, R/3)$ of T (the circumsphere of the medial tetrahedron t of T) is the homothetic transform $(G, -1/3)$ of the circumsphere (O, R) of T. The sphere $(\omega, R/3)$ is also the homothetic transform $(M, 1/3)$ of (O, R). Consequently, the lines MG_a, MG_b, MG_c, MG_d meet (O, R) at the points A_2, B_2, C_2, D_2, diametrically opposite the vertices A, B, C, D, and also at the points A_3, B_3, C_3, D_3. Hence

$$MA_2 = 3MG_a = 4Mg_a.$$

The point A_3 is on the sphere described on Ag_a as a diameter. The power of M with respect to this sphere is

$$Mg_a \cdot MA_3 = \frac{MA_2 \cdot MA_3}{4} = \frac{\overline{MO}^2 - R^2}{4}, \cdots \tag{11}$$

thus proving the theorem.

COROLLARY. *That the spheres described on* Ag_a, Bg_b, Cg_c, Dg_d *as diameters are orthogonal to* (M, σ) *follows from* (9), (10), *and* (11).

THEOREM IX. *In an orthocentric group of five tetrahedrons* $ABCDH$ *the orthocenter of any one of the orthocentric tetrahedrons* $T\equiv ABCD$, $T_a\equiv HBCD$, $T_b\equiv HCDA$, $T_c\equiv HDAB$, $T_d\equiv HABC$ *coincides with the radical center of the orthocentric spheres of the four other tetrahedrons.*

For this configuration the Monge points of tetrahedrons T and T_i coincide with the orthocenter H and the vertices A, B, C, D of the orthocentric tetrahedron T.

COROLLARY. *The radical planes of each of the orthocentric spheres of tetrahedrons T_i associated with the orthocentric sphere of T coincide with the planes of the faces of T, and the six radical axes of the orthocentric spheres of a pair of tetrahedrons T_i associated with the orthocentric sphere of T coincide with the edges of T.*

(c) Other spheres.

THEOREM X. *In a tetrahedron $T \equiv ABCD$ the spheres described on the director circles of the inscribed Steiner ellipses of the triangles BCD, CDA, DAB, ABC as great circles are orthogonal to the spheres described on MA, MB, MC, MD as diameters.*

The power of the centroid G_a of the face BCD with respect to the sphere (A_1) of center A_1 described on MA as a diameter has for its value

$$P(G_a) = \frac{4\overline{G_aA_1}^2 - \overline{MA}^2}{4} = \frac{\overline{AG_a}^2 + \overline{MG_a}^2 - \overline{MA}^2}{2}.$$

By formulas (1) to (6) the above expression may be reduced to

$$P(G_a) = \frac{a^2 + b'^2 + c'^2}{18} = \rho_a^2.$$

Analogous formulas hold for $P(G_b)$, $P(G_c)$ and $P(G_d)$, thus verifying the proposition.

COROLLARY. *In the medial tetrahedron t the vertices G_a, G_b, G_c, G_d are situated in the radical planes of the Longchamps sphere (M, ρ) associated with each of the spheres (A_1), (B_1), (C_1), (D_1).*

THEOREM XI. *In a tetrahedron T the circumsphere intersects the spheres $(G_a, \rho_a\sqrt{2})$ in great circles.*

From (3) and (6) it follows that

$$R^2 - \overline{OG_a}^2 = \frac{a^2 + b'^2 + c'^2}{9} = 2\rho_a^2, \cdots.$$

7. Coaxal pencils of spheres. The spheres (G_a'), (g_a'), (A_1) described on AG_a, Ag_a', AM as diameters have the points A and A_3 in common; AA_3 is their radical axis. Since the sphere (G_a, ρ_a) is orthogonal to (A_1) and to the Longchamps sphere (M, ρ) of the medial tetrahedron t, the radical plane of (A_1) and (M, ρ) perpendicular to A_1M, passes through G_a and intersects AA_3 at the radical center A_4 of (A_1), (M, ρ), (G_a'), (g_a'), (O, R). A_4M meets AG_a at right angles at A_5, since M is the orthocenter of the triangle AG_aA_4. Hence A_5 is on the orthocentric sphere (L),

$$A_4A_3 \cdot A_4A = A_4M \cdot A_4A_5,$$

the spheres (O, R), (L), (M, ρ), whose centers O, L, M are collinear, belong to a coaxial pencil (F) of spheres. Their common radical plane (π), perpendicular to OG, contains the point A_4 as well as the analogous points B_4, C_4, D_4. Furthermore, since the Monge point M has the same power with respect to the spheres (G_a') and (G_a, ρ_a) by Theorem IV, the radical plane of these spheres, perpendicular to $G_a' G_a$, contains the line A_4A_5. The line A_4A_5 is situated also in the radical plane of the spheres (G_a, ρ_a) and (G, σ). Hence the sphere (G, σ) is a fourth sphere of the coaxal pencil (F). If the radical plane (π) of (G, σ) and (O, R) intersects OG at K, we have

$$- \sigma^2 - (\overline{GO}^2 - R^2) = 2\overline{GO} \cdot \overline{GK}.$$

Since

$$\sigma^2 = \sum \frac{a^2 + a'^2}{48}, [13]$$

and

$$\overline{GO}^2 = R^2 - \sum \frac{a^2 + a'^2}{16},$$

we find

$$\overline{GK} = \sum \frac{a^2 + a'^2}{48GO} \cdot$$

The point ω, the center of the circumsphere $(\omega, R/3)$ of the medial tetrahedron t (twelve-point sphere of T), is on OG, and

$$\frac{G\omega}{GO} = - \frac{1}{3}$$

$$K\omega = KG - \frac{GO}{3}$$

$$KO = KG + GO.$$

As $\overline{KG}^2 - \sigma^2 = \overline{K\omega}^2 - R^2/9$, K has the same power with respect to (G, σ) and $(\omega, R/3)$. We conclude that $(\omega, R/3)$ belongs also to the coaxal pencil (F).

THEOREM XII. *The great circles cut from the spheres (G_a, ρ_a), (G_b, ρ_b), (G_c, ρ_c), (G_d, ρ_d) by the diametral planes of these spheres perpendicular to MG_a, MG_b, MG_c, MG_d are on a sphere (m, r) which belongs to the coaxal pencil (F).*

If m is the Monge point of t we observe that $mG_a = MA/3$, $mG_b = MB/3$, $mG_c = MC/3$, $mG_d = MD/3$. Formulas (3) and (5) show that

$$\overline{mG_a}^2 + \rho_a^2 = \frac{[R^2 + \sum (a^2 + a'^2)/4]}{9} = \overline{mG_b}^2 + \rho_b^2 = \cdots = r^2.$$

The common centroid G of T and t is at an algebraic distance d_g from the radical plane of (m, r) and (G, σ) such that

$$\overline{Gm}^2 - r^2 + \sigma^2 = 2\overline{Gm}\cdot d_g = \frac{2\overline{GO}\cdot d_g}{3}\cdot$$

Hence $d_g = \sum(a^2+a'^2)/48\overline{OG} = \overline{GK}$; so (m, r) belongs to (F).

COROLLARY. *In an orthocentric tetrahedron T the director circles of the inscribed Steiner ellipses of the triangular faces of T are on the same sphere, whose center is the orthocenter of t.*

THEOREM XIII. *For a tetrahedron T the circumsphere, the twelve-point sphere, the orthocentric sphere, the director sphere of the inscribed Steiner ellipsoid, the Longchamps sphere of the medial tetrahedron t* [13] *and the sphere (m, r) belong to the same coaxal pencil (F).*

Note 1. The homothetic transform $(G, -3)$ of this configuration is another coaxal pencil of spheres to which we have called attention [13], adding $(M, 3r)$ as the transform of (m, r).

Note 2. This theorem generalizes and completes a known proposition relative to a triangle [14].

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A NOTE ON PURE RECURRENCE RELATIONS

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1. Introduction. The pure recurrence relations of many of the classical polynomials, for example, the Laguerre [5], and in particular those of the hypergeometric type, such as the Jacobi [5], Legendre [5], and so on, are readily obtained by the method exhibited in this paper. Many of the recently studied polynomials, as Rice's $H_n(\xi, p, v)$, [4] and [3], Bateman's $J_n^{u,v}$, [1] and [3], and Bateman's $Z_n(t)$ [1], lend themselves to this treatment. It is particularly useful for polynomials which do not form an orthogonal set, and for which there are not available the extremely simple methods of the theory of orthogonal polynomials.

2. Pure recurrence relation for Bateman's $Z_n(t)$. To illustrate we use the polynomial

$$Z_n(t) = {}_2F_2(-n, n+1; 1, 1; t) = \sum_{r=0}^{\infty} \frac{(-n)_r(n+1)_r t^r}{(r!)^3},$$

in which $(a)_r = a(a+1) \cdots (a+r-1)$; $(a)_0 = 1$. To simplify the exposition we let

$$Z_n(t) = \sum_{r=0}^{\infty} \beta_r$$

where, of course,

$$\beta_r = \frac{(-n)_r(n+1)_r t^r}{(r!)^3}.$$

We list a sequence of the $Z_k(t)$ and $tZ_k(t)$ $(k = n, n-1, \cdots)$ in which for clarity we exhibit each in both its explicit form and also in a form involving β_r as defined above. Thus

$$Z_n(t) = \sum_{r=0}^{\infty} \frac{(-n)_r (n+1)_r t^r}{(r!)^3} = \sum_{r=0}^{\infty} \beta_r$$

$$Z_{n-1}(t) = \sum_{r=0}^{\infty} \frac{(-n+1)_r(n)_r t^r}{(r!)^3} = \sum_{r=0}^{\infty} \frac{n-r}{n+r}\beta_r$$

$$tZ_{n-1}(t) = \sum_{r=1}^{\infty} \frac{(-n+1)_{r-1}(n)_{r-1} t^r}{(r-1)!^3} = \sum_{r=0}^{\infty} \frac{-r^3}{(n+r)(n-1+r)}\beta_r$$

$$Z_{n-2}(t) = \sum_{r=0}^{\infty} \frac{(-n+2)_r(n-1)_r t^r}{(r!)^3} = \sum_{r=0}^{\infty} \frac{(n-r)(n-1-r)}{(n+r)(n-1+r)}\beta_r$$

$$tZ_{n-2}(t) = \sum_{r=1}^{\infty} \frac{(-n+2)_{r-1}(n-1)_{r-1} t^r}{(r-1)!^3} = \sum_{r=0}^{\infty} \frac{-r^3(n-r)}{(n+r)(n-1+r)(n-2+r)}\beta_r$$

$$Z_{n-3}(t) = \sum_{r=0}^{\infty} \frac{(-n+3)_r(n-2)_r t^r}{(r!)^3} = \sum_{r=0}^{\infty} \frac{(n-r)(n-1-r)(n-2-r)}{(n+r)(n-1+r)(n-2+r)}\beta_r$$

If the coefficients of β_r in the above series are written with a least common denominator, and the numerators are polynomials of degree at most four in r, then a linear combination of the six series would have as numerator of its coefficient of β_r a polynomial of degree four in r. Such a linear combination would leave five undetermined constants with which to make the five coefficients in the fourth degree polynomial vanish. This suggests that for $n \geqq 3$ there exists a linear recurrence relation of the form

$$Z_n(t) + (A + Bt)Z_{n-1}(t) + (C + Dt)Z_{n-2}(t) + EZ_{n-3}(t) = 0, \tag{1}$$

in which A, B, C, D, and E are rational functions in n and are independent of t. To determine the coefficients in (1) we replace the several $Z_k(t)$ and $tZ_k(t)$ by their respective forms where the factor β_r occurs explicitly under the summation sign. Equating coefficients of β_r, we have for $r \geqq 0$

$$\begin{aligned} &1 + \frac{A(n-r)}{(n+r)} - \frac{Br^3}{(n+r)(n-1+r)} + \frac{C(n-r)(n-1-r)}{(n+r)(n-1+r)} \\ &- \frac{Dr^3(n-r)}{(n+r)(n-1+r)(n-2+r)} + \frac{E(n-r)(n-1-r)(n-2-r)}{(n+r)(n-1+r)(n-2+r)} = 0. \end{aligned} \tag{2}$$

Clearing the above expression of fractions gives the following identity in r:

$$\begin{aligned} &(r+n)(r+n-1)(r+n-2) + A(-r+n)(r+n-1)(r+n-2) \\ &\quad - Br^3(r+n-2) + C(-r+n)(-r+n-1)(r+n-2) \\ &\quad - Dr^3(-r+n) + E(-r+n)(-r+n-1)(-r+n-2) = 0. \end{aligned} \tag{3}$$

From this identity we can readily determine the coefficients in (1). Substituting the values thus obtained and clearing the result of fractions, we get, for $n \geqq 3$, the four-term recurrence relation:

$$\begin{aligned} &n^2(2n-3)Z_n - (2n-1)[3n^2 - 6n + 2 - 2(2n-3)t]Z_{n-1} \\ &\quad - (2n-3)[3n^2 - 6n + 2 + 2(2n-1)t]Z_{n-2} - (2n-1)(n-2)^2 Z_{n-3} = 0. \end{aligned} \tag{4}$$

3. A generalized hypergeometric polynomial with one parameter. We use a special case of a set of generalized hypergeometric polynomials [**2**], namely,

$$f_n(a; -; x) = {}_3F_2(-n, n+1, a; \tfrac{1}{2}, 1; x) = \sum_{r=0}^{\infty} \frac{(-n)_r(n+1)_r(a)_r x^r}{(\frac{1}{2})_r r! r!} \tag{5}$$

(a is independent of n and does not equal one or one-half) as a second example to depict the method in a more concise form. In the array below, the quantity

following the colon is the factor by which

$$\frac{(-n)_r(n+1)_r(a)_r x^r}{(\frac{1}{2})_r r!r!}$$

must be multiplied under the summation sign to give the respective expressions summed for each of the f_k and xf_k ($k=n, n-1, \cdots$) where f_k is an abbreviation for $f_k(a; -; x)$. Then

$$f_n: 1$$

$$f_{n-1}: \frac{(-r+n)}{(r+n)}$$

$$xf_{n-1}: \frac{-r^2(r-\frac{1}{2})}{(r+n)(r+n-1)(r+a-1)}$$

$$f_{n-2}: \frac{(-r+n)(-r+n-1)}{(r+n)(r+n-1)}$$

$$xf_{n-2}: \frac{-r^2(r-\frac{1}{2})(-r+n)}{(r+n)(r+n-1)(r+n-2)(r+a-1)}$$

$$f_{n-3}: \frac{(-r+n)(-r+n-1)(-r+n-2)}{(r+n)(r+n-1)(r+n-2)}.$$

It is easily seen that when the above fractions are written with a least common denominator, neither it nor any numerator exceeds degree four in r. Hence, a linear combination of these fractions, with five constants to be determined, equated to zero gives an identity in r from which we can readily find the coefficients of a pure recurrence relation involving four of the f_k. For $n \geqq 3$

$$\begin{aligned} nf_n - [(3n-2) - 4(n-1+a)x]f_{n-1} \\ + [(3n-4) - 4(n-1-a)x]f_{n-2} - (n-2)f_{n-3} = 0. \end{aligned} \tag{6}$$

It is worth noting that for the polynomial

$$f_n(-;-;x) = \sum_{r=0}^{\infty} \frac{(-n)_r(n+1)_r x^r}{(\frac{1}{2})_r r!r!} = \lim_{a\to\infty} f_n\left(a; -; \frac{x}{a}\right)$$

the recurrence relation

$$nf_n - (3n-2-4x)f_{n-1} + (3n-4+4x)f_{n-2} - (n-2)f_{n-3} = 0 \tag{7}$$

can be obtained from (6) by a limiting process. The reader may also derive (7) by using the method of this note.

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THE NUMBER OF REPRESENTATIONS OF AN INTEGER AS THE SUM OF A PRIME AND A k-FREE INTEGER

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1. Introduction. T. Estermann* has discussed the representation of an integer as the sum of a square and a square-free number. Making use of a somewhat similar method, I shall establish the following result.

THEOREM 1. *Let k be any integer greater than* 1, *and H any positive number. Then every sufficiently large integer n can be represented as the sum of a prime and a k-free integer.*** *For $n \to \infty$ the number $T(n)$ of such representations is given by*

$$T(n) = \prod_{p \nmid n}\left(1 - \frac{1}{p^{k-1}(p-1)}\right)\mathrm{Li}\, n + O\left(\frac{n}{\log^H n}\right),$$

where

$$\mathrm{Li}\, n = \int_2^n \frac{du}{\log u},$$

and the O-constant depends at most on k and H.

2. Notation. Our notation is as follows. The symbols $\mu(n)$, $\phi(n)$ denote, as usual, the functions of Möbius and Euler, and $\mu_k(n)$ is defined as 1 or 0 according as n is or is not k-free.

The letter x denotes a certain function of n (to be fixed later) which tends to infinity with n; other small letters denote positive integers, and p is reserved for primes.

The highest common factor of a and b is denoted by (a, b).

The O-notation refers to the passage $n \to \infty$, and O-constants depend at most on k and H.

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* Einige Sätze über quadratfreie Zahlen, Math Annalen, vol. 105, 1931, pp. 654–662, §1.

** An integer is called k-free if it is not divisible by the kth power of any prime.

3. Proof of Theorem 1. Since

$$\mu_k(m) = \sum_{a^k b=m} \mu(a),$$

we have

$$\begin{aligned} T(n) &= \sum_{m+p=n} \mu_k(m) = \sum_{a^k b+p=n} \mu(a) \\ &= \sum_{a^k b+p=n, a\leqq x} \mu(a) + \sum_{a^k b+p=n, a>x} \mu(a) \\ &= \sum_1 + \sum_2, \end{aligned}$$

say. To evaluate $\sum_1$ we use the well known result* that, for $(d, q)=1$,

$$\sum_{p<n, p\equiv d (\mathrm{mod}\, q)} 1 = \frac{1}{\phi(q)} \mathrm{Li}\, n + O\left(\frac{n}{\log^{2H} n}\right).$$

Hence we have

$$\begin{aligned} \sum_1 &= \sum_{a\leqq x} \mu(a) \sum_{p<n, p\equiv n (\mathrm{mod}\, a^k)} 1 \\ &= \sum_{a\leqq x, (a,n)=1} \mu(a) \left\{\frac{1}{\phi(a^k)} \mathrm{Li}\, n + O\left(\frac{n}{\log^{2H} n}\right)\right\} + O(x) \\ &= \mathrm{Li}\, n \sum_{(a,n)=1} \frac{\mu(a)}{\phi(a^k)} + O\left(\frac{n}{\log n} \sum_{a>x} \frac{1}{\phi(a^k)}\right) + O\left(\frac{nx}{\log^{2H} n}\right) + O(x) \\ &= \prod_{p\nmid n} \left(1 - \frac{1}{\phi(p^k)}\right) \mathrm{Li}\, n + O\left(\frac{n}{\log n} \sum_{a>x} \frac{\log\log a}{a^k}\right) + O\left(\frac{nx}{\log^{2H} n}\right) + O(x) \\ &= \prod_{p\nmid n} \left(1 - \frac{1}{p^{k-1}(p-1)}\right) \mathrm{Li}\, n + O\left(\frac{n \log\log x}{x \log n}\right) + O\left(\frac{nx}{\log^{2H} n}\right) + O(x). \end{aligned}$$

Furthermore

$$\left|\sum_2\right| \leqq \sum_{a^k b+n=p, a>x} 1 \leqq \sum_{a^k b<n, a>x} 1 = O\left(\frac{n}{x^{k-1}}\right) = O\left(\frac{n}{x}\right).$$

Thus we have

$$\begin{aligned} T(n) = \prod_{p\nmid n} \left(1 - \frac{1}{p^{k-1}(p-1)}\right) \mathrm{Li}\, n + O\left(\frac{n \log\log x}{x \log n}\right) \\ + O\left(\frac{nx}{\log^{2H} n}\right) + O(x) + O\left(\frac{n}{x}\right), \end{aligned}$$

* See, e.g., J. G. van der Corput, Sur l'hypothèse de Goldbach pour presque tous les nombres pairs, *Acta Arith.*, vol. 2, 1937, pp. 266–290, Footnote 4.

and Theorem 1 follows on putting $x = \log {}^H n$.

4. Another result. In his paper referred to previously, Estermann also obtained an asymptotic formula for the number of square-free integers not exceeding n and having the form a^2+l, where l is a given non-zero integer. Correspondingly, we have the following result.

THEOREM 2. *Let l be any non-zero integer, k any integer greater than* 1, *and H any positive number. Let, moreover, $U(n)$ denote the number of k-free integers $m \leqq n$, having the form $m = p + l$. Then, as $n \to \infty$,*

$$U(n) = \prod_{p \nmid l} \left(1 - \frac{1}{p^{k-1}(p-1)}\right) \mathrm{Li}\, n + O\left(\frac{n}{\log^H n}\right),$$

where the O-constant depends at most on k, l, and H.

The proof of this result is very similar to that of Theorem 1, and may be left to the reader.

MATHEMATICS

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Mathematics is one component of any plan for liberal education. Mother of all the sciences, it is a builder of the imagination, a weaver of patterns of sheer thought, an intuitive dreamer, a poet. The study of mathematics cannot be replaced by any other activity that will train and develop man's purely logical faculties to the same level of rationality. Through countless dimensions, riding high the winds of intellectual adventure and filled with the zest of discovery, the mathematician tracks the heavens for harmony and eternal verity. There is not wholly unexpected surprise, but surprise nevertheless, that mathematics has direct application to the physical world about us. For mathematics, in a wilderness of tragedy and change, is a creature of the mind, born to the cry of humanity in search of an invariant reality, immutable in substance, unalterable with time. Mathematics is an infinity of flexibles forcing pure thought into a cosmos. It is an arc of austerity cutting realms of reason with geodesic grandeur. Mathematics is crystallized clarity, precision personified, beauty distilled and rigorously sublimated. The life of the spirit is a life of thought; the ideal of thought is truth; everlasting truth is the goal of mathematics.

MATHEMATICAL NOTES

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A SIMPLE PROOF THAT, FOR ODD $p>1$, ARC COS $1/p$ AND π ARE INCOMMENSURABLE

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The purpose of this note is to give a short and elementary proof that, for p an odd integer greater than 1, $\theta = \text{arc cos } 1/p$ and π are incommensurable. The case $p=3$ is of importance in the theory of equivalence of polyhedra by dissection, and proofs for this special case, not so simple as the following, have been given [**1**, **2**, **3**]. On the other hand a more general theorem, embracing ours, but utilizing more advanced notions, has been given by D. H. Lehmer [**4**].

The proof proceeds by contradiction. Assume the existence of two relatively prime integers r and s such that

$$r\theta = s\pi. \tag{1}$$

Consider the binomial expansion

$$(\cos\theta + i\sin\theta)^r = \sum_{n=0}^{r}\binom{r}{n} i^n \cos^{r-n}\theta \sin^n\theta. \tag{2}$$

Taking the imaginary parts of both members, and remembering (1), we find

$$\begin{aligned} 0 = \sin r\theta &= \binom{r}{1}\cos^{r-1}\theta\sin\theta \\ &\quad + \sum_{n=1}^{[(r-1)/2]}\binom{r}{2n+1}(-1)^n\cos^{r-(2n+1)}\theta\sin^{2n+1}\theta, \end{aligned}$$

where $[(r-1)/2]$ denotes the largest integer not greater than $(r-1)/2$. Substituting $\cos\theta = 1/p$, $\sin\theta = p^{-1}(p^2-1)^{1/2}$, we find

$$0 = p^{-r}(p^2-1)^{1/2}\left\{r + \sum_{n=1}^{[(r-1)/2]}\binom{r}{2n+1}(-1)^n(p^2-1)^n\right\}.$$

Since p is an odd integer greater than 1, the quantity in braces must vanish, and r must be even inasmuch as each of the terms in the following sum is even. Since r and s were chosen as relatively prime, s must then be odd. Thus $\cos(r\theta/2) = \cos(s\pi/2) = 0$. Now, in equation (2), replace r by $r/2$ and take the real parts of both members, obtaining

$$0 = \cos (r\theta/2) = \cos^{r/2} \theta + \sum_{n=1}^{[r/4]} (-1)^n \binom{r/2}{2n} \cos^{r/2-2n} \theta \sin^{2n} \theta$$
$$= p^{-r/2} \left\{ 1 + \sum_{n=1}^{[r/4]} (-1)^n \binom{r/2}{2n} (p^2 - 1)^n \right\}.$$

But this equation is impossible because each term of the sum is even and the number in the braces is thus odd and cannot vanish. Therefore our original assumption (1) is untenable, and the theorem is proved.

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ON A PROBLEM OF ERDÖS

V. L. KLEE, JR., University of Virginia

In his neat solution of a problem proposed by Erdös, Lampek has shown [this MONTHLY, vol. 55, 1948, p. 103] that if $x_k = (k!)^2/\phi(k!)$, then $\phi(x_k) = k!$, ϕ being Euler's function. We have $x_1 = 1$, $x_2 = 2^2$, $x_3 = 2 \cdot 3^2$, $x_4 = 2^3 \cdot 3^2$, $\cdots$, and are led to notice that for each k the following result holds.

(1) *Either x_k has the property that each of its primes factors appears to the power two or higher, or $x_k = 2y_k$, where y_k is an odd integer having this property.*

Thus we are led to make the following definitions: P_i is the collection of all integers m such that exactly i distinct primes appear to the first power in the canonical factorization of m; $S_i(n)$ is the number of solutions $x \in P_i$ of the equation, $\phi(x) = n$.

Then (1) implies

$$S_0(k!) \geqq 1. \tag{2}$$

I conjecture that also $S_1(k!) \geqq 1$, although I am not able to prove this. It may be of interest to note that for $2 \leqq k \leqq 6$, $S_1(k!)$ is the $(k-1)$th Fibonacci number; however, there is no obvious reason for expecting this to hold for larger values of k.

Investigating the functions S_i for more general arguments, we arrive at the following results:

$$S_0(n) \leqq 1; \tag{3}$$

$$\sup_n S_2(n) = \sup_n S_3(n) = \infty. \tag{4}$$

It follows from (2) and (3) that $S_0(k!)=1$.

It seems probable that $\sup_n S_j(n)=\infty$ for $j\geqq 1$.

Proof of (1). Observe that if k is prime, then $x_k=k^2x_{k-1}/(k-1)$, while $x_k=kx_{k-1}$ if k is composite. From this it follows by induction that if p is an odd prime and $p<k$, then $p^2\mid x_k/k$, and (1) is an obvious consequence.

Proof of (3). Suppose that $y=\prod p_i^{a_i}\prod v_k^{c_k}$ and $z=\prod q_j^{b_j}\prod_{v_k}^{d_k}$ are in P_0, the p's, v's, and q's denoting distinct primes. (The products involved may be empty, in which case they are defined to have the value 1.) Then if $\phi(y)=\phi(z)$ and $y\neq z$, we have

$$\prod p_i^{a_i-1}\mid\prod(q_j-1)<\prod q_j^{b_j-1}\mid\prod(p_i-1)<\prod p_i^{a_i-1},$$

where the inequalities are justified by the fact that all the exponents are $\geqq 2$. This clearly is a contradiction, so we must have $y=z$ and the proof is complete.

Proof of (4). This is an immediate consequence of results of Erdös, who has shown (Quart. Journ. Math., vol. 7, 1936, pp. 16 and 227) that the number of solutions of $\phi(x)=n$ which are products of two (three) distinct primes is unbounded as $n\to\infty$.

SOME EQUATIONS INVOLVING EULER'S TOTIENT FUNCTION

Leo Moser, University of Manitoba

We give some new results concerning certain equations involving $\phi(n)$ which have previously been discussed by V. L. Klee, Jr. [1].

The equation $\phi(n)=\phi(n+1)$ has the following solutions for $n<10{,}000$: $n=1$, 3, 15, 104, 164, 194, 255, 495, 584, 975, 2204, 2625, 2834, 3255, 3705, 5186, 5187. The last four entries are new and were found from Glaisher's table [2]. Klee noted that for $3<n<3000$ the odd numbers of n and $n+1$ are divisible by 15 but $n=5187$ shows that this is not always the case. Note that $n=5186$ is a solution of $\phi(n)=\phi(n+1)=\phi(n+2)$. P. Erdös mentioned the existence of such a solution in [3] and conjectured that for every k, $\phi(n)=\phi(n+1)=\phi(n+2)=\cdots=\phi(n+k)$ is solvable.

The equation $\phi(n)=\phi(n+2)$ is satisfied by $n=2(2p-1)$ if both p and $2p-1$ are odd primes, and by $n=2^{2^a+1}$ if $2^{2^a}+1$ is a Fermat prime. Other solutions with $n<10{,}000$ found from [2] are: $n=7$, 70, 308, 572, 635, 728, 910, 1015, 1330, 2132, 2170, 2590, 2695, 4292, 4338, 5950, 9100. Five of the entries under 3000 were overlooked in [1] while all those over 3000 are new.

The equation $\phi(n)+2=\phi(n+2)$ is satisfied if n and $n+2$ are primes, if n has the form $4p$ where p and $2p+1$ are primes, and if $n=2M_p$ where M_p is a Mersenne prime $M_p=2^p-1$. This last class of solutions was overlooked in [1]. We can show that there are no other solutions with $n<100{,}000$. To do this we make use of the following three lemmas:

1. *If $\phi(n)+2=\phi(n+2)$ then at least one of n and $n+2$ is of the form p^α or $2p^\alpha$ where p is a prime of the form $4r+3$.*

Proof: If neither n nor $n+2$ is of this form then $\phi(n)$ and $\phi(n+2)$ would both be divisible by 4, so that their difference could not be 2.

2. *The case* $\alpha=1$ *leads to the classes of solutions mentioned above.*

Proof: If n is a prime then $\phi(n)=n-1$, while if n is composite then n has a prime factor $\leqq\sqrt{n}$, so that $\phi(n)\leqq n(1-1/\sqrt{n})=n-\sqrt{n}$. Hence if one of n and $n+2$ is prime so is the other. If $n=2p$, then $\phi(n)=p-1=(n-2)/2$, so that $\phi(n)+2=\phi(n+2)$ would imply $\phi(n+2)=(n+2)/2$, in which case $n+2$ is clearly a power of 2. If $n+2=2p'$ then $\phi(n+2)=p'-1=n/2$, so that $\phi(n)=n/2-2$, and n must be of the form $4p$.

3. *We have* $\alpha\neq 2$.

Proof: If $n=p^2$, then $\phi(n)=n-\sqrt{n}$, while if $n+2$ is composite (but clearly not a square), then $\phi(n+2)<(n+2)(1-1/\sqrt{n})^2<\phi(n)+2$. Similarly we can dispose of the cases $n=2p^2$, $n+2=p^2$ and $n+2=2p^2$.

This leaves relatively few numbers $<10^6$ to be examined and these can be tested directly.

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CONGRUENCES FOR SETS OF PRIMES

P. A. CLEMENT, University of California, Los Angeles

1. Introduction. Wilson's function $P_1(n)$ is the function $P_1(n)\equiv(n-1)!+1$. By Wilson's theorem the condition $P_1(n)\equiv 0 \bmod n$ is necessary and sufficient in order that an integer $n>1$ be prime. In this note we find a congruence condition, similar to the above, for twin primality, and we indicate a method which furnishes a condition for sets of prime numbers of any prescribed type.

2. Twin primes. We shall establish the following result:

THEOREM. *A necessary and sufficient condition that two integers,* n *and* $n+2$, $n>1$, *both be prime is that*

$$4[(n-1)!+1]+n\equiv 0 \bmod n(n+2). \tag{1}$$

Proof. The sufficiency is obvious as divisions by n and $n+2$ separately reduce either to Wilson's theorem or to a simple modification of it.

The necessity follows as easily, but we wish to indicate how (1) may be obtain directly. Thus, with n and $n+2$ both primes, we have

$$(n-1)!+1\equiv 0 \bmod n, \tag{2}$$

$$(n+1)!+1\equiv 0 \bmod (n+2). \tag{3}$$

Reducing the factorial of (3) mod $(n+2)$ and rewriting as an equation we obtain

$$2[(n-1)!]+1 = k(n+2), \qquad k \text{ some integer;} \tag{4}$$

then, using (2), we must have

$$2k+1 \equiv 0 \bmod n. \tag{5}$$

Substitution of (5) in (4) determines the congruence of the theorem.

It may be noted that if 1 is considered the first prime, then the restriction $n>1$ can be deleted from the above theorem.

3. Further congruences. By analogous procedure, now using (1), we find that three positive integers n, $n+2$ and $n+6$, are a prime triple if and only if

$$4320[4(\overline{n-1}!+1)+n]+361n(n+2) \equiv 0 \bmod n(n+2)(n+6). \tag{6}$$

As stated, 1 is admitted as the first prime; if desired this may be obviated by requiring $n>1$. A similar congruence may be obtained for the other possible class of prime triples given by integers n, $n+4$, and $n+6$.

We indicate a less laborious method than that of the theorem for obtaining (6). By a modification of Wilson's theorem, $n+6$ is prime if and only if

$$720(n-1)!+1 \equiv 0 \bmod (n+6). \tag{7}$$

Then using (1) we write

$$A[4(\overline{n-1}!+1)+n]+Bn(n+2) \equiv 0 \bmod n(n+2)(n+6),$$

and seek integers A and B so that this congruence mod $(n+6)$ reduces to a multiple of (7). This gives (5) immediately, and the process can be applied in this recursive fashion to prime sets of any prescribed type.

4. Prime quadruples. Let $P_2(n)$ be the function on the left of (1), and $P_3(n)$ be the left side of (6). We then have

$$P_2(n) = 4P_1(n)+n,$$

and

$$P_3(n) = 4320P_2(n)+361n(n+2).$$

The four positive integers n, $n+2$, $n+6$, $n+8$ may each be prime, the set then being a prime quadruple consisting of two sets of twin primes. For the function associated with this set, $P_4(n)$, we find

$$P_4(n) = 224P_3(n)+111n(n+2)(n+6).$$

The congruence condition

$$P_4(n) \equiv 0 \bmod n(n+2)(n+6)(n+8)$$

is necessary and sufficient for the set to be a prime quadruple. By (1) a like condition is presented by the two congruences

$$P_2(n) \equiv 0 \bmod n(n+2)$$
$$P_2(n+5) \equiv 0 \bmod (n+6)(n+8).$$

As an exercise one might show that these two sets of conditions actually are equivalent.

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College and Institute for Advanced Study

All material for this department should be sent to C. B. Allendoerfer, Institute for Advanced Study, Princeton, New Jersey.

LOGARITHMIC INTEGRATION*

H. F. MACNEISH, University of Miami

1. Introduction. Logarithmic differentiation is a device by means of which complicated products, quotients, and exponential functions may be differentiated with much less algebraic manipulation than is required by the use of the standard formula. We recall the rule for logarithmic differentiation:

$$\frac{dU(x)}{dx} = U(x)\frac{d}{dx}\ln U(x) \tag{1}$$

Applying this to a numerical example we have:

$$\frac{d}{dx}\frac{\sqrt{x^2+1}}{\sqrt[3]{x^3+1}} = \frac{x-x^2}{\sqrt{x^2+1}\,(x^3+1)^{4/3}}. \tag{2}$$

The integration of the answer, however, cannot be accomplished by standard methods. By integration we mean, as usual, the expression of the integral in a finite number of terms containing only elementary functions. In this note we outline a method for integrating certain expressions of this form, and we call the method "logarithmic integration."

2. Case I. Here we are concerned with a method of integrating certain ex-

* This paper was delivered before the New Haven meeting of the Association in September 1947 as a portion of the symposium "How to Solve it."

pressions of the type: $f(x) = [F_1(x)]^{p_1}[F_2(x)]^{p_2}R(x)$ where $F_1(x)$, $F_2(x)$, and $R(x)$ are polynomials and p_1 and p_2 are non-integral rational numbers. The extension to more factors $F_i(x)$ will be apparent. According to a theorem of Abel: If $\int f(x)dx$ is algebraic in x, then it is rational in $f(x)$ and x. (See J. F. Ritt "*Integration in Finite Terms*" p. 31.) This suggests that we consider as the simplest possibility:

$$\int f(x)dx = k[F_1]^{p_1+1}[F_2]^{p_2+1} + c \tag{3}$$

where k and c are constants.

Using logarithmic differentiation we have that:

$$f(x) = k[F_1]^{p_1}[F_2]^{p_2}\{(p_1 + 1)F_1'F_2 + (p_2 + 1)F_1F_2'\}.$$

Hence formula (3) holds if:

$$R(x) = k\{(p_1 + 1)F_1'F_2 + (p_2 + 1)F_1F_2'\}. \tag{4}$$

Consequently if (4) holds:

$$\int [F_1]^{p_1}[F_2]^{p_2}Rdx = k[F_1]^{p_1+1}[F_2]^{p_2+1} + c. \tag{5}$$

Let us apply this to the example:

$$I = \int \frac{x^2 + 4x - 3}{(x + 1)^{1/2}(x^2 + 1)^{4/3}}\,dx$$

$$F_1 = x + 1; \qquad F_2 = x^2 + 1; \qquad R = x^2 + 4x - 3$$

$$p_1 = -1/2; \qquad p_2 = -4/3$$

$$(p_1 + 1)F_1'F_2 + (p_2 + 1)F_1F_2' = \frac{-x^2 - 4x + 3}{6}.$$

Hence (4) is satisfied with $k = -6$. Therefore

$$I = \frac{-6\sqrt{x + 1}}{\sqrt[3]{x^2 + 1}} + c.$$

If condition (4) fails to hold, the integration of $f(x)$ is not necessarily impossible, even as an algebraic function. For other rational combinations of x and $f(x)$ may be tried; for example consider expressions of the form $Q(x)[F_1]^{p_1+1}[F_2]^{p_2+1}$ where $Q(x)$ is a polynomial. The general question of the existence and determination of such an algebraic integral has been solved by Liouville (see Ritt, p. 32), but we shall not enter into this matter here. The question as to whether $\int f(x)dx$ is expressible in terms of elementary functions other than algebraic ones in an unsolved problem.

3. Case II. As a further illustration of this method let us consider the integral of $g(x)$ where $g(x) = [F(x)]^p R(x)$, $F(x)$ and $R(x)$ being polynomials and p a non-integral fraction. We assume here that the integral has the form:

$$\int g(x)dx = (ax + b)[F(x)]^{p+1} + c \tag{6}$$

where a, b, and c are constants. Differentiating we obtain:

$$[F(x)]^p R(x) = [F(x)]^p \{aF(x) + (p + 1)(ax + b)F'(x)\}. \tag{7}$$

Hence formula (6) works if we can find a and b such that:

$$R(x) = aF(x) + (p + 1)(ax + b)F'(x). \tag{8}$$

We illustrate with a numerical example:

$$I = \int \frac{5x^3 - 3x^2 + 6x - 3}{\sqrt{2x^3 + 4x - 1}} dx.$$

Then we must find a and b such that:

$$\begin{aligned} 5x^3 - 3x^2 + 6x - 3 &= a[2x^3 + 4x - 1] + \tfrac{1}{2}(ax + b)(6x^2 + 4) \\ &= 5ax^3 + 3bx^2 + 6ax + 2b - a. \end{aligned}$$

This is consistent in a and b, giving $a = 1$ and $b = -1$. Hence:

$$I = (x - 1)\sqrt{2x^3 + 4x - 1} + c.$$

SOME INTEGRAL FORMULAS

H. W. Smith, Oklahoma A. and M. College

When we orthonormalize the set of harmonic functions $r^n \cos n\theta$, $r^n \sin n\theta$ over the square of side 2 with center at the origin and sides parallel to the coordinate axes, integrals of the type

$$\int \sec^k \theta \cos p\theta d\theta, \qquad \int \sec^k \theta \sin p\theta d\theta$$

$$\int \csc^k \theta \cos p\theta d\theta, \qquad \int \csc^k \theta \sin p\theta d\theta$$

with appropriate limits must be computed.

Formulas 369–372 of Pierce's tables may be used as reduction formulas for these integrals, but they are given there as the sum of two other integrals of the same form.

To obtain reduction formulas that involve only one integral in the right members for the first two of these we integrate

$$\int \sec^k \theta e^{ip\theta} d\theta$$

by parts, to obtain

$$(1) \qquad \int \sec^k \theta e^{ip\theta} d\theta = 1/ip \sec^k \theta e^{ip\theta} - k/ip \int \sec^k \theta \tan \theta e^{ip\theta} d\theta.$$

Integrating by parts again we find

$$\int \sec^k \theta \tan \theta e^{ip\theta} d\theta = 1/ip \sec^k \theta \tan \theta e^{ip\theta} - \frac{k+1}{ip} \int \sec^{k+2} \theta e^{ip\theta} d\theta$$
$$+ k/ip \int \sec^k \theta e^{ip\theta} d\theta.$$

Substitute this result into the right member of (1), solve for $\int \sec^{k+2} \theta e^{ip\theta} d\theta$ and separate the real and imaginary parts to obtain:

$$\int \sec^{k+2} \theta \cos p\theta d\theta = \frac{\sec^k \theta}{k(k+1)} [k \tan \theta \cos p\theta + p \sin p\theta]$$
$$+ \frac{k^2 - p^2}{k(k+1)} \int \sec^k \theta \cos p\theta d\theta.$$

$$\int \sec^{k+2} \theta \sin p\theta d\theta = \frac{\sec^k \theta}{k(k+1)} [k \tan \theta \sin p\theta - p \cos p\theta]$$
$$+ \frac{k^2 - p^2}{k(k+1)} \int \sec^k \theta \sin p\theta d\theta.$$

The last two integrals can be found in the samé way and are:

$$\int \csc^{k+2} \theta \cos p\theta d\theta = \frac{-\csc^k \theta}{k(k+1)} [k \cot \theta \cos p\theta - p \sin p\theta]$$
$$+ \frac{k^2 - p^2}{k(k+1)} \int \csc^k \theta \cos p\theta d\theta.$$

$$\int \csc^{k+2} \theta \sin p\theta d\theta = \frac{-\csc^k \theta}{k(k+1)} [k \cot \theta \sin p\theta + p \cos p\theta]$$
$$+ \frac{k^2 - p^2}{k(k+1)} \int \csc^k \theta \sin p\theta d\theta.$$

The case appearing most frequently in the particular problem above is that in which $k=p=2n$, so that in this case the coefficient of the integral on the right is zero.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 844. *Proposed by Orrin Frink, Pennsylvania State College*

Sum the series

$$1 + 1/5! + 1/10! + 1/15! + \cdots + 1/(5n-5)! + \cdots .$$

E 845. *Proposed by Joseph Rosenbaum, Hartford, Connecticut*

It is required to write the fifteen combinations of a, b, c, d in a sequence such that any two adjacent terms of the sequence shall differ by a single letter. How many such sequences are there? How can they be written down?

E 846. *Proposed by H. J. Hamilton, Pomona College*

The following is typical of many characterizations of the principal part of an infinitesimal which are to be found in elementary calculus texts.

"If an infinitesimal consists of two or more terms of different orders, the term of lowest order is called the *principal part* of the infinitesimal."

Show that this is not definitive and give a valid definition.

E 847. *Proposed by Albert Newhouse, University of Houston*

Let a, b, A be the given parts of a triangle in the ambiguous case. Show that the area of the triangle is given by

$$K = \tfrac{1}{2}b^2 \sin A\left[\cos A \pm (a^2 - b^2 \sin^2 A)^{1/2}\right].$$

E 848. *Proposed by Leo Moser, University of Manitoba*

Prove that every integer greater than 10^5 can be expressed as the sum of two abundant numbers.

E 849. *Proposed by C. W. Trigg, Los Angeles City College*

The area of a triangle is to the area of the triangle determined by the points of contact of its incircle (or excircle) as its circumdiameter is to its inradius (or exradius).

E 850. *Proposed by L. C. Hsu, National Tsing-Hua University, Peiping, China*

Three persons, A, B, and C, in rotation throw a pair of dice. If the points 6, 7, 8 are consecutively gotten by A, B, C, then A is declared winner; if the points 7, 8, 6 are consecutively gotten by B, C, A, then B is declared winner; if

the points 8, 6, 7 are consecutively gotten by C, A, B, then C is declared winner. Find their chances of winning.

SOLUTIONS

Pythagorean Triangles with Equal Perimeters

E 812 [1948, 248]. *Proposed by Monte Dernham, San Francisco, California*

Find the shortest perimeter common to two different primitive Pythagorean triangles.

Solution by Fritz Herzog, Michigan State College. To each primitive Pythagorean triangle corresponds bi-uniquely a pair of integers s, t with $s>t>0$, $(s, t)=1$, and $s\not\equiv t \pmod 2$, such that the sides of the triangle are $2st$, s^2-t^2, s^2+t^2. (See Hardy and Wright, *An Introduction to the Theory of Numbers*, Theorem 225, p. 189.) The perimeter of the triangle is $2su$, where $u=s+t$, and the above conditions on s and t read, in terms of s and u:

$$(1) \qquad s < u < 2s; \qquad (s, u) = 1; \qquad u \equiv 1 \pmod 2.$$

The problem is, therefore, to find the smallest positive integer y which admits of two different representations $y=su$, where s and u satisfy (1). Let $y=su$ and $y=s'u'$ be two such representations and assume that $u<u'$ and hence $s>s'$. Let c/d be the reduced form of the fraction $u/u'=s'/s$ and write $u=bc$, $u'=bd$, $s=ad$, $s'=ac$. The conditions (1) on s, u and s', u' together with $u<u'$ are easily shown to be equivalent to the following conditions on the four positive integers a, b, c, d:

$$(2) \qquad d/2c < a/b < c/d < 1,$$

$$(3) \qquad a,\ b,\ c,\ d \text{ mutually relatively prime},$$

$$(4) \qquad b \equiv c \equiv d \equiv 1 \pmod 2.$$

Since $y=abcd$ it remains to solve the inequality (2) with the restrictions (3) and (4) and such that $y=abcd$ is a minimum.

We first conclude from (2) that $1/2<a/b<1$. Hence the values of a and b which give the smallest possible value of ab are $a=2$, $b=3$. Let a and b be so chosen. Then from (2), $3/4<c/d<1$ and, in view of (3) and (4), the values of c and d which give the smallest possible value of cd (after the choice $a=2$, $b=3$) are $c=11$, $d=13$. We thus arrive at the solution $a=2$, $b=3$, $c=11$, $d=13$, which yields $y=858$.

To show that there is no solution of (2), (3), and (4) which would result in a smaller value of y, we observe first that, by (2), $1/2<c^2/d^2<1$. The only values of c and d satisfying this inequality as well as (3) and (4) and yielding a value cd smaller than $11\cdot 13$ are (i) $c=9$, $d=11$, (ii) $c=7$, $d=9$, and (iii) $c=5$, $d=7$. However, as was shown above, in these three cases b could not equal 3 so that, by (3) and (4) $b\geqq 5$ in (i) and (ii) and $b\geqq 9$ in (iii). Thus by (2) we obtain for all

three of the above choices of c and d

$$y = abcd > (bd/2c)bcd = (bd)^2/2 \geqq (9\cdot 5)^2/2 > 858.$$

Thus the smallest number having the property required in the proposed problem is $2\cdot 858=1716$, which is the perimeter of the primitive Pythagorean triangles 364, 627, 725 and 748, 195, 773.

Also solved by W. E. Buker, William Douglas, L. S. Kennison, Sidney Kravitz, Roger Lessard, D. W. Matlack, F. L. Miksa, and the proposer.

Several solvers resorted to more or less laboriously constructed tables. Miksa found thirty-six perimeters, less than 20000, which are common to two different primitive Pythagorean triangles. He also found the perimeter 14280 as the only one, less than 20000, which is common to three primitive Pythagorean triangles. C. S. Ogilvy found 240 as the smallest perimeter common to two non-primitive Pythagorean triangles. This result was also given by Buker who, in addition, found 1680 as the smallest perimeter common to three non-primitive Pythagorean triangles.

Editorial Note. For other problems involving Pythagorean triangles see E 18, E 67, E 73, E 283, E 324, E 327, E 380, and E 828.

Magic Square as a Determinant

E 813 [1948, 248]. *Proposed by C. W. Trigg, Los Angeles City College*

Let S be the sum of the integer elements of a magic square of order three, and let D be the value of the square considered as a determinant. Show that D/S is an integer.

I. *Solution by R. J. Walker, Cornell University.* Let

$$\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}$$

have the magic sum $N=S/3$. Then

$$\begin{aligned} N = (a+e+i) + (d+e+f) + (g+e+c) \\ - (a+d+g) - (c+f+i) = 3e, \end{aligned}$$

and $S=9e$. Hence, adding rows and columns,

$$D = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ 3e & 3e & 3e \end{vmatrix} = \begin{vmatrix} a & b & 3e \\ d & e & 3e \\ 3e & 3e & 9e \end{vmatrix} = \begin{vmatrix} a & b & e \\ d & e & e \\ 1 & 1 & 1 \end{vmatrix} S.$$

II. *Solution by Monte Dernham, San Francisco.* Every magic square of the third order may be written in a form equivalent to

$$\begin{array}{lll} m+x & m-x-y & m+y \\ m-x+y & \quad m & m+x-y \\ m-y & m+x+y & m-x \end{array}$$

(See Kraitchik, *Mathematical Recreations*, p. 148.) Considering the square as a determinant, expanding and simplifying, we find its value to be

$$9m(y^2 - x^2);$$

and, since $S=9m$, and all elements are integers, D/S is an integer.

If the square be transformed by rotation, the foregoing result remains unaffected; likewise, if it be transformed by reflection in a mirror, except that in the latter case x^2 and y^2 are interchanged.

III. *Remarks by the Proposer.* The property $D/S=I$, an integer, may be extended to magic squares of higher orders with certain restrictions.

In the absence of a qualifying adjective, such as pandiagonal, semi-nasik, multiplicative, *etc.*, there is a certain looseness in the use of the term "magic square." Sometimes the term is defined as a square array of integers with the property that the elements of each row, of each column, and of each of the principal diagonals have equals sums (*cf.* E 791). As shown by the preceding proofs, $D/S=I$ for all third order squares of this broad type.

An nth order square, for which $D/S=I$, may have this property spoiled without loss of its magical nature by adding the proper integer to each of n properly chosen elements. Let the coordinates of an element be determined by its row and column. Then, if n is even, the square may be spoiled by adding an integer x to the n elements $(1, 1)$, $(2, n)$, $(3, 2)$ $(4, 2)$, $\cdots$, $(n, n-1)$. If $n>3$ is odd, the square may be spoiled by adding x to the n elements $(1, 1)$, $(2, n)$, $(3, n-1)$, $(4, 2)$, $(5, 3)$, $\cdots$, $(n, n-2)$. For example, consider the squares:

$$\begin{array}{rrrr} \underline{15} & 6 & 9 & 4 \\ 2 & 13 & 8 & \underline{11} \\ 7 & \underline{12} & 1 & 14 \\ 10 & 3 & \underline{16} & 5 \end{array} \qquad \begin{array}{rrrrr} \underline{17} & 24 & 1 & 8 & 15 \\ 23 & 5 & 7 & 14 & \underline{16} \\ 4 & 6 & 13 & \underline{20} & 22 \\ 10 & \underline{12} & 19 & 21 & 3 \\ 11 & 18 & \underline{25} & 2 & 9 \end{array}$$

The fourth order square has $D/S=-96$. When x is added to each of the underlined elements the derived square is still magic, but now

$$D/S = (x + 16)(x^2 + 4x - 24)/4,$$

which is an integer only if x is even. The fifth order square has $D/S=15600$. When x is added to each of the underlined elements we find

$$D/S = (x^4 + 25x^3 + 60x^2 - 3475x - 78000)/5,$$

which is an integer only when x is a multiple of 5. It may be conjectured that if x is prime to n, D/S, is not an integer.

More frequently, an nth order "magic square" is defined as above with the restriction that its elements are the first n^2 integers. Occasionally this restriction is weakened to include any n^2 integers in arithmetic progression. Under this latter condition D/S may be examined more profitably. Let the difference in the arithmetic progression be d, and let Y be the sum of the elements of any row or column. Then

$$S = nY = (n^2/2)[2a + (n^2 - 1)d],$$

whence

$$Y = (n/2)[2a + (n^2 - 1)d] = n[a + (d/2)(n^2 - 1)].$$

If n is odd, n^2-1 is even, and $Y=nk$, where k is an integer. If n is even and d is even, then again $Y=nk$. In the determinant add the elements of the first $n-1$ columns to those of the nth column. In the derived determinant add the elements of the first $n-1$ rows to those of the nth row. Every element of the nth row and nth column is now $Y(=nk)$ except the common element of this row and column, which is nY. Hence Y and n may be factored out, so that $D=nYD'$ and $D/S=D'$, which is an integer since the elements of the determinant D' are integers. It will be observed that no use has been made of the sum of the diagonals in the proof.

The case where n is even and d is odd offers more difficulty, for then $Y=nk/2$, where k is odd, so that

$$D = \begin{vmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} = \begin{vmatrix} \cdot & \cdot & \cdot & Y \\ \cdot & \cdot & \cdot & Y \\ \cdot & \cdot & \cdot & Y \\ Y & Y & Y & nY \end{vmatrix} = (Yn/2) \begin{vmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 \\ k & k & k & 2 \end{vmatrix} = YnD'/2.$$

Then, in order that D/S may be an integer, D' must be even.

In the evaluation of these determinants the rows may be interchanged at random, as may be the columns, since only the absolute value is of concern. Since n is even and d is odd, there will be $n^2/2$ even integers (e) and $n^2/2$ odd integers (o) in the array. If in any pair of columns (or rows) the odd and even integers occurs in the same sequence we shall say that the pair is "matched," *e.g.*,

$$\begin{matrix} o & o & e & o & e & e & o & e \\ o & o & e & o & e & e & o & e \end{matrix}.$$

If in any pair of columns (or rows) the odd and even integers occur completely

out of sequence we shall say that the pair is "mismatched," *e.g.*,

$$\begin{matrix} o & o & e & o & e & e & o & e \\ e & e & o & e & o & o & e & o \end{matrix}.$$

If in a pair of columns (or rows) reference is made to two integers in the same row (or column) they will be called a "couple."

If the pair under consideration are rows, interchange the columns and rows. If a pair of columns are matched or matched except for a single couple, rearrange the columns so that neither of the pair is the nth column. Rearrange the rows so that the exceptional couple falls in the nth row. Convert D into $YnD'/2$. In D' add the elements of one of the pair to the elements of the other of the pair, which will then contain all even elements, so D' is even. (All even-ordered magic squares which I have examined fall in this category.)

If the pair is mismatched or mismatched except for a single couple, prepare the determinant as in the case of the matched pair. Convert D into $YnD'/2$. In D' successively add each column of the pair to the nth column, which will then contain all even elements, so D' is even.

It may be noted further that if n is of the form $4m$, then Y is even, so in each of the columns, rows, and diagonals the odd and even elements are each even in number. If n is of the form $4m+2$, then Y is odd, so each column, row, and diagonal contains an odd number of odd elements.

When $n=4$, all elements of a particular row or column will be like or there will be two odd and two even elements. There are 6 permutations of *eeoo*, and 15 combinations of distinct permutations taken 4 at a time. (No combination containing two like permutations need be considered since they constitute a matched pair for which D' is even.) Only three of these combinations contain an even number of odd and an even number of even elements in every column. Hence, by interchanging rows, all possible squares devoid of matched pairs may be converted into one of the following:

$$\begin{matrix} e & e & e & e \\ o & o & o & o \\ e & e & o & o \\ o & o & e & e \end{matrix} \qquad \begin{matrix} e & e & o & o \\ e & o & e & o \\ o & e & o & e \\ o & o & e & e \end{matrix} \longrightarrow \begin{matrix} e & e & o & o \\ e & o & o & e \\ o & e & e & o \\ o & o & e & e \end{matrix} \longrightarrow \begin{matrix} e & o & e & o \\ e & o & o & e \\ o & e & e & o \\ o & e & o & e \end{matrix}$$

Each of these squares contains a mismatched pair, so D' is even and $D/S=I$ for all fourth order squares composed of 16 elements in arithmetic progression. (This fact has been established in another way by this writer in a note on *Determinants of fourth order magic squares*, this MONTHLY, November, 1948.) Since the sum of the elements of the diagonals has not been employed, the squares need be magical only in their columns and rows.

When $n=6$, an arrangement is possible which may be magical in rows, columns, and diagonals, although I have no numerical example of this, namely:

$$\begin{matrix} o & e & e & o & o & e \\ o & o & e & e & o & e \\ e & o & o & e & o & e \\ o & e & o & e & e & o \\ e & o & e & e & o & o \\ e & e & o & e & o & o \end{matrix}$$

When this is converted into $YnD'/2$ and D' is expanded by minors, D' is found to be odd. Hence this approach will not confirm the integer nature of D/S for magic squares with $n>4$ and even and d odd without calling upon some of the other properties of these magic squares.

Also solved by R. V. Andree, W. G. Brady, J. C. Eaves, R. W. Frankel, R. E. Harton, Roger Lessard, N. S. Mendelsohn, Leo Moser, S. T. Parker, and T. L. Reynolds. Several solvers gave partial generalizations.

Location of a Minimum Area

E 814 [1948, 248]. *Proposed by Sidney Kravitz, New York, New York*

Given the curve $y=e^x/x$. Consider all areas under the curve, over the x-axis, and between two ordinates one unit apart. Locate the boundary lines of the area which is a minimum.

Solution by D. W. Matlack, North American Aviation, Inc. Let the area under consideration be A. Then

$$A = \int_x^{x+1} y dx,$$

and

$$dA/dx = y\Big|_x^{x+1} = \frac{e^{x+1}}{x+1} - \frac{e^x}{x}.$$

Setting $dA/dx=0$ we find $x=1/(e-1)$, $x+1=e/(e-1)$.

Also solved by W. G. Brady, Karl Itkin, J. N. P. Lawrence, Roger Lessard, Julius Liebleim, Leo Moser, C. S. Oglivy, S. T. Parker, and the proposer.

Ogilvy stated the general theorem: *Of all areas bounded by the curve $y=f(x)$, the x-axis, and two ordinates n units apart, the maximum (or minimum) is that whose left-hand ordinate is the solution of x of the equation $f(x+n)-f(x)=0$.* It is interesting to note that no matter how unsymmetrical or irregular the curve may be between x and $x+n$, the two bounding ordinates of the required area are of equal length.

J. N. Eastham pointed out that this problem occurs as Ex. 12, p. 249, in Reddick and Miller, *Advanced Mathematics for Engineers*, 2nd ed.

Several solvers showed that the minimum of the given curve occurs at $x=1$.

The area in question is approximately equal to 2.83 square units.

Marble in Bowl

E 815 [1948, 248]. *Proposed by J. S. Miller, Dillard University*

A marble rolls in a hemispherical bowl. Find its period.

Solution by J. C. Miller, University of California, Berkeley. Denote the radius of the bowl by R, the radius and mass of the marble by r and m respectively, and the acceleration of gravity by g. Then in spherical coordinates, with the pole directed downwards, the kinetic and potential energies T and V are

$$T = (7/10)m(R - r)(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta)$$

and

$$V = mg(R - r)(1 - \cos \theta).$$

The Lagrangian equations of motion are thus

$$\ddot{\theta} - \dot{\phi}^2 \sin \theta \cos \theta + [5g/7(R - r)] \sin \theta = 0 \tag{1}$$

and

$$d(\dot{\phi} \sin^2 \theta)/dt = 0. \tag{2}$$

Equation (2) integrates immediately to give

$$\dot{\phi} \sin^2 \theta = \text{constant} = a.$$

Substitution of this solution in equation (1) gives a first integral

$$\dot{\theta}^2 + a^2/\sin^2 \theta - [10g/7(R - r)] \cos \theta = \text{constant} = b. \tag{3}$$

To avoid elliptic integrals keep θ small and replace $\sin \theta$ by θ, and $\cos \theta$ by $1-\theta^2/2$. Equation (3) may then be integrated for time to give

$$t = \int_0^\theta \theta(- a^2 + c\theta^2 - \omega^2\theta^4)^{-1/2} d\theta,$$

where

$$c = a^2/\theta_0^2 + \omega^2\theta_0^2 \quad \text{and} \quad \omega^2 = 5g/7(R - r),$$

provided the motion starts from rest at $\theta=\theta_0$. The period Ω will be the time required for θ to go from θ_0 to $-\theta_0$ and back, so that

$$\begin{aligned}
\Omega &= 4\int_0^{\theta_0} \theta(- a^2 + c\theta^2 - \omega^2\theta^4)^{-1/2} d\theta \\
&= (2/\omega) \arcsin \left[(2\omega^2\theta^2 - c)(c^2 - 4a^2\omega^2)^{-1/2}\right]\Big|_0^{\theta_0} \\
&= \pi/\omega - (2/\omega) \arcsin \left[c(c^2 - 4a^2\omega^2)^{-1/2}\right].
\end{aligned}$$

For $\dot{\phi}=0$, *i.e.* $a=0$, the period will be

$$2\pi/\omega - 2\pi[7(R-r)/5g]^{1/2},$$

which is the time for one complete oscillation in a plane through the polar axis. For large θ the time integral will be an elliptic integral of the first kind.

Also solved by Roger Lessard, who pointed out that this problem may be found in Timoshenko and Young, *Engineering Mechanics*, problem 534, page 463. It is there shown that the system is equivalent to a simple pendulum whose length is $7(R-r)/5$.

Editorial Note: This problem is not to be confused with that where a particle *slides* from the edge to the bottom of a smooth hemispherical bowl of radius R. This latter problem furnishes a nice application of the gamma function. The time of descent of the particle is

$$t = [\Gamma(1/4)/\Gamma(3/4)][\pi R/8g]^{1/2}.$$

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4325. *Proposed by Orrin Frink, Pennsylvania State College*

Show that on every simple closed plane curve there are four points which are the vertices of a square.

4326. *Proposed by R. J. Walker, Cornell University*

Prove or disprove: 128 is the only power of 2 of two or more digits each of which is a power of 2.

4327. *Proposed by R. J. Walker, Cornell University*

In attempting to solve the preceding problem we tried to show that for some positive integer r no power of 2 had its last r digits all powers of 2. Show that this attempt had to fail; in particular, show that for each r there exist powers of 2 each of whose last r digits is either 1 or 2.

4328. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Given a triangle ABC whose altitudes are AA', BB', CC'. Prove that the Euler lines of the triangles $AB'C'$, $BC'A'$, $CA'B'$ are concurrent on the nine-point circle at a point P which is such that one of the distances PA', PB', PC' equals the sum of the other two.

4329. *Proposed by A. W. Goodman, Rutgers University*

Let θ be an irrational number, $a=e^{i\theta\pi}$. Prove that

$$f(z) = \sum_{n=0}^{\infty} a^{n^2} z^n$$

has the unit circle as a natural boundary.

4330. *Proposed by Paul Erdös, Syracuse University*

Let $a_1 < a_2 < \cdots$ be an infinite sequence of integers. Prove that there exists either an infinite subsequence in which no integer divides another or an infinite subsequence where each integer is a multiple of the preceding one.

SOLUTIONS

A Permutation Problem

3731 [1935, 255]. *Proposed by Raphael Robinson, University of California at Berkeley*

In how many ways can a_1 1's, a_2 2's, $\cdots$, a_n n's be arranged, so that in reading from the beginning, none of the $(k+1)$'s are reached until at least one of the k's has been reached?

Solution by W. D. Smith, Student, Rutgers University. All desired arrangements may be obtained in the following way: (A) place a 1 first and put the other 1's in any selection of a_1-1 out of the $a_1+a_2+\cdots+a_n-1$ remaining places; then (B) place a 2 in the first remaining vacant place and put the other 2's in any selection out of the $a_2+a_3+\cdots+a_n-1$ remaining places; then (C) continue in the same fashion with the 3's, 4's, $\cdots$ until the $(n-1)$'s have been placed; and finally (D), the a_n n's will fit into the remaining a_n places in just one way. The number of ways in which each step can be taken is clearly:

$$\text{(A)} \quad \binom{a_1+a_2+\cdots+a_n-1}{a_1-1}, \qquad \text{(B)} \quad \binom{a_2+a_3+\cdots+a_n-1}{a_2-1},$$

$$\text{(C)} \quad \binom{a_3+\cdots+a_n-1}{a_3-1}, \cdots, \binom{a_{n-1}+a_n-1}{a_{n-1}-1},$$

where $\binom{x}{y}$ is the number of combinations of x things taken y at a time. Their product is the total number of ways of forming the desired arrangement:

$$\prod_{k=1}^{n} \binom{a_k+a_{k+1}+\cdots+a_n-1}{a_k-1}. \tag{1}$$

This result may also be put in the following form:

$$\frac{(a_1 + a_2 + \cdots + a_n)!}{\prod_{k=1}^{n} (a_k + a_{k+1} + \cdots + a_n)\cdot(a_k - 1)!}.$$

If instead of a_k k's $(k=1, 2, \cdots, n)$ we had been given a_k members of a kth class distinguishable one from another, the total number of ways would be given by (1) multiplied by $a_1!a_2! \cdots a_n!$.

A Locus of Radical Centers

4200 [1946, 225]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Given in a plane the triangle ABC and the fixed point P which is center of a variable circle (P). Find the locus of the radical center of the circles passing through A, B, C, respectively, which have with (P) the sides BC, CA, AB as radical axes. Consider the analogous problem for a tetrahedron and a sphere with fixed center, and show that the locus is a twisted cubic through the vertices and centroid of the tetrahedron.

*Solution by R. Bouvaist, Vincelles, Saône-et-Loire, France.** Let ABC be the triangle of reference in a system of trilinear coördinates. A circle (P) is given by

$$(P) \equiv (ayz + bxz + cxy) + K(ax + by + cz)(ux + vy + wz) + \lambda(ax + by + cz)^2 = 0.$$

A circle (A), having with (P) the radical axis AB, is given by

$$(P) + \mu x(ax + by + cz) = 0.$$

It goes through A if $Kau+\lambda a^2+\mu a=0$, whence $\mu=-(Ku+\lambda a)$ and

$$(A) \equiv (P) - (Ku + \lambda a)(x)(ax + by + cz).$$

The radical center of (A) and the analogous circles (B) and (C) is given by

$$(Ku + \lambda a)x = (Kv + \lambda b)y = (Kw + \lambda c)z.$$

It describes a conic whose equation is

$$\begin{vmatrix} u & a & 1/x \\ v & b & 1/y \\ w & c & 1/z \end{vmatrix} = 0$$

or

$$yz(cv - bw) + xz(aw - cu) + xy(bu - av) = 0.$$

This conic is easily seen to be circumscribed about the triangle ABC and to pass

* Translated from the French by O. J. Ramler, Catholic University of America, Washington, D. C.

through its center of gravity. It could be defined as follows:

If O is the center of the circle ABC, let U be the point in which a perpendicular to OP meets the ideal line; let V be the isogonal conjugate of U, then V is on the circle ABC; then the above conic is the triangular inverse of the trilinear polar of V.

In the case of a tetrahedron $ABCD$, let $(S)=0$ be the equation of the circumsphere and let $(\pi)\equiv Ax+By+Cz+Dt$. Then a sphere (P) is given by

$$(P) \equiv (S) + K(\pi)(ux + vy + wz + st) + \lambda(\pi)^2 = 0.$$

An arbitrary sphere, having with (P) the radical plane BCD is given by $(P)+\mu x(\pi)=0$. It goes through A if $\mu=-(Ku+\lambda A)$, where A is the area of BCD, B is the area of ACD, $\cdots$. The radical center of the four analogous spheres is given by

$$x(Ku + \lambda A) = y(Kv + \lambda B) = z(Kw + \lambda C) = t(Ks + \lambda D).$$

Its locus is the twisted cubic which is circumscribed about the given tetrahedron and which is the intersection of the cones

$$\begin{vmatrix} u & A & 1/x \\ v & B & 1/y \\ w & C & 1/z \end{vmatrix} = 0, \qquad \begin{vmatrix} u & A & 1/x \\ v & B & 1/y \\ s & D & 1/t \end{vmatrix} = 0.$$

This cubic goes through the center of gravity of the tetrahedron.

Divisibility of Factorial Numbers

4252. [1947, 287]. *Proposed by Paul Erdös, Syracuse University*

It is well known that $2n!/n!(n+1)!$ is always an integer. Prove that for every k there are infinitely many n's such that $2n!/n!(n+k)!$ is an integer.

Solution by Fritz Herzog, Michigan State College. Let k be a fixed integer greater than unity. For integral m and integral $q>1$, we put

$$F(m, q) = [2m/q] - [m/q] - [(m + k)/q], \tag{1}$$

where $[x]$ represents the greatest integer which does not exceed x. It then follows from a well known formula in number theory that $(2n)!/n!(n+k)!$, for integral $n \geqq 0$, is an integer if and only if

$$\sum_{a=1}^{\infty} F(n, p^a) \geqq 0 \tag{2}$$

for all primes p.

We shall need the following facts concerning $F(m, q)$. In the first place, we have obviously $F(m, q)=F(m', q)$ when $m\equiv m'$ (mod q). Secondly, we show that

(3) $$F(m, q) \geqq 0 \qquad \text{for } q \geqq 2k \text{ and for all } m.$$

By the above, it suffices to assume that $0 \leqq m < q$, so that $[m/q] = 0$. For $0 \leqq m < q-k$ we have $[2m/q] = 0$ or 1 and $[(m+k)/q] = 0$; for $q-k \leqq m < q$ we have $[2m/q] = 1$ and $([m+k)/q] = 1$. Thus (3) follows from (1). Thirdly we show that

(4) $$F(m, q) = 1 \qquad \text{for } q \geqq 2k + 2 \text{ and for } m \equiv -k - 1 \pmod q.$$

For we have $F(m, q) = F(-k-1, q)$ and it is easily verified that $[(-2k-2)/q] = [(-k-1)/q] = [(-1)/q] = -1$. Thus (4) follows from (1).

From (3) we conclude that (2) holds for all n when p is a prime greater than $2k$. Let p be a prime less than $2k$, let A be such that $p^{A+1} \geqq 2k$ and let $n \equiv -k - 1 \pmod{p^A}$. Then, by (3), $F(n, p^a) \geqq 0$ when $a > A$ and, by (4), $F(n, p^a) = 1$ when $2k+2 \leqq p^a \leqq p^A$. Thus we may choose A sufficiently large, with $n \equiv -k - 1 \pmod{p^A}$, so that there will be enough values of a for which $F(n, p^a) = 1$ to cancel out the effects of those values of a less than $2k+2$ for which perhaps $F(n, p^a) < 0$, whence (2) will hold for p, the particular prime under consideration. We now apply this process to each prime $p < 2k$, obtaining in each case an exponent $A = A(p)$. Let P be the product of all prime powers $p^{A(p)}$ thus obtained. Then (2) holds for $n \equiv -k-1 \pmod P$ and for all primes p. Hence $(2n)!/n!(n+k)!$ is an integer for at least every $n \equiv -k-1 \pmod P$.

Also solved by P. T. Bateman, Robert Breusch, R. H. Bruck, and N. J. Fine.

Editorial Note. The Proposer added the following remarks. (a) It can be shown that the values of n for which $(2n)!/n!(n+k)!$ is not an integer have density zero. More precisely, the number of $n \leqq x$ for which $(2n)!/n!(n+k)!$ is not an integer is less than $x^{1-\epsilon}$ for large x. (b) Balakran proved (Journal of the Indian Mathematical Society, vol. 1) that for infinitely many n, $(2n)!/(n+1)! \cdot (n+1)!$ is an integer. A more difficult problem is to decide whether there are infinitely many n for which $(2n)!/(n+2)!(n+2)!$ is an integer.

Limit of a Sequence

4255 [1947, 346]. *Proposed by G. Polya, Stanford University.*

A sequence $[x_n]$ is defined recursively, in terms of two numbers x_0 and x_1, by the formula

$$x_n = \frac{(n-1)g}{1 + (n-1)g} x_{n-1} + \frac{1}{1 + (n-1)g} x_{n-2},$$

where g is a given positive quantity. Find an expression for the limit of x_n as $n \to \infty$. (This generalizes problem E 694 (1945, 516) which corresponds to the special case $g = 1$.)

Solution by R. C. Buck, Brown University. We solve a somewhat more general problem first. Let $p_n > 1$ be a given sequence of positive numbers such that

$$\lim_{n\to\infty} \prod_{k=2}^{n} p_k = +\infty.$$

Define the sequence $\{x_n\}$ by

$$x_n = x_{n-1}(p_n - 1)/p_n + x_{n-2}/p_n.$$

Then $\{x_n\}$ converges to the number $x_1-(x_1-x_0)K$, where K is a constant depending only on $\{p_n\}$.

To see this, we rewrite the recursion formula as

$$x_n - x_{n-1} = (x_{n-1} - x_{n-2})/(-p_n);$$

by iteration, this becomes

$$x_n - x_{n-1} = (-1)^{n-1}(x_1 - x_0)/p_2p_3 \cdots p_n,$$

and summing,

$$x_n = x_1 + (x_1 - x_0)\sum_{k=2}^{n} (-1)^{k-1}/p_2p_3 \cdots p_k.$$

By virtue of the assumptions on p_n, the series

$$\sum_{n=2}^{\infty} (-1)^n/p_2p_3 \cdots p_n$$

is a convergent alternating series. If we denote its sum by K, then the sequence $\{x_n\}$ is convergent to the limit $x_1-(x_1-x_0)K$.

Other expressions for K are possible. Thus, by a familiar transformation, K can be expressed as the continued fraction

$$K = \frac{1}{p_2 +}\,\frac{p_2}{p_3 - 1 +}\,\frac{p_3}{p_4 - 1 +}\,\frac{p_4}{p_5 - 1 +}\cdots.$$

In the special case of the proposed problem, $p_n=1+(n-1)g$ and K can also be expressed as a definite integral. Let

$$F(x) = \frac{x}{1+g} - \frac{x^2}{(1+g)(1+2g)} + \frac{x^3}{(1+g)(1+2g)(1+3g)} - \cdots.$$

Then, $(1+x)F(x)+gx\,F'(x)=x$, and solving this we have

$$F(x) = c(xe^x)^{-c}\int_0^x (ue^u)^c du,$$

where $c=1/g$. Hence

$$K = F(1) = ce^{-c}\int_0^1 (ue^u)^c du.$$

Also solved by P. T. Bateman, N. J. Fine, J. G. Herriot, Aaron Herschfeld, J. Lehner and A. M. Peiser, J. F. Locke, Leo Moser, and the Proposer.

Editorial Note. The Proposer's solution is in the form

$$\lim_{n\to\infty} x_n = x_0 + (x_1 - x_0)\int_0^1 \exp\left(\frac{t^g - 1}{g}\right) dt.$$

Circles Orthogonal to a Sphere

4256 [1947, 346]. *Proposed by N. A. Court, University of Oklahoma*

Given a sphere orthogonal to two circles lying in two distinct planes. If the center of the sphere is conjugate, with respect to one of the circles, to the point in which the plane of that circle cuts the axis of the other circle, the same is true of the center of the sphere, if the roles of the two circles are interchanged.

Note. A circle is orthogonal to a sphere if the plane of the circle cuts the sphere along a great circle orthogonal to the given circle (see, for instance, the Proposer's *Modern Pure Solid Geometry*, p. 138, art. 416).

Solution by P. D. Thomas, U. S. Coast and Geodetic Survey, Washington, D. C. Let the sphere have center O, radius R: and the circles have centers O_1, O_2 and radii r_1, r_2, respectively. The axis of (O_1) meets the plane of (O_2) in the point P_2. The corresponding point in the plane of (O_1) is P_1. From the given orthogonal property,

$$\overline{OO_1}^2 = r_1^2 + R^2, \qquad \overline{OO_2}^2 = r_2^2 + R^2. \tag{1}$$

Since the axes are perpendicular to the planes of their circles,

$$\overline{P_1P_2}^2 = \overline{P_1O_1}^2 + \overline{P_2O_1}^2 = \overline{P_1O_2}^2 + \overline{P_2O_2}^2, \tag{2}$$

$$\overline{P_1O}^2 = \overline{OO_2}^2 + \overline{P_1O_2}^2, \qquad \overline{P_2O}^2 = \overline{OO_1}^2 + \overline{P_2O_1}^2. \tag{3}$$

Consider first the point P_1. The powers of P_1 with respect to the great circle section of the sphere (O) and with respect to its orthogonal circle (O_1) are respectively

$$p_1 = \overline{P_1O}^2 - R^2, \qquad p_2 = \overline{P_1O_1}^2 - r_1^2. \tag{4}$$

From the first of equations (1), equation (2), and the second of equations (3) find

$$p_2 = \overline{P_1O_1}^2 - r_1^2 = \overline{P_1O_2}^2 + \overline{P_2O_2}^2 - \overline{P_2O}^2 + R^2. \tag{5}$$

From the second of equations (1) and the first of equations (3) find

$$p_1 = \overline{P_1O}^2 - R^2 = r_2^2 + \overline{P_1O_2}^2. \tag{6}$$

The condition that P_1 be the conjugate of O with respect to (O_1) is that the powers of (4) should be equal. With $p_1=p_2$, (5) and (6) become

$$\overline{P_2O}^2 - R^2 = \overline{P_2O_2}^2 - r_2^2,$$

which is exactly the condition that P_2 should be the conjugate of O with respect to (O_2), which proves the theorem.

Also solved by Ou Li, and the Proposer.

Triangle and Orthocenter

4258 [1947, 346]. *Proposed by H. F. Sandham, Trinity College, Ireland*

Prove that the necessary and sufficient condition that four non-collinear points are such that each is the orthocenter of the other three, is

$$\pm\, 34\cdot 42\cdot 23 \pm 41\cdot 13\cdot 34 \pm 12\cdot 24\cdot 41 \pm 23\cdot 31\cdot 12 = 0,$$

where rs denotes the distance between the rth and sth points, and three of the signs differ from the fourth.

Solution by the Proposer. Let the four points be designated by a, b, c, d in a system of complex coördinates, and let 23, 31, and 12 be perpendicular to 41, 42, 43, respectively. Then

$$(1)\qquad \begin{gathered}(c-b) = ip(a-d), \qquad (a-c) = iq(b-d),\\ (b-a) = ir(c-d),\end{gathered}$$

where p, q, r are real.* Put $\Delta=(c-d)(b-d)(a-d)$. Then

$$(2)\qquad \begin{aligned}&A \equiv (b-c)(c-d)(d-b) = ip\Delta, \quad B \equiv (c-a)(a-d)(d-c) = iq\Delta,\\ &C \equiv (a-b)(b-d)(d-a) = ir\Delta, \quad D \equiv (b-a)(a-c)(c-b) = -ipqr\Delta.\end{aligned}$$

Note that $|A| = 34\cdot 42\cdot 23$, $|B| = 41\cdot 13\cdot 34$, $|C| = 12\cdot 24\cdot 41$, $|D| = 23\cdot 31\cdot 12$; and that

$$(3)\qquad A + B + C + D = 0$$

is an identity, being the expansion according to the first row of

$$\begin{vmatrix} 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ a & b & c & d\\ a^2 & b^2 & c^2 & d^2\end{vmatrix} = 0.$$

Equations (2) and (3) imply

$$(4)\qquad p + q + r - pqr = 0.$$

* It is easy to show that p, q, r are the tangents of the angles of triangle 123, so that (4) is the familiar identity $\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma = 0.$

Also from (2) it is easy to see that the ratios of A, B, C, D are all real, whence

$$\pm |A| \pm |B| \pm |C| \pm |D| = 0, \tag{5}$$

and, regardless of the algebraic signs of p, q, r, three of the signs in (5) must differ from the fourth.

Conversely, having given (5) with three signs differing from the fourth, comparison with (3) shows that the closed quadrilateral whose sides are parallel and equal in modulus to the quantities A, B, C, D must collapse into a line, whence the ratios of A, B, C, D are all real. If p, q, r be defined as in (1), then $(D/A)(B/C) = -q^2$ is real, so that q is real or imaginary. Then q, $r=q(C/B)$, and $p=q(A/B)$ are alike all real or all imaginary. If they are all imaginary, it is easy to see that the points 1, 2, 3, 4 are collinear; if they are all real, (1) shows that 23, 31, 12 are respectively perpendicular to 41, 42, 43, as required.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

A Textbook of Mathematical Analysis. By R. L. Goodstein. New York, Oxford University Press, 1948. 12+475 pages. $9.00.

The author's confessed aim is to present a logically correct and clear exposition of the differential and integral calculus, which shall still be so simple that it can be "appreciated by any student with a School Certificate knowledge of arithmetic and algebra." He begins with the numbers representable as terminating decimals, and brings in the other real numbers as limits of decimal sequences, that is, as infinite decimals. He is able to develop many of the ideas and theorems concerning infinite sequences and series in his very first chapter of 26 pages. The discussions of continuity and differentiation, introduced in Chapters II and III respectively, are restricted to uniform continuity, and uniform differentiability. Thus every derivative considered is continuous. The indefinite integral (antiderivative) is introduced in Chapter VIII, and numerous formulas are developed in a space of 21 pages. The integral as limit of a sum is discussed in Chapter XIV, with a proof of its existence for uniformly continuous functions. The final chapter XVII on double integrals includes proofs of the equality of repeated integrals, of Green's theorem, and of the formula for transformation of double integrals. There are other chapters on the definition and properties of the elementary transcendental functions, differential geometry of plane curves, differential equations, Taylor's series, and maxima and minima. A proof of the

existence of the limit of a convergent sequence of decimals is given in an appendix. The book closes with 37 pages of examples, of a purely mathematical nature, followed by 101 pages of solutions. A few corrections are listed following the index.

The reviewer does not agree with the view indicated in the author's preface that a presentation of the calculus based to a considerable extent on geometrical intuition necessarily leaves the student "the servant, and never the master, of a fundamental technique." A student's mastery of a subject is a relative matter, and can be increased by a book only if that book attracts the student's interest and stimulates his imagination and intellectual activity. In the reviewer's experience, young students, even those destined to become mathematicians, do not usually begin with an interest in the logical nature of number or the logical proofs of function theory. On the other hand, the budding scientist is usually ready to learn about the tools of mathematical thought, how they work, and what they will do. Thus it would seem that an introduction to the calculus should include correct definitions of the notions of limit, derivative, antiderivative, and the definite integral as the limit of a sum, with ample explanations of the meaning of these notions as well as practice in their use, but without undue emphasis on proofs. Logical proofs and a complete statement of the assumptions used in theorems, should be included when they can be made sufficiently simple. A complete definition of the real number system and derivation of its properties, and other basic logical ideas and proofs of function theory, can be absorbed much more rapidly and effectively by the student after he has had considerable experience with the formal and manipulative side of mathematics.

A natural accompaniment of the representation of the real numbers as decimals is the author's use of the notation $0(n)$ to indicate a number less than $1/10^n$. This notation replaces the ϵ, δ notation in the definitions of limit and continuity, and in fact throughout the book. There are other points at which the terminology is unconventional. This may cause some confusion for the student who pursues mathematics further. The style of printing does not make it easy to pick out the theorems or other significant points—only occasionally is an important result set in bold face. The book seems to be carefully written, and may be useful as supplementary reading for the teacher of calculus even though he is not likely to want it for a class text.

L. M. GRAVES

Essentials of Analytic Geometry. By D. R. Curtiss and E. J. Moulton. Boston, D. C. Heath and Co., 1947. 10+259 pages. $2.80.

This is a revision of the authors' *Analytic Geometry*. Rather extensive changes have been made including the deletion of several chapters and the inclusion of many new exercises. The goal in making the revision was to present the essentials, as the name implies, in a form that could be presented in a four or five semester hour course.

The selection of topics is in general the usual one including a brief treatment

of solid analytic geometry. The latter subject would be more helpful as preparation for the calculus if some treatment were given to solids which are bounded by two or more surfaces. The last chapter, entitled *Systems of Coordinates*, consists of four sections, two on translation and rotation of axes in three space, and two on cylindrical and spherical coordinates. The first two of these sections hardly seem to be "most essential as preparation for the calculus and for engineering courses." Perhaps it is intended that the five pages comprising this chapter might be referred to subsequently in somewhat the same way that a student in this course would refer to the seven pages of formulas from algebra and trigonometry which are included at the end of the book.

There are some other features of the book which deserve comment. One is the introduction of polar coördinates in the first chapter along with rectangular coördinates. Thereafter each topic is discussed in terms of both coördinate systems. As to whether this is desirable or not is probably a matter of taste. Another feature is the lists of supplementary exercises at the ends of most of the chapters. In this connection one wonders why the authors did not go one step further and give references to some other books. Surely, students should be encouraged to use the library.

The authors are to be commended for their attention to accuracy of statement. As examples, on page 4, a distinction is made between a segment and the measure of a segment and on page 24 between the trigonometric functions of angles and of numbers. Figures are well drawn and the book would be quite attractive except for the rather small print in the exercises and examples.

R. H. BARDELL

College Algebra. By T. S. Peterson. New York, Harper and Brothers, 1947. 8+334 pages. $2.50.

In college algebra, the unending controversy between the radicals and the conservatives treats of both the material to be covered and the manner of covering it. The conservative would teach all the traditional topics and drill them thoroughly, in order that the student may become a skilled manipulator able to handle any algebraic difficulty which comes his way. The radical, on the other hand, would select a limited number of basic topics, present them to show their relation to the rest of mathematics, stress algebra as a method of thinking, and try to give unity to the whole.

This book clearly belongs to the former class. It resembles in detail the standard college algebra of the past fifty years. It begins with about 100 pages of review of elementary algebra; it discusses in a minimum of space the underlying principles of each topic and codifies them into Rule 1, Rule 2, $\cdots$; it has many problems following each article; it covers all the traditional topics from rules of signs to infinite series, with induction, inequalities, Horner's method, the cubic and the quartic, and partial fractions given careful attention. All this is condensed, by means of a concise style and small type, into 298 pages.

Some of the more recent additions to the subject have been included. Sig-

nificant digits are given two pages; interest and annuities rate a 13-page chapter; while statistics and curve fitting occupy the last 15 pages. In this respect the author has been more successful in bringing the book up to date than in the problems, where new cars cost $900, beginning salaries are $1200 per year, cars travel 30 and 24 miles an hour, factory employes earn $2.50 a day and fast trains get up a speed of 60 miles an hour.

The Preface suggests that the book would be useful (1) for students with only one year of algebra behind them; (2) as a terminal course in mathematics; (3) as preparatory for majors in mathematics and science. To the reviewer, it seems (1) too brief, too dull, and too difficult in the first 100 pages for the average student with only one year of algebra; (2) lacking in unity and not sufficiently rich in interpretation and in the breadth of its applications to other fields for a terminal course; and (3) while it might develop skilled manipulators, its problems are so routine and formal that it neglects development of the student's powers of analysis.

However, to a student with sufficient background, with an overwhelming determination to master college algebra, and with an inspiring teacher to provide motivation, this text would surely give a thorough knowledge of the traditional material of college algebra and make him a master of algebraic manipulation.

R. C. HUFFER

Fundamentals of Business Mathematics. By W. R. VanVoorhis and C. W. Topp. New York, Prentice-Hall, 1948. 8+454 pages. $3.75.

Designed to meet the needs of students of business, this book aims to provide the kind and amount of mathematical background needed in the study of finance, commerce, accountancy, business statistics, and related topics. Beginning with a brief review of arithmetic and elementary algebra, the text proceeds to discuss elementary statistical methods, percentage, simple interest and discount, probability, exponents and radicals, logarithms, progressions, problems of finance, and equations. Each of these topics is treated in a simple, clear manner that should appeal to the student. Ample drill problems have been carefully selected and excellent self-tests are given at the end of each chapter.

The reviewer noted but two minor criticisms. On page 9, *factors* are confused with *divisors*. On page 29 it is stated incorrectly (or ambiguously), "This procedure of rounding off the answer to the number of decimal places appearing in the least accurate number entering into the operation is followed in all arithmetic calculations involving approximate numbers."

In the reviewer's opinion, this should prove to be a successful text, particularly in schools and colleges of business administration and in junior colleges.

H. D. LARSEN

Elements of Nomography. By R. D. Douglass and D. P. Adams. New York, McGraw-Hill Book Co., 1947. 9+209 pages. $3.50.

The purpose of this text is to teach the practical construction of elementary

alignment charts, especially to those who study without benefit of instructor. Besides a certain maturity, it presupposes, for the most part, only a knowledge of algebra, plane geometry and logarithms. It should prove satisfactory for class room instruction of engineering students as well as for self-instruction.

For three variables the authors list six types under which most of the charts fall. Type II is the addition nomogram (three parallel straight supports); type III, the multiplication nomogram (two parallel straight supports crossed by a third straight support, the so-called N- or Z-chart); type V, the hexagonal nomogram (three straight supports concurrent in a finite point); type VI, the circular nomogram (one straight support crossing two coincident circular supports); types I, IV are special cases of II, V in which one support bisects, respectively, the distance and the angle between the other two. The main portion of the book thus concerns certain nomograms of genus zero and one of genus two. The final chapter constructs two nomograms of genus one, compounds nomograms of the various types to give nomograms for more than three variables, discusses other related graphical devices and gives charts for solving quadratic, cubic and trinomial equations.

The subject is developed by solving problems illustrative of the various types. The stress is upon mechanical details of the solution, which is on the whole reduced to a series of carefully described steps. Printed forms to be used in solving are given in several cases, with a word of caution against using them before mastering the principles on which they are based. Questions of sign and disposition, which often puzzle the beginner, are dissected at great length. The drawings are large scale, numerous and excellent. The discussion of the technique for the actual drafting is unusually complete.

Although the notation is admirably consistent, it is perhaps excessive and cumbersome. On page 21, for example, ten symbols, two of which involve a subscript on a subscript and also a superscript, are used in the relatively simple operation of adjusting the modulus and the zero of a scale. The exclusion of such tools as analytic geometry and determinants, which have had great heuristic value in the development of the subject and which are available to the advanced undergraduate, handicaps the text for use in a strictly mathematical course as does also the total absence of references to the pertinent literature. The answer to this is perhaps that our overcrowded curriculum has no room for such a course. Since apart from and in spite of its applications nomography is a fascinating branch of mathematics, this state of affairs would indeed be unfortunate, but it would at least justify the authors in limiting their objective, which they seem to have attained with a high measure of success.

J. M. Thomas

NEW BOOK RECEIVED

Eleven and Fifteen Place Tables of Bessel Functions of the First Kind, to All Significant Orders. By E. Cambi. New York, Dover Publications, 1948. 5+154 pages. $3.95.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS 1947–48

Pi Mu Epsilon, Louisiana State University

Some of the papers presented at the semi-monthly meetings of *Louisiana Alpha* Chapter of *Pi Mu Epsilon* include:

Some early discoveries in modern mathematics, by Dr. F. A. Rickey

The geometric aspects of the derivative, by Ernest Ikenberry

Approximation of $N!$ for large N, by Dr. D. B. Sumner

Geometric remarks on differential equations, by Dr. Solomon Lefschetz of Princeton University

Mathematics in European universities, by Dr. Valdas Merkys

The Mayan system of numeration, calendar, and astronomy, by M. B. Smith.

Some elementary mathematics used in the study of electronics, by Donald Shipp.

The officers for the year 1947–48 were: President, Wendell Craft; Vice-President, Elinor Ernst; Secretary, Lillian Hudson; Treasurer, M. B. Smith; Faculty Advisor, Prof. H. T. Karnes.

Mathematics Club, Oberlin College

The activities of the *Mathematics Club* of Oberlin College included the following talks:

Pi, by Thomas Morgan

Number systems and the game of Nim, by Emery Thomas

Vectors at sea, by Charles Petree

Men of mathematics, by graphical analysis, by Mary Wright and Jeanne Taylor

An infinity of paradoxes, by Ruth Berger

Number theorems, by Lester Arnold

The foundations of geometry, by Joe Solomon

Ballistic pendulums, by Bruce Clark

Plane surveying, by Robert Keesey

Modern mathematics, by Walter Wood.

Prof. S. W. McCuskey of Case Institute of Technology spoke on the subject *Astronomical statistics* at the annual club banquet.

Officers for 1947–48 were: President, Lester Arnold; Vice-president, Ruth Berger; Secretary-treasurer, Mary Wright; Publicity chairman, Evelyn Schmidt; Social Chairman, Rosalind Monastersky; Faculty Advisor, Prof. E. P. Vance.

Officers for 1948–49 are: President, Emery Thomas; Vice-President, Joe Solomon; Secretary-treasurer, Evelyn Schmidt; Publicity Chairman, Ann Johnson; Social Chairman, Jeanne Taylor; Faculty Advisor, Prof. R. W. Wagner.

Mathematics Club, Sampson College

The Mathematics Club of Sampson College functions to promote interest in mathematics and in its application. Aiming for this goal, the club presented the following lectures:

Dimensional analysis, by Mr. Petrie
Mathematics and chemistry, by Jack Bulloff
What is a real number?, by Mr. R. G. Albert
Ballistics, by Edwin Peck
Mathematics in music, by Dr. Paul Squires
Cartography, by Isidore Greenberg
Plane geometry and the nine-point circle, by Edmund Frankel
Vector analysis, by Winton Laubach
Mathematics in mechanical optics, by H. C. Fish
Magic, by Jack Martin
Mathematics of tool and die making, by William Krasnow.

An intra-club mathematical contest was held during one of the meetings.

The novel event of the year was the presentation of Mack Martin's play, *How to catch a tortoise*, enacted by Eugene Hochdanner, Otto Fabian, Jack Martin and Howard Fish. The core of the play was a discussion between a modern philosopher and Zeno (returned to earth) upon the paradoxical nature of motion.

The schedule of future club activities includes a treasure hunt in mathematics books, a mathematics exhibition, and a mathematics party. Five minute reports by students on books in mathematics will be featured in future meetings.

The Mathematics Club boasts its own library room with blackboard, books and periodicals for informal study. Plans for a mathematical mural and a cabinet of geometrical models have been initiated.

The members elected to officers' positions at the second meeting were: President, Eugene Hochdanner; Vice-president, Robert Forrest; Secretary, Howard Fish; Publicity Director, Gilbert Friedenreich; Science Council Representative, A. W. Vandewinckel; Program Committee, William Krasnow and Arthur Siegel; Faculty Advisor, Mr. Richmond G. Albert.

Mathematics-Physics Club, College of Saint Teresa

Applications of mathematics to various professions was the theme of the year's programs of the Mathematics-Physics Club of College of Saint Teresa. Guest speakers devoted evenings to architecture, engineering, biophysics, and chess. Members of the club gave talks on *Bridges*, *Body mechanics in nursing arts*, *The sextant*, and *Celestial navigation*. Mathematics films also had their part in the programs. The year closed with a mathematical picnic at which the year's theme was put into effect in a recreational way.

Officers for the years were: President, Betty Gervais; Vice-president, Genevieve Suprenant; Secretary, Mary Jane Dyer; Treasurer, Joan Koch.

Kappa Mu Epsilon, Upsala College

The papers presented at the regular monthly meetings of the *New Jersey Alpha* Chapter of *Kappa Mu Epsilon* were:

Mathematics at Shrivenham University and other English Colleges, by Dr. D. R. Davis of Montclair State Teachers College

An introduction to nomography, by Marjorie Cohen

Different methods of proving the Pythagorean theorem, by Frances Rischmuller.

Nomographic chart for quadratic equations, by Robert Wallace

The body of mathematics as we have it today, by Dr. M. A. Nordgaard, faculty advisor

Relationship of mathematics to philosophy, by June Davidson

Theory of probability, by James Gill and Martin Monroe.

At the Ninth Annual Banquet Dr. Nathan Lazar of Teachers College, Columbia University, spoke on *The development of the abacus*. The members of the fraternity joined the Mathematics Club in a trip to Hayden Planetarium.

The officers during the year 1947–48 were: President, Marjorie Cohen; Vice-President and Treasurer, June Davidson; Recording Secretary, Frances Rischmuller; Corresponding Secretary, Dr. M. A. Nordgaard.

The officers for the year 1948–49 are: President, Frances Rischmuller; Vice-President, Robert Wallace; Recording Secretary, James Gill; Treasurer, Martin Monroe; Corresponding Secretary, Dr. M. A. Nordgaard.

Mathematics Club, Montana State University

The *Mathematics Club* of Montana State University held ten regular meetings during the year 1947–48. Faculty members of the mathematics and physics departments and undergraduate students presented the following papers:

Diophantine equations, by Mr. Coffee

Atomic research, by Dr. Miller

History of pi, by Don Marshall

Probability and games of chance, by Dr. Ostrom

Curve fitting, by Dr. A. S. Merrill

Mathematics in occupations, by Frank McElwain

One to one correspondence, by Dr. Lennes

Rubber sheet geometry, by Paul Rygg.

There were two social meetings during the year, one at the home of Dr. and Mrs. A. S. Merrill during the winter quarter, and the other was the annual Mathematics-Chemistry Club picnic in the spring.

The officers for 1947–48 were: President, Don Marshall; Vice-President, Jean Popham; Secretary-Treasurer, Veronica Kreitel; Faculty advisor, Dr. T. G. Ostrom.

The officers for 1948–49 are: President, Paul Rygg; Vice-President, Veronica Kreitel; Secretary-Treasurer, Elsie Taylor; Faculty advisor, Dr. T. G. Ostrom.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

ACKNOWLEDGMENT

The editor of the MONTHLY wishes to make grateful acknowledgment of the services rendered by the following persons who have refereed papers: C. B. Allendoerfer, Norman Anning, W. L. Ayres, E. F. Beckenbach, L. M. Blumenthal, H. W. Brinkmann, W. E. Byrne, G. F. Carrier, E. W. Chittenden, Paul Civin, A. B. Coble, Nathaniel Coburn, C. J. Coe, Richard Cohn, Richard Courant, H. S. M. Coxeter, H. T. Davis, Douglas Derry, P. S. Dwyer, H. T. Engstrom, P. Erdös, H. W. Eves, L. R. Ford, Tomlinson Fort, J. S. Frame, Orrin Frink, W. H. Gage, J. W. Givens, L. M. Graves, N. G. Gunderson, F. S. Harper, G. E. Hay, E. R. Heineman, M. R. Hestenes, T. H. Hildebrandt, R. T. Hood, G. B. Huff, H. K. Hughes, R. D. James, Fritz John, L. S. Johnston, P. S. Jones, Fred Kiokemeister, H. D. Larsen, D. H. Lehmer, J. N. Livingood, C. C. MacDuffee, Saunders MacLane, H. B. Mann, W. T. Martin, N. H. McCoy, A. D. Michal, Norman Miller, C. N. Moore, Marston Morse, Ivan Niven, Rufus Oldenburger, C. D. Olds, Oystein Ore, C. G. Phipps, George Pólya, G. B. Price, E. D. Rainville, O. J. Ramler, J. F. Randolph, W. T. Reid, P. R. Rider, G. de B. Robinson, J. B. Rosser, I. M. Sheffer, H. L. Smith, C. E. Springer, E. P. Starke, J. L. Synge, A. H. Taub, R. M. Thrall, R. S. Underwood, H. S. Vandiver, R. J. Walker, J. L. Walsh, S. W. Warschawski, Louis Weisner, Charles Wexler, R. L. Wilder, H. E. Wolfe, M. A. Zorn.

PREDOCTORAL FELLOWSHIP OF SIGMA DELTA EPSILON

The Fellowship Board of Sigma Delta Epsilon, Graduate Women's Scientific Fraternity, announces that applications for the Sigma Delta Epsilon Research Fellowship in Science should be submitted before February 1, 1949. Application blanks may be secured from Dr. Virginia Barton, 7 Chemistry Annex, University of Illinois, Urbana, Illinois. Women of outstanding ability, who are predoctoral candidates in the mathematical, physical, or biological sciences and who need financial assistance to further a well-defined research project, are eligible candidates.

HARRY BATEMAN RESEARCH FELLOWSHIPS

Applications are invited for these post-doctorate fellowships in Pure Mathematics at the California Institute of Technology. Recipients will devote the major part of their time to research. In addition, they are expected to teach one upperclass course in mathematics. The stipend is $3600 for the academic year. For application blanks and further information, address the Dean of the Faculty, California Institute of Technology, Pasadena 4, California. Applications must be returned before March 1, 1949.

RESEARCH GRANTS OF THE INSTITUTE FOR ADVANCED STUDY

The School of Mathematics, Institute for Advanced Study, will allocate a small number of stipends to gifted young mathematicians and mathematical physicists to enable them to study and do research work at Princeton during the academic year 1949–50. Candidates must have given evidence of ability in research comparable at least with that expected for the Ph.D. degree. Blanks for application may be obtained from the School of Mathematics, Institute for Advanced Study, Princeton, New Jersey and are returnable by February 1, 1949.

ORGANIZED RESERVE RESEARCH GROUPS OF THE UNITED STATES ARMY

The Department of the Army has established a program of particular interest to mathematicians and other scientists, who hold Reserve commissions in the Army, and who are professionally engaged in teaching or research and development. The objectives of the program are to: (1) maintain the useful affiliation of mathematicians and other scientists with the Organized Reserve Corps, (2) provide peacetime Reserve assignments for these officers, enabling optimum utilization of their education, experience and skills, (3) furnish mobilization assignments which will fully utilize their talents, and (4) adequately prepare these officers for mobilization.

Reserve officers who are currently engaged in civilian research, college or university teaching, or industrial research or development, or who in the past have had specific research experience are eligible to make application for assignment to an Organized Reserve Research and Development Group. A group may be organized in any locality where there are twenty or more qualified officer scientists who desire to participate in the program. A subgroup may be organized with ten qualified members.

Inquiry about organization of an Organized Reserve Research and Development Group or about assignment to a group already organized should be made of the Unit Instructor, ORC, or of the Senior Army Instructor, ORC, in the locality in which the officer resides.

PERSONAL ITEMS

Professor Charles Blanc of the University of Lausanne and Professor Albert Pfluger of the Swiss Federal Institute of Technology have been elected President and Vice-President, respectively, of the Swiss Mathematical Society.

Professor L. M. Graves of the University of Chicago has received the President's Certificate of Merit for his work on the panel of applied mathematics of the National Defense Research Committee.

Associate Professor E. H. C. Hildebrandt has been elected President of the National Council of Teachers of Mathematics. He is also National Secretary-Treasurer of Pi Mu Epsilon and Secretary of the Mathematics Section of the Central Association of Science and Mathematics Teachers.

Professor George Polya of Stanford University received an honorary degree

from the Swiss Federal Institute of Technology, Zurich, on November 15, 1947.

Professor Hassler Whitney of Harvard University has received an honorary degree of Doctor of Science from Yale University.

It has been announced by the U. S. Army that the following mathematicians will serve as an advisory committee especially with regard to speed calculation: Dr. Hendrik Bode, Bell Telephone Laboratories; Professor Richard Courant, New York University; Professor John von Neumann, Institute for Advanced Study; Professor J. B. Rosser, Cornell University; Professor H. P. Robertson, California Institute of Technology; Professor J. J. Stoker, New York University; Professor E. R. Lorch of Columbia University will be mathematics adviser to the research group.

Alabama Polytechnic Institute announces the promotions of Associate Professor Ernest Williams to a professorship and Assistant Professor S. L. Thompson to an associate professorship.

Arizona State College reports: Associate Professor R. B. Lyon, who is on sabbatical leave, is at the University of Texas; Dr. Lee Byrne, formerly of Purdue University, has been appointed Special Lecturer in Mathematics.

At Boston College, Dr. R. J. Marcou has been promoted to a professorship; Dr. H. G. Haefeli and Dr. T. S. Motzkin have been appointed to associate professorships.

Butler University makes the following announcements: Professor H. E. Crull, head of the Department of Mathematics, is now also Director of the University College; Mr. William Fuller and Miss Jane Uhrhan have been appointed to instructorships.

Colorado State College of Education announces the appointment of Mr. K. E. Hansen to an instructorship. A very successful Summer Conference on Science, Mathematics and Social Studies was held on July 9–10, 1948.

At De Paul University, Mr. Chester Pachuki and Mr. Arthur Saastad have been appointed to instructorships.

Drake University announces the promotions of Associate Professor B. E. Gillam to a professorship and Instructor R. W. Gardner to an assistant professorship.

Duke University announces the appointments of the following instructors: J. R. Garrett, M. P. Jarnagin, T. D. Reynolds, all formerly graduate students at Duke University.

Emory University reports: Professor C. G. Latimer has been appointed Chairman of the Department of Mathematics; Mr. C. R. Partington of Purdue University has been appointed to an instructorship.

Harvard University makes the following announcements: Assistant Professor George Mackey has been promoted to an associate professorship; Dr. Oscar Goldman of Princeton University, Dr. John Gurland of the University of California in Berkeley and Dr. Raymond Redheffer of the Massachusetts Institute of Technology have been appointed Benjamin Peirce Instructors of Mathematics; Professor D. V. Widder is on sabbatical leave during 1948–49 and is spending

the year in California. Professor Garrett Birkhoff was abroad during the spring term and summer as a Guggenheim Fellow; he lectured in the London Mathematical Society and at the Institut Henri Poincaré. Professor Mackey delivered an invited address at a conference held at the University of Chicago on Topological Groups during the spring of 1948.

Johns Hopkins University announces: Dr. W. L. Chow of the University of Shanghai and the Institute for Advanced Study has been appointed to an associate professorship; Dr. L. G. Peck of New York University and the Institute for Advanced Study has been appointed Assistant Professor; Mr. E. D. Carey of New York University, Dr. M. L. Juncosa of Cornell University, and Doctors E. A. Coddington, C. R. Putnam, Sylvan Wallach of Johns Hopkins University have been appointed to instructorships. Professor F. D. Murnaghan has resigned to accept an offer from the Brazilian government; he will assume his new duties in Brazil in January 1949. Professor B. L. van der Waerden has resigned to resume his duties with the Royal Dutch Oil Company.

Lehigh University reports the following: Associate Professor A. E. Pitcher has been promoted to a professorship; Mr. Paul Meier of Princeton University has been appointed to an assistant professorship; Mr. Michael Tikson has been appointed to a graduate assistantship; Assistant Professor H. G. Means has retired.

Louisiana State University makes the following announcements: Assistant Professor L. I. Wade of Duke University has been appointed Professor and Head of the Department of Mathematics; Mr. B. B. Townsend of the University of Texas has been appointed to an assistant professorship; Professor I. C. Nichols has retired with the title of Professor Emeritus.

New York University announces these promotions: Associate Professor H. R. Cooley to a professorship; Assistant Professor H. E. Wahlert to an associate professorship; Dr. Joseph Keller who has been connected with the Institute of Mathematics and Mechanics to a research assistant professorship; Instructors Beatrice G. Edison and David Gans to assistant professorships. The Pi Mu Epsilon Mathematics Contest was held at Washington Square College on March 13, 1948 with representatives of one hundred eighty-three schools participating.

Northwestern University reports the following: Instructor D. H. Potts has been promoted to an assistant professorship; Mr. E. W. Banhagel of Wayne University, Mrs. Helen Betz, Morton Junior College, and Mr. W. M. Boothby, Rackham Fellow at University of Michigan, have been appointed to instructorships; Professor W. T. Reid served as visiting professor at the University of California at Los Angeles during the second Summer Session of 1948. Assistant Professor M. E. Wescott has been elected Chairman of Board of Editors of Industrial Quality Control.

At Purdue University the following appointments to instructorships have been made: Mrs. Nadine Johannesen of Wayne University, Miss Ruth King and Mr. James McKnight of Purdue University, Dr. Harold Shniad, University of

California at Los Angeles, and Mr. Erskine St. Clair, University of Michigan.

Rice Institute announces the promotion of Dr. H. D. Brunk to an assistant professorship.

Sam Houston State Teachers College reports: Mr. M. W. Wells of Texas Agricultural and Mechanical College has been appointed to an assistant professorship; Associate Professor B. M. Wall is on leave and is studying at the University of Texas.

At Simmons College a separate Department of Mathematics has been organized effective with this academic year. Miss Sybil Tanenbaum, formerly at Brown University, has been appointed to an instructorship.

Syracuse University announces the following: Assistant Professor F. W. Borgward and Professor I. S. Carroll have retired with the title of Professor Emeritus; Dr. O. O. Pardee has been appointed to an instructorship. Professor P. C. Rosenbloom has returned from Sweden where he spent eighteen months on a Guggenheim fellowship. Professor Abe Gelbart has returned after a year at the Institute for Advanced Study.

United States Military Academy makes the following announcements: Colonel W. W. Bessell, Jr. has been appointed Professor and Head of the Department of Mathematics; Colonel C. P. Nicholas has been appointed to a professorship; Lieutenant Colonel R. C. Yates has been promoted to an associate professorship; Lieutenant Colonel J. T. Honeycutt and Lieutenant Colonel A. W. Oberbeck have been promoted to assistant professorships. Brigadier General Harris Jones, formerly head of the Department of Mathematics, is now Dean of Academic Board; Colonel W. N. Underwood is District Engineer on Guam; Lieutenant Colonel J. S. B. Dick is overseas.

United States Naval Academy announces the following promotions: Assistant Professors J. M. Holme, J. F. Milos, R. C. Morrow, V. N. Robinson, S. S. Saslaw, and C. W. Seekins to associate professorships; Instructors B. H. Buikstra, A. R. Craw, M. V. Gibbons, J. R. Gorman, E. C. Gras, F. P. Kowalewski, K. F. McLaughlin, G. J. Mann, Joseph Milkman, Robert Molly, N. O. Niles, A. J. Pejsa, J. W. Popow, R. W. Rector, R. C. Simpson, H. K. Sohl, A. H. Streinbrenner, M. F. Stilwell, W. J. Strange, G. R. Strohl, Jr., E. G. Swafford, C. E. Thompson, J. A. Tierney, E. C. Watters, Jr., J. H. White, and H. Wierenga to assistant professorships.

At the United States Naval Postgraduate School, Assistant Professor R C. Campbell of the University of Pennsylvania and Assistant Professor B. J. Lockhart of the University of Michigan have been appointed to assistant professorships; Mr. N. F. Spraggins, formerly at Stanford University, has been appointed to an instructorship.

University of Chicago reports: Visiting Professor Andre Weil has been promoted to a professorship; Assistant Professor Leopoldo Nachbin of the University of Brazil and Professor Candido Lima Da Silva Dias of the University of Sao Paulo have been appointed Research Associates. Instructor E. J. Akutowicz is now an Atomic Energy Post Doctoral Fellow; Instructor J. B. Crabtree is with

the ONR in London.

University of Colorado announces the appointment of Mr. Burrowes Hunt, Jr. and Mr. Gideon Culpepper of the University of Colorado to instructorships.

University of Florida announces: Mr. Kenneth Walters of Lehigh University has been appointed to an instructorship; Mr. L. H. Potter, graduate student of the University of Florida, has been appointed to an acting instructorship; Mr. M. J. Cleveland, Mr. A. J. Owens, Mr. J. M. Robertson, and Mr. W. B. Stovall, Jr. have been appointed to teaching assistantships; Mr. J. A. Adkinson, Mr. J. R. Duffett, Mr. G. B. Findley, and Mrs. Orpha Ingwalson have been appointed to graduate assistantships.

University of Georgia makes the following announcements: Professor Tomlinson Fort has been appointed Distinguished Service Professor; Associate Professor J. A. Ward was promoted to a professorship; Professors W. V. Parker and J. A. Ward have been given special research grants by the University of Georgia; new members of the staff are Associate Professor G. M. Conwell and Instructor James Bercos; Professor G. B. Huff is on leave of absence for the current year and is at Harvard University. The Department of Mathematics has been authorized to offer the Ph.D. Degree.

At the University of Houston, Associate Professor A. A. Aucoin has been promoted to a professorship and Assistant Professor Albert Newhouse has been promoted to an associate professorship.

University of Kansas reports: Associate Professor P. O. Bell has been promoted to a professorship; Dr. I. N. Herstein, Dr. G. K. Overholtzer, Mr. Robert Marceau, and Miss Minnie Stewart have been appointed to instructorships; Associate Professor H. E. Jordan has retired with the title of Associate Professor Emeritus. Mr. S. G. Kneale, graduate student, has been awarded a scholarship at Harvard University.

The University of Illinois announces: Professor W. G. Madow of the University of North Carolina has been appointed Professor of Mathematical Statistics; Professor Loo Keng Hua, Institute for Advanced Study, has been appointed Visiting Professor; Dr. Lowell Schoenfeld, Harvard University, and Dr. Irving Reiner, formerly of the Institute for Advanced Study, have been appointed to assistant professorships; Assistant Professor F. E. Hohn of the University of Maine, Mr. R. G. Langebartel of the University of Illinois, and Mrs. Irma M. Reiner of Temple University have been appointed to instructorships; Instructors W. A. Ferguson, Joseph Landin, B. E. Meserve, and M. E. Munroe have been promoted to assistant professorships.

Dr. H. L. Alder, instructor at the University of California at Berkeley, is now an instructor at the University of California at Davis.

Associate Professor H. C. Ayres of the United States Naval Academy has accepted a position as professor and head of the Department of Mathematics of Jersey City Junior College.

Dr. L. J. Burton has been appointed to an assistant professorship at Bryn Mawr College.

Mr. P. F. Cauffman of Lehigh University has been appointed Professor and Head of the Department of Mathematics of State Teachers College, Salisbury, Maryland.

Dr. J. H. Curtiss, chief of the National Applied Mathematics Laboratories, has assumed temporary additional duties as acting chief of the Institute for Numerical Analysis at the University of California at Los Angeles.

Assistant Professor M. P. Fobes of the College of Wooster has been promoted to the headship of the Department of Mathematics.

Associate Professor T. J. Higgins of the Illinois Institute of Technology has been appointed to a professorship in the Department of Electrical Engineering of the University of Wisconsin.

Mr. K. R. Jones, who has been associated with the NEPA Project, Oak Ridge, is now employed at Ordnance Research Number One, University of Chicago.

Dr. Samuel Karlin has been appointed Research Instructor for the current academic year at California Institute of Technology.

Mr. A. W. Kaufman is now an instructor at Aurora College.

Assistant Professor R. R. Kuebler of Dickinson College has been promoted to an associate professorship.

Mr. J. V. Limpert of Syracuse University has been appointed to an assistant professorship at St. Lawrence University.

Dr. Eugene Lukacs has a position as statistician at the United States Naval Ordnance Test Station, Inyokern, California.

Mr. K. E. McLachlan of Baylor University has been promoted to an assistant professorship.

Mr. H. W. Morrow, Jr. has been appointed Instructor in Mechanics at the University of Florida.

Mr. L. R. Norwood of Yale University is now employed as a mathematician at the United States Army Signal Corps Laboratory, Fort Monmouth, New Jersey.

Miss Gloria Olive of the University of Arizona has accepted an appointment as instructor at Idaho State College.

Mr. R. V. Person has been appointed to an instructorship at American University.

Reverend Albeni Poitras of St. Joseph's University, New Brunswick, Canada has been made Dean of Studies.

Mr. R. M. Schmied, graduate fellow at Brown University, has been appointed to an instructorship at Tulane University.

Assistant Professor Abraham Schwartz of Pennsylvania State College is now an instructor at the College of the City of New York.

Mr. R. R. Seeber has been made Senior Staff Member with the International Business Machines Corporation, New York City.

Mr. W. H. Simons of the University of British Columbia has been promoted to an assistant professorship.

Mr. M. L. Stein of the University of California at Los Angeles has accepted a position as mathematician at the Institute for Numerical Analysis, U. C. L. A.

Mr. H. W. Stephens of the University of Florida has accepted an instructorship at the University of Maryland.

Professor Gabor Szego of Stanford University was on leave of absence during the Summer and Autumn Quarters for scientific work in Europe.

Associate Professor Alexander Tartler, Drexel Institute of Technology, has been promoted to a professorship.

Mrs. Maria Weber has been appointed to an instructorship at Goucher College.

Professor C. O. Williamson of the College of Wooster has been promoted to the headship of the Department of Applied Mathematics.

Mr. H. W. Zeoli, formerly a secondary school teacher at Hyannis, Massachusetts, has been appointed to an assistant professorship at Central Michigan College.

Professor Emeritus G. W. Finley of Colorado State College of Education died on May 17, 1948.

Professor Emeritus J. H. M. Wedderburn of Princeton University died early in October, 1948.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following seventy-six persons have been elected to membership on applications duly certified:

M. K. Al-Ghita, M.S. (Michigan) Graduate Student, University of Michigan, Ann Arbor, Mich.

H. W. Baeumler, B.S. (Buffalo S.T.C.) Instructor, University of Buffalo, N. Y.

D. Y. Barrer, B.S. (New York University) Graduate Asst., Northwestern University, Evanston, Ill.

R. C. F. Bartels, Ph.D. (Wisconsin) Asst. Professor, University of Michigan, Ann Arbor, Mich.

Rene Baus, Jr., B.S. (Tulane) Graduate Student, Tulane University, New Orleans, La.

Mildred E. Blackman, B.S. (St. Ambrose) Instructor, St. Ambrose College, Davenport, Iowa

May C. Blackstock, M.A. (Brown) Instructor, University of Tennessee, Knoxville, Tenn.

Daniel Block, B.A. (Yeshiva) Instructor, Yeshiva University, New York, N. Y.

W. H. Blum, M.A. (West Virginia) 505 Kruger Street, Elm Grove, W. Va.

Mamie I. Braswell, M.A. (George Peabody) Asso. Professor, Alabama College for Women, Montevallo, Ala.

R. C. Brown, Jr., M.S. (West Virginia) Instructor, West Virginia University, Morgantown, W. Va.

R. K. Brown, B.Sc. (Muhlenberg) Teaching Asst., Rutgers University, New Brunswick, N. J.

EMALOU BRUMFIELD, M.A. (Ohio State) Instructor, Kent State University, Kent, Ohio

J. M. CAMERON, M.S. (North Carolina State College) Mathematician, National Bureau of Standards, Washington, D. C.

W. F. CARNES, A.M. (Harvard) Tutor, St. John's College, Annapolis, Md.

J. G. CHRISTIANO, M.S. (Pittsburgh) Asst. Professor, University of Pittsburgh, Pa.

J. L. COLLINS, B.S. (St. Mary's) Instructor, St. Ambrose College, Davenport, Iowa

BROTHER DAMIAN CONNELLY, F.S.C., Ph.D. (Catholic University) Asst. Professor, LaSalle College, Philadelphia, Pa.

R. R. CROXTON, M.Ed. (South Carolina) Instructor, University of South Carolina, Columbia, S. C.

R. A. DEUTSCH, B.S. (Long Island) Electrical Engineer, Kellex Corporation, New York, N. Y.

P. W. EDMONSON, M.A. (Michigan) Teacher, Dearborn Junior College, Mich.

C. G. GOA, D.Ing. (Venezuela) Engineer, Ministry of Public Works, Caracas, Venezuela

RUTH E. GOODMAN, Ph.D. (Pennsylvania) Asst. Professor, Duquesne University, Pittsburgh, Pa.

RUBY M. GRIMES, A.M. (Illinois) Asst. Professor, North Dakota Agricultural College, Fargo, N. D.

G. B. HAGGERTY, M.A. (Bucknell) Asst. Professor, Rhode Island State College, Kingston, R. I.

EXA O'D. HARDIN, B.S. (Prairie View College) Teacher, Phillis Wheatley High School, Houston, Tex.

R. C. HASELTINE, M.A. (Pennsylvania State College) Chairman of Dept., Pennsylvania State College (Swarthmore Center), Swarthmore, Pa.

HELEN M. HEATER, M.S. (West Virginia) Instructor, West Virginia University, Morgantown, W. Va.

P. F. HULTQUIST, B.A. (Colorado) Assistant, University of Wisconsin, Madison, Wis.

R. Y. IWANCHUK, M.A. (Columbia) Asst. Professor, Kent State University, Kent, Ohio

AYRLENE McG. JONES, M.A. (Texas) Asst. Professor, Mississippi Southern College Hattiesburg, Miss.

H. J. JONES, M.A. (Texas) Teacher, Leuzinger High School, Lawndale, Calif.

J. J. JONES, M.A. (George Peabody) Asst. Professor, Mississippi Southern College, Hattiesburg, Miss.

R. L. JONES, M.S. (South Carolina) Adjunct Professor, University of South Carolina, Columbia, S. C.

ABRAHAM KARRASS, Boys High School, Brooklyn, N. Y.

F. L. KIOKEMEISTER, Ph.D. (Wisconsin) Asso. Professor, Mount Holyoke College, S. Hadley, Mass.

E. H. LARGUIER, S.J., Ph.D. (Michigan) Professor, Spring Hill College, Mobile, Ala.

J. A. LA RUE, B.A. (West Virginia) Teaching Fellow, West Virginia University, Morgantown, W. Va.

T. H. LEE, M.A. (North Carolina) Adjunct Professor, University of South Carolina, Columbia, S. C.

E. J. LOWRY, M.Sc. (Nebraska) Professor, Hastings College, Neb.

W. C. LOWRY, M.Ed. (Ohio University) Instructor, Kent State University, Kent, Ohio

L. J. MATTESON, JR., A.B. (Colgate) Actuarial Trainee, Mutual Life Insurance Co., New York, N. Y.

J. E. McGAUGHY, B.S. (Chicago) Instructor, Lawrence College, Appleton, Wis.

J. S. MINAS, Student, Wayne University, Detroit, Mich.

B. N. MOYLS, Ph.D. (Harvard) Asst. Professor, University of British Columbia, Vancouver, B. C.

T. F. MULCRONE, S.J., M.S. (Catholic University) Asst. Professor, Spring Hill College, Mobile, Ala.

BERNICE ORSHANSKY, B.A. (Hunter) Mathematics Editor, Educational Testing Service, Cooperative Test Division, New York, N. Y.

S. J. PAGANO, M.A. (Washington University) Instructor, Missouri School of Mines, Rolla, Mo.

L. G. RIGGS, M.S. (Syracuse) Instructor, Northwestern University, Evanston, Ill.

G. B. ROBISON, M.A. (Columbia) Asst. Professor, Sampson College, N. Y.

REV. E. J. ROCHE, M.S. (Notre Dame) Professor, St. Dunstan's University, Charlottetown, P. E. I.

W. G. ROULEAU, A.B. (Catholic University) Statistical Clerk, National Bureau of Standards, Washington, D. C.

R. R. SEEBER, JR., A.B. (Harvard) Senior Staff Member, International Business Machine Corp., New York, N. Y.

C. J. SENSALE, Student, New Jersey State Teachers College, Montclair, N. J.

J. R. SEWELL, B.S. (Rollins) Graduate Student, Northwestern University, Evanston, Ill.

E. I. SHAPIRO, M.A. (Cornell) Instructor, Brooklyn College, N. Y.

R. L. SHIVELY, M.S. (Michigan) Teaching Fellow, University of Michigan, Ann Arbor, Mich.

C. E SHOTWELL, B.A. (Tusculum) Asst. Instructor, University of Missouri, Columbia, Mo.

A. T. SKINNER, M.A. (Columbia) Asst. Professor, Sampson College, N. Y.

R. E. SMART, B.S. (Ohio State) Engineer, Anchor Hocking Glass Corporation, Lancaster, Ohio

J. L. SOLOMON, Student, Oberlin College, Ohio

R. L. SPENCER, M.A. (Michigan) Instructor, Dearborn Junior College, Mich.

W. L. STAMEY, B.A. (Colorado State) Asst. Instructor, University of Missouri, Columbia, Mo.

R. H. STARK, Ph.D. (Northwestern) Instructor, Northwestern University, Evanston, Ill.

R. F. STEWARD, B.S. (Wheaton) Teaching Asst., New Jersey College for Women, New Brunswick, N. J.

R. B. STILES, A.B. (Middlebury) Instructor, Vanderbilt University, Nashville, Tenn.

J. C. THURMAN, M.A. (Vanderbilt) Asst. Professor, Vanderbilt University, Nashville, Tenn.

J. T. VALLANDINGHAM, A.B. (Georgetown) Instructor, Cumberland College, Williamsburg, Ky.

JOHNNIE VAN, P.O. Box 27, Hastings, Fla.

F. E. VAN BERGEN, D.Sc. (Brussels) Teacher, Royal Athenaeum of St. Niklaas, St. Niklaas-Waas, Belgium

A. A. VUYLSTEKE, Student, Wayne University, Detroit, Mich.

J. G. WENDEL, Ph.D. (California Tech.) Instructor, Yale University, New Haven, Conn.

W. D. WILLIAMS, M.S. (Illinois) Instructor, Hannibal La Grange College, Hannibal, Mo.

E. H. WOHLER, M.A. (Toledo) Asst. Professor, Bowling Green State University, Ohio

J. E. WOOD, M.A. (Colorado State) Director of Aviation, Scottsbluff Junior College, Neb.

L. G. WORTHINGTON, M.A. (North Texas State) Head of Department, John Tarleton College, Stephenville, Tex.

THE APRIL MEETING OF THE KANSAS SECTION

The thirty-second annual meeting of the Kansas Section of the Mathematical Association of America was held at Mount St. Scholastica College in Atchison, on Saturday, April 10, 1948. Sessions were held in the morning and afternoon. Sister Helen Sullivan presided at these sessions.

The attendance was one hundred forty-one including the following forty-four members of the Association: R. W. Babcock, Wealthy Babcock, H. H. Barnett, Florence Black, Frances Breneman, C. H. Brown, Virginia L. Chatelain, L. E. Curfman, Lucy T. Dougherty, Paul Eberhart, Albert Furman, W. H. Garrett, Laura Z. Greene, Edison Greer, J. R. Hanna, W. C. Janes, L. E. Laird, C. F. Lewis, Anna Marm, Margaret E. Martinson, Thirza A. Mossman, C. V. Newsom, Sister Jeanette Obrist, S. T. Parker, O. J. Peterson, Rev. P. S. Pretz, G. B. Price, O. M. Rasmussen, C. B. Read, C. A. Reagan, L. M. Reagan, E. S. Robbins, R. G. Sanger, Robert Schatten, J. A. G. Shirk, G. W. Smith, R. G. Smith, Sister Helen Sullivan, C. B. Tucker, Gilbert Ulmer, Frances E. Walsh, E. B. Wedel, A. E. White, P. M. Young.

At the business meeting the following officers were elected for next year: Chairman, R. G. Sanger, Kansas State College; Vice-Chairman, R. G. Smith, Kansas State Teachers College, Pittsburg; Secretary-Treasurer, Anna Marm, Bethany College.

The following papers were presented:

1. *Establishing our mathematical perspective*, by Professor C. V. Newsom, Oberlin College.

A fundamental problem of education is that of perfecting the powers of a person, especially insofar as he is enabled to comprehend and control his environment. This requires an understanding of the methods employed in attempting to bridge the gap between the mind of man and the conduct of nature. The systematic construction of various models, including symbolic systems, that are useful in coordinating and predicting events in nature characterized the scientific reformation of the seventeenth century. Since that time the use of the symbolic patterns of mathematics has grown, and virtually every science acknowledges the importance of mathematical procedures. These facts make it essential that early courses in mathematics should make clear the true role of mathematics. It should be emphasized that (a) mathematics is manmade, and, in general, a mathematical pattern fits any segment of nature only approximately, although the fit may be amazingly good; (b) the utility of mathematics would be quite limited except for the generality inherent in its formalizations; (c) abstract mathematical systems employ the method of postulational thinking, and every attempt should be made to acquaint students with the mathematical method and the desire for rigor. A corollary of these conditions is the fact that the mathematician, or at least the teacher of mathematics, must be extremely versatile in his preparation and interests.

2. *Panel: Mathematics and the development of civilization.*

This discussion consisted of five parts. First the role of mathematics in physical sciences was treated by G. B. Price of the University of Kansas. He remarked that mathematics is concerned with the study of structure. The role of mathematics in the physical sciences is to supply, through the study of the structure of mathematical systems, mathematical models which can be used to explain physical phenomena and to predict results which have not been observed. Three important examples are Newton's model of the universe, Fourier's model for the study of the flow of heat, and Einstein's model of the universe.

Then Frances M. Breneman of Washburn University dealt with the role of mathematics in the business world. Direct application of mathematics to accounting, interest, and annuities was mentioned. The place of mathematics in a liberal arts education and the importance of this training in the business world was considered. Emphasis was placed upon the importance of sound reasoning and the role of formal mathematics in developing this type of thinking.

Mathematics in its relation to learning was discussed by Frances E. Walsh of St. Louis University. It was remarked that the title of this paper might well be *A Defense of the Mathematics Scholar*. The author advocates an attempt to elevate the masses to accept the criteria of the scholar. This necessitates the creation of scholars among whom the mathematician holds a prominent position because he deals with a speculative science. Intrinsically mathematics has much to offer the scholar. In considering the study of pure mathematics the following points illustrate the contribution of mathematics in the training of scholars: (1) The study of mathematics from a historical point of view gives one a cultural background; (2) The pursuit of mathematics gives the student an understanding and an appreciation of what constitutes rigorous thinking; (3) The process of deductive reasoning provides training in the ability to relate precise statements so as to formulate valid conclusions.

O. J. Peterson of Kansas State Teachers College at Emporia then discussed the general attitudes to be developed in our future mathematics teachers. He presented a number of desirable general attitudes consistent with present-day educational aims, and indicated how these attitudes would aid in promoting general citizenship competency and professional or vocational success.

The merits and limitations of the mathematical method were then treated by P. M. Young of Kansas State College. Abstraction, generalization, logical structure and rigor were discussed as merits of the mathematical method. Utility of results, adherence to logical principles, and the danger of overformalization are among the limitations inherent in the method.

3. *Recognition and constructions of conic sections without the use of the discriminant*, by Albert Furman, Kansas State College.

The equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{1}$$

can be written in the form

$$y = -\frac{hx + f \pm A}{b}$$

where

$$A = (h^2 - ab)x^2 + 2(hf - bg)x + f^2 - bc.$$

The graph of (1) can thus be constructed as the composite $y = y_1 + y_2$, where $y_1 = -(hx+f)/b$ and $y_2 = \pm A$. The latter equation represents the conic section

$$(h^2 - ab)x^2 - b^2y^2 + 2(hf - bg)x + (f^2 - bc) = 0. \tag{2}$$

Classification may be effected by use of the principle that equations (1) and (2) agree as to type and degeneracy or non-degeneracy.

4. *Vector methods in analytic geometry*, by L. M. Reagan, University of Wichita.

It was pointed out that vector methods, widely used with great effectiveness in Mechanics and other branches of applied mathematics are equally effective in the development of many branches of pure mathematics, and there is a growing tendency to use them, especially in Trigonometry and Geometry, including Analytic Geometry. Some texts for use in standard courses in Analytic Geometry and Calculus, based on vector methods, have appeared recently, and while not widely adopted as yet, have a great deal of merit. The simplicity of vector methods in Analytic Geometry was illustrated with several examples. In particular, the development of the standard, intercept, slope and y-intercept forms of the straight line, condition for collinearity of three points, condition that four points lie in a plane, equation of the circle in various positions, and equation of a tangent to a circle, were treated by use of vectors.

5. *A characterization of a sphere*, by Robert Schatten, University of Kansas.

Point-sets whose all plane sections are Jordan curves are considered in the three-dimensional euclidean space. In particular, a point-set whose every plane section is the circumference of a circle characterizes the surface of a sphere.

ANNA MARM, *Secretary*

THE APRIL MEETING OF THE TEXAS SECTION

The annual meeting of the Texas Section of the Mathematical Association of America was held at The Rice Institute, April 23–24, 1948. Professor R. S. Underwood, Chairman of the Section, presided.

The following members of the Association were present: Ina Mae Bramblett, H. E. Bray, H. D. Brunk, J. W. Calkin, J. A. Daum, Alice C. Dean, E. H. Hanson, H. M. Hardy, E. R. Heineman, N. C. Hunsaker, H. A. Luther, Hazel Mason, Harlan C. Miller, Edith Morgan, Albert Newhouse, J. W. Querry, C. B.

Rader, L. W. Ramsey, Dorothy Rees, C. R. Sherer, M. M. Slotnick, F. W. Sparks, E. J. Stulken, Jennie Tate, H. E. Taylor, W. B. Temple, F. E. Ulrich, R. S. Underwood, and T. J. White.

The following officers were elected for next year: Chairman, F. E. Ulrich, The Rice Institute; Vice Chairman, H. J. Ettlinger, University of Texas; Secretary-Treasurer, C. R. Sherer, Texas Christian University. The next meeting will be held at Denton, Texas under the joint sponsorship of North Texas State Teachers College and Texas State College for Women.

An informal dinner was held at The Rice Institute, April 23 at 7 P.M. Dr. H. A. Wilson, Professor Emeritus of Physics, The Rice Institute, gave a very interesting address.

The following papers were presented:

1. *The application of group theory to the normal vibration of a cubic crystal*, by Dr. W. V. Houston, President, The Rice Institute.

This paper was delivered by special invitation.

2. *On a differential equation of infinite order*, by Dr. H. D. Brunk, The Rice Institute.

3. *College level problems from seismic prospecting*, by Dr. E. J. Stulken, Geophysical Service, Inc.

4. *Some applications of extended analytic geometry*, by Dr. R. S. Underwood, Texas Technological College.

5. *The exact square root of any positive number by construction*, by P. J. C. Lauback, Westminister College, introduced by the Secretary.

6. *Freshman mathematics for business students*, by C. R. Sherer, Texas Christian University.

7. *The training of mathematics teachers for secondary schools*, by Dr. E. H. Hanson, North Texas State Teachers College.

C. R. SHERER, *Secretary*

APRIL MEETING OF THE MISSOURI SECTION

The annual meeting of the Missouri Section of the Mathematical Association of America was held at the University of Kansas City, Kansas City, Missouri, on April 23, 1948. Professor P. R. Rider presided.

About sixty persons were in attendance, including the following twenty-eight members of the Association: L. M. Blumenthal, C. H. Brown, C. B. Burcham, L. H. Cutting, C. E. Denny, W. C. Doyle, D. H. Erkiletian, Jr., G. M. Ewing, D. G. Ewy, C. A. Johnson, L. O. Jones, J. C. Koken, Sister M. Pachómia Lackay, L. E. Laird, Walter Leighton, C. W. Mathews, E. F. Moore, S. T. Parker, A. D. Pierson, G. B. Price, R. M. Rankin, P. R. Rider, J. S. Rosen, R. G. Sanger, G. W. Smith, R. G. Smith, C. B. Tucker, G. B. Van Schaack.

The University of Kansas City was host at a luncheon, at which Dean Norman Royall welcomed those present.

The following officers were elected for the coming year: Chairman, P. R.

Rider, Washington University; Vice-Chairman, W. C. Doyle, Rockhurst College; Secretary, C. W. Mathews, Washington University.

The program was arranged by Professor J. S. Rosen of the University of Kansas City. The following papers were presented:

1. *Almost periodic functions*, by Mr. Asger Aaboe, Washington University, introduced by Professor Walter Leighton.

The theory of almost periodic functions of a real variable, created by Harold Bohr, is briefly outlined.

2. *Convergence regions for continued fractions*, by Professor W. J. Thron, Washington University, introduced by Dr. C. W. Mathews.

The historical development of the subject is outlined. This includes the contributions of Worpitzky, A. Pringsheim, H. B. Van Vleck, H. S. Wall, W. Leighton, and the author. The second part of the paper consists of a brief discussion of the principal methods of proof used.

3. *The definition of the Dirac δ-function*, by Professor G. M. Ewing, University of Missouri.

The so-called Dirac δ-function not only appears in the literature as far back as Kirchhoff, but is introduced in many recent books, usually with an apology, followed by the statement that nevertheless valid results are obtained by following the stated conventions. The present paper introduces a certain class K of functions $\delta_\epsilon(t)$, and points out that the limits of a number of integrals involving δ_ϵ provide interpretations for familiar statements about the δ-function. Such results are independent of the choice of δ_ϵ in K. Linear differential equations $F(D)x=\delta_\epsilon(t)$ are also considered.

4. *The exponential function in applied science*, by Professor Herman Betz, University of Missouri, introduced by Professor L. M. Blumenthal.

The assumptions usually made in describing certain theorems occurring in the natural sciences by means of the exponential function can be considerably relaxed. This paper indicates how this may be done, even in undergraduate courses.

5. *Rational right triangles*, by S. G. Campbell, University of Kansas City, introduced by Professor J. S. Rosen.

The speaker discussed the problem of finding a method of obtaining a series of primitive rational right triangles, the ratios of whose sides are increasingly close approximations to the ratios of the sides of any given irrational right triangle. The method used involves use of a theorem by Euler for expressing the roots of a quadratic equation in terms of a continued fraction. He also dealt with the problem of finding a general method of locating the series of all primitive rational right triangles with a given relation between the sides (as a given difference between any two sides.) The method developed uses a series (recurrent) with scale which is also satisfied by consecutive coefficients generated by a particular rational fraction.

6. *A theorem on determinants*, by Professor G. B. Price, University of Kansas.

This paper contains a simple proof of the following theorem and some of its generalizations: If (a_{ik}), $i, k=1, 2, \cdots, n$, is any matrix of complex numbers such that

$$|a_{ii}| > \sum_{k\neq i} |a_{ik}|, \qquad i = 1, 2, \cdots, n$$

then the determinant $|a_{ik}| \neq 0$. This theorem has been rediscovered repeatedly ever since Levy

published the first proof of it in 1881—two first discoveries are reviewed in the 1947 volume of *Mathematical Reviews*. The theorem is one of considerable importance in both pure and applied mathematics, and its history emphasizes the need for a better dissemination of known results.

7. *The importance of computational technique in applied mathematics*, by Y. L. Luke, Midwest Research Institute, Kansas City, Missouri, introduced by Professor J. S. Rosen.

The complexity of many problems in applied mathematics makes important the need for computational technique. Some examples met in actual practice are given. The need for courses in applied mathematics in the undergradutate school is emphasized.

8. *A new trigonometric shifting theorem*, by Professor Eugene Stephens, Washington University, introduced by Professor P. R. Rider.

We may shift sin ax, cos ax, sinh ax, cosh ax across a linear differential operator of the form $F(D)=\sum a_nD^n$ in the following manner:

$$\text{(A)}\qquad F(D)\cdot\sin ax \equiv \sin ax\cdot F(D+aC),\qquad C\equiv\cot ax$$
$$\text{(B)}\qquad F(D)\cdot\cos ax \equiv \cos ax\cdot F(D-aT),\qquad T\equiv\tan ax$$
$$\text{(C)}\qquad F(D)\cdot\sinh ax \equiv \sinh ax\cdot F(D+aC_h),\qquad C_h\equiv\coth ax$$
$$\text{(D)}\qquad F(D)\cdot\cosh ax \equiv \cosh ax\cdot F(D+aT_h),\qquad T_h\equiv\tanh ax$$

If in the right hand side of these operator identities there appears anywhere C^2, T^2, C_h^2, T_h^2, each can be replaced by -1. These are considered as constants and as commutative with the operator D. The resulting operations on a given subject are simplified by the elementary operational theorems. In the final results the C, T, C_h, T_h are replaced by their respective trigonometric forms.

9. *The real representation of imaginary loci*, by L. E. Laird, Kansas State Teachers College.

This paper presents a method of representation of the pairs of complex numbers (complex points) which satisfy an equation $w=f(z)$, where $w=u+iv$ and $z=x+iy$. The method sets up a one-to-one correspondence between the complex points and real lines in space by use of Plucker's line coordinates. If a functional relationship is assumed between x and y, a single infinity of real lines is determined which constitute a ruled surface.

10. *The convergence in probabilities of statistical sequences*, by Dr. Maria Castellani, University of Kansas City, introduced by Professor J. S. Rosen.

The theory of fitting a curve to a given statistical sequence of data may be considered as a special type of convergence in probabilities. The purpose of the paper is to show how the several definitions of convergence may suit the statistical data, and how results obtained may be tested. The continuity in probabilities of statistical sequences is also considered with special regard to the case in which the sequence converges to a parabola or to a sinusoid.

11. *Mathematics placement tests at the University of Kansas*, by Professor G. W. Smith, University of Kansas.

Professor Smith points out the need for some kind of a placement test in mathematics, and explains the plan used at the University of Kansas. He points out that in September 1946 about twenty per cent of the entering freshmen were affected by the test. The classes were made more nearly uniform, and the college algebra course was strengthened.

12. *Preparation for college mathematics*, by Professor W. C. Doyle, Rockhurst College.

Professor Doyle conducted a panel discussion on ways and means to encourage high schools to strengthen their mathematics programs. It was remarked that colleges should better coordinate what they expect of freshmen, and make better use of placement tests.

P. R. RIDER, *Secretary*

APRIL MEETING OF THE ROCKY MOUNTAIN SECTION

The thirty-first annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at Colorado A. and M. College, Fort Collins, Colorado, April 23 and 24, 1948. Professor H. T. Guard presided at all the sessions.

Among the eighty-one persons who registered were the following thirty members of the Association: C. F. Barr, D. L. Barrick, W. G. Brady, J. R. Britton, F. M. Carpenter, A. G. Clark, G. S. Cook, A. B. Farnell, F. N. Fisch, H. T. Guard, Leota C. Hayward, I. L. Hebel, Ruth I. Hoffman, LeRoy Holubar, Burrowes Hunt, J. A. Hurry, C. A. Hutchinson, A. J. Kempner, Claribel Kendall, A. J. Lewis, M. L. Madison, A. E. Mallory, W. K. Nelson, Greta Neubauer, K. L. Noble, Nathan Schwid, S. R. Smith, L. C. Snively, V. J. Varineau, Lillie C. Walters.

At the business meeting, the officers elected for the coming year were: Chairman, Professor I. L. Hebel, Colorado School of Mines; Vice-Chairman, Professor A. J. Lewis, University of Denver; Secretary-Treasurer, Professor J. R. Britton, University of Colorado. Professor A. J. Lewis was also elected Sectional Governor for a term of three years. A resolution commending Professor Abraham Wald for the excellence of his invited addresses was unanimously adopted.

The program of papers presented was as follows:

1. *A method of defining the real number system*, by Robert Howerton, University of Denver, introduced by A. J. Lewis.

2. *A slow-motion algorithm*, by Burrowes Hunt, University of Colorado.

The euclidean algorith for two relatively prime integers $a>b$ which leads to the equations

$$a = q_1 b + r_1, \qquad b = q_2 r_1 + r_2, \cdots$$

is modified by taking each $q_i = 1$. This algorithm terminates if and only if a and b are successive integers of the Fibronacci sequence. The least positive remainder is 1 if and only if, as a regular continued fraction, $a/b = (1; 1, \cdots, 1, k)$, k being an arbitrary positive integer. If $a/b = (1; 1, \cdots, 1_n, a_0, a_1, \cdots, a_k)$, and $(a_0; a_1, \cdots, a_k) = q/r$, the algorithm gives a least positive remainder r on the nth step.

3. *A note on expansion of determinants*, by Professor W. R. Eikelberger, University of Denver, introduced by A. J. Lewis.

4. *An approximation to the solution of a non-linear partial differential equation*, by Professor Nathan Schwid, University of Wyoming.

The differential equation of heat conduction

$$c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(K \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(K \frac{\partial u}{\partial z}\right)$$

is non-linear when the conductivity K is a function of u. Here the diffusivity $K/c\rho$ is taken as $\alpha^2+\beta^2u$, where the ratio β^2/α^2 is small. The heat is considered as flowing in the x direction only in a plate of finite width. The solution of the resulting non-linear equation is approximated by a modification of a method given by Kirchoff about 1890 in the *Annalen der Physik* for an analogous problem involving flow of heat in one direction in a semi-infinite solid with the same type of diffusivity as above.

5. *Expansion of functions in combinations of generalized hypergeometric functions*, by Professor Leonard Bristow, University of Wyoming, introduced by C. F. Barr.

The purpose of this paper is to obtain the expansion of a suitable arbitrary function of a real variable in a series of solutions of a self adjoint differential equation of the Cauchy or Euler type containing a parameter. There are one-point boundary conditions (taken to be at $x=1$) together with regularizing conditions at the regular singular point (taken to be at $x=0$) of the differential equation. A Green's function is obtained. Fourier series, Fourier-Bessel, and Dini expansions in Bessel functions are obtained as special cases.

6. *Introduction to sequential analysis*, by Professor Abraham Wald, Columbia University.

7. *Principles of sequential analysis*, by Professor Abraham Wald, Columbia University.

These papers by Professor Wald were invited addresses.

8. *Ivory Towers*, by Miss Ruth I. Hoffman, Denver, Colorado.

There is need for college mathematics teachers to step down and become acquainted with the content of secondary mathematics, the problems and factors that influence the type of courses offered, and the quality of these courses. The university people should know of the valiant struggle that mathematics teachers in secondary education are making to keep up with modern educational trends, and to meet the needs of the present student body while still teaching sound mathematics, and even showing the beauty, as well as the usefulness, of mathematics.

9. *On certain equations involving radicals*, by Harlan Bartram, University of Colorado, introduced by A. J. Kempner.

The following problem was discussed: Given

$$\sqrt{a+bi}+\sqrt{c+di}=f, \qquad i=\sqrt{-1}.$$

If a, b, c, and d are real, and the signs of the radicals are properly chosen, when will f be real? The necessary and sufficient condition for this was found to be that $(b^2-d^2)^2=4(a-c)(ad^2-cb^2)$.

10 *Some teaching devices in undergraduate mathematics*, by Professor S. R. Smith, University of Wyoming.

Experience has shown that the majority of students entering college have difficulty in mathematics courses. At least part of this difficulty is due to lack of organization of their work, particularly in the solution of problems, and to the interpretation of the solutions found. Teaching devices, not necessarily new, are suggested to facilitate the solution of systems of quadratic equations, the discussion and sketching of plane curves in analytic geometry, and the application of the first and second derivatives in calculus.

11. *The book says so*, by Professor A. E. Mallory, Colorado State College of Education.

In this paper attention is called to the importance of teaching the connection between the solution of equations and the concept of a function.

12. *What constitutes good mathematics for undergraduates?* by Professor A. G. Clark, Colorado A. and M. College.

The author used the article, *Can We Teach Good Mathematics to Undergraduates?* by R. G. Helsel and T. Radó, which appeared in the January, 1948 issue of this MONTHLY, as the basis for his discussion. He agreed in part with the opinions of Helsel and Radó, but the extent of the agreement was dependent upon the connotation given the word "elegant," a term which mathematicians seem to have appropriated. The concept of "efficient" mathematics for undergraduates was presented.

13. *Report on entrance requirements,* by Professor A. J. Kempner, University of Colorado.

This paper was a brief report on the discussions which were held and the resolutions which were passed at the meetings of the Mathematical Association at Athens, Georgia, and the National Council of Teachers of Mathematics at Indianapolis, Indiana, in connection with the problems of lowering college entrance requirements and standards in general, and those pertaining to mathematics in particular.

Following a short discussion of the last paper, the following resolution was unanimously adopted: The Rocky Mountain Section of the Mathematical Association of America approves whole-heartedly the recent action of the Mathematical Association and the National Council of Teachers of Mathematics in expressing their desire for the closest cooperation in the critical problems confronting secondary and college mathematics.

J. R. BRITTON, *Secretary*

CALENDAR OF FUTURE MEETINGS

Joint Meeting with American Society for Engineering Education, Troy, New York, June 20–21, 1949.

Thirty-first Summer Meeting, Boulder, Colorado, August 29–30, 1949.

Thirty-third Annual Meeting, New York City, December 30, 1949.

ALLEGHENY MOUNTAIN West Virginia University, Morgantown, May 7, 1949

ILLINOIS, Bradley University, Peoria, May 13–14, 1949

INDIANA, University of Notre Dame, Spring, 1949

IOWA, Drake University, Des Moines, April 15–16, 1949

KANSAS, Manhattan, April 2, 1949

KENTUCKY

LOUISIANA-MISSISSIPPI, University of Mississippi, Oxford, Spring, 1949

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK Brooklyn College, April 9, 1949

MICHIGAN

MINNESOTA

MISSOURI

NEBRASKA, Lincoln, May, 1949

NORTHERN CALIFORNIA

OHIO, Ohio State University, Columbus, April 2, 1949

OKLAHOMA

PACIFIC NORTHWEST, Oregon State College, Corvallis, March 25–26, 1949

PHILADELPHIA, Haverford College, November 26, 1949

ROCKY MOUNTAIN, Colorado School of Mines, Golden, April, 1949

SOUTHEASTERN, University of Alabama, University, March 18–19, 1949

SOUTHERN CALIFORNIA, John Muir Junior College, Pasadena, March 12, 1949

SOUTHWESTERN

TEXAS, Denton, Spring, 1949

UPPER NEW YORK STATE, University of Buffalo, April 30, 1949

WISCONSIN, Lawrence College, Appleton, May 14, 1949

CANADIAN
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Journal Canadien de Mathématiques

The chief languages of the *Journal* are English and French.

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All other correspondence should be addressed to the *Managing Editor,* G. de B. Robinson, University of Toronto.

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Outline of the History of Mathematics

By

RAYMOND CLARE ARCHIBALD

Professor of Mathematics, Emeritus, Brown University

Sixth edition, revised and enlarged

The Second

HERBERT ELLSWORTH SLAUGHT

MEMORIAL PAPER

Published as a supplement to the AMERICAN MATHEMATICAL MONTHLY

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VOLUME 56 NUMBER 1

PART II

Outline of the History of Mathematics

RAYMOND CLARE ARCHIBALD

"Mad Mathesis alone was unconfin'd,
Too mad for mere material chains to bind,
Now to pure Space lifts her ecstatic stare,
Now running round the circle finds it square."
POPE

Number 2
of the
HERBERT ELLSWORTH SLAUGHT
MEMORIAL PAPERS

JANUARY 1949

ONE DOLLAR A COPY

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

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PREFACE

This *Outline* originally consisted of two Lectures delivered at the University of Minnesota in September 1931, at a Summer School for Engineering Teachers organized by The Society for the Promotion of Engineering Education. In February 1932 the Society published an edition of 1030 copies of the *Outline* as number 18 of its *Selected Papers*. By the end of 1933 this edition was exhausted, and since it was contrary to the policy of the Society to publish second editions of such *Papers*, all rights in the publication were made over to the Mathematical Association of America, Inc., which felt that it might be rendering a service to teachers of the History of Mathematics, and to others, by publishing a second revised and enlarged edition of the *Outline*. This appeared in July 1934.

The similarly revised and enlarged third, fourth and fifth editions appeared successively in June 1936, February 1939, and June 1941. And now with the fifth edition entirely exhausted The Mathematical Association is about to publish the largest edition yet, 7500 copies, and to send it forth as one of the *Papers* in memory of Professor Slaught—one of my most highly prized friends for the last twenty years of his life. No longer hampered by changes in standing type I have devoted my best efforts to making this Memorial worthy. Every statement in the main text was carefully reconsidered, every reference in the Literature List and Notes (LLN) was once more consulted; as a result many errors were eliminated. Moreover it has been my endeavor to present the very latest results of scholarly research, and in the 328 LLN greatly to augment the list of topics discussed, and sources where material may be found for notably amplifying the text. This material ought to be of special value to the large group of teachers of the History of Mathematics in this country. According to a survey in *Scripta Mathematica*, v. 2, pp. 161–165, 1934, courses in this field were offered during the years 1932–34 in 161 institutions: Liberal Arts Colleges and Universities (125), Teachers' Colleges and Normal Schools (32), and Technical Schools (4).

The main text of this edition has undergone very many changes but it has been only slightly enlarged, through such things as the increase of guiding material connected with Babylonian and Arabic mathematics, the transfer from LLN to the text of remarks on achievements of PLATO, PACIOLI, STEVIN, KEPLER, and SACCHERI, and through the addition, among others, of notes about THEON, PROCLUS, and JAMES GREGORY. That references to astronomy are desirably more numerous than in earlier editions will be observed.

There are a good many passages where current histories are amplified, or changed, or corrected. Perhaps the most striking one is the placing of HERON, MENELAUS, and DIOPHANTUS together in the first century of our era. On most questions, and especially those which are debatable, authorities are always noted. It is not claimed that DIOPHANTUS certainly flourished in the first century, but merely that evidence definitely points to this as a possibility, while denying the correctness of arguments assigning him to the third century.

Should the scholar happen to turn these pages he should understand that in many LLN excessive details and numerous odd bits of information were not assembled for him, but intended to help the teacher in elaborating and vivifying his material for class use. The Index of Names has been improved by the inclusion of many dates.

In preparing this edition I have had the privilege of constant consultation with my colleague and friend Professor OTTO E. NEUGEBAUER, the greatest living authority on ancient mathematics and astronomy. My debt to him in this connection is indeed very great. Here, as in earlier editions and in other publications, I am also deeply indebted to my friend Mr. S. A. JOFFE, who added materially to a desirable final form of the printer's manuscript and proof by very numerous suggestions for improvement.

I should be grateful to any one informing me of inaccuracies found in this *Outline*.

R. C. A.

Brown University
Providence, Rhode Island

INTRODUCTORY NOTE

An attempt is here made to give indications of the development of mathematics before the nineteenth century, and to refer briefly to some developments of the nineteenth and twentieth centuries in connection with topics usually discussed at undergraduate colleges. Chinese and Japanese mathematics are not considered in such a skeleton survey, and the reference to mathematics of the Hindus is brief.

It was felt that consideration of only what was most important from a purely mathematical point of view would not be desirable in the survey, partly because many of the most important things cannot, in a few sentences, be made intelligible to one wholly unfamiliar with the subject, and partly because the quotation of complicated results in a lecture is not possible. Hence many minor items in the development of the science have been introduced in order that interest may be sustained through knowing the historical setting for certain simply stated, or well-known, results.

While an endeavor has been made to present the results of the latest investigations, no claim for originality can be made; phrases, quotations, and adaptations from many sources have been used constantly. In the Literature List and Notes all such sources are indicated and in the text many references are given to them in order that the reader may if he choose inform himself more fully concerning matters discussed. We indicate the chief worthwhile sources in English (many of them popular in style); but a scholarly grasp of any field would, however, necessitate the consultation of literature in other languages, and many sources of this kind have been listed.

SYNOPSIS

Following SMITH (see Literature List and Notes [4] = LLN 4) I have used such an abbreviation as −1850 for 1850 B.C. in the synopsis which follows, and have associated with the name of each mathematician a single definite date indicating a year about when he flourished or when he published some outstanding work.

I. History of Mathematics before 1600 A. D.

A. Babylonian and Egyptian Mathematics and Astronomy 3100 B.C. to 1 B.C.

Babylonian: Geometry and algebra, solution of quadratic and cubic equations, tables, compound interest, square root approximations, series texts and simultaneous equations, astronomy, NABURIANU and KIDINNU, −2000 to −1

Egyptian: Great pyramid, −2900; obelisks; timepieces; Moscow papyrus −1850; RHIND papyrus −1650; astronomy

B. Greek Mathematics 600 B.C. to 600 A.D.

THALES (−600): Geometry

PYTHAGOREANS (−540 to −400): Geometry, arithmetic, music

Famous problems (−450):

(*a*) Trisection of an angle: HIPPIAS of Elis (−425) with quadratrix, NICOMEDES (−240) with conchoid

(*b*) Squaring the circle: DINOSTRATUS (−350) and NICOMEDES (−240) with quadratrix, ARCHIMEDES (−225) with his spiral

(*c*) Duplication of the cube: HIPPOCRATES of Chios (−460) with two mean proportionals, ARCHYTAS (−400) with surfaces, DIOCLES (−180) with cissoid

ZENO of ELEA (−450): paradoxes

PLATO (−380): mathematics important in training mind

THEAETETUS (−375): Incommensurables, regular solids

EUDOXUS (−370): Incommensurables, method of exhaustion, astronomy

MENAECHMUS (−350): Conic sections

EUDEMUS (−335): historian of science

AUTOLYCUS of Pitane (−330): Spherical geometry [75]

EUCLID (−300): Elements, perfect numbers, conics, optics

ARISTARCHUS of Samos (−287): Copernican system

ARCHIMEDES (−240): Geometry, arithmetic, mechanics, hydrostatics

ERATOSTHENES (−230): Size of earth, sieve

APOLLONIUS of Perga (−225): Conic sections, plane loci, tangencies, epicycles, eccentrics

HIPPARCHUS (−140): Trigonometry, solar and lunar movement, star catalogue

THEODOSIUS of Bithynia (between −180 and −25): Spherical geometry [75]

HERON of Alexandria (+75): Pneumatica, cube root, torus volume

DIOPHANTUS of Alexandria (perhaps 75): Diophantine equations

MENELAUS of Alexandria and Rome (100): Spherical geometry and trigonometry

PTOLEMY of Alexandria (150): Trigonometry, projections, lunar and planetary theory, star catalogue, geography, music, optics

PAPPUS of Alexandria (300): Mathematical Collection, Commentaries on Euclid's Elements, bk. 10, and Ptolemy's *Almagest*

THEON of Alexandria (390): Edition of Euclid's Elements, commentary on the *Almagest* assisted by his daughter Hypatia

PROCLUS (470): Valuable commentary on Euclid's Elements and his Hypotyposis of Astronomical Hypotheses

Hindu Mathematics before 600: [113]

C. Hindu, Arabic, Persian Mathematics 600 to 1200

BRAHMAGUPTA (630): Area of an inscribed quadrilateral

BHĀSKARA (1150): Linear and quadratic indeterminate equations

Hindu-Arabic numerals (−300 to +876)

AL-KHOWÂRIZMÎ (820): Algebra, numeration, algorism, tables, astronomy

ABÛ'L WEFÂ (980): Geometric constructions with ruler, and compasses having a fixed opening, trigonometry, tables

OMAR KHAYYÀM (1100): Solutions of cubic equations, calendar

D. European Mathematics 1200 to 1600

LEONARDO of Pisa (Fibonacci) (1202): *Liber Abaci, Liber Quadratorum, Practica Geometriae, Flos*

NASÎR ED-DÎN AL-TÛ'SÎ (1250): First independent work on trigonometry, commentary Euclid's Elements

ULUGH BEG (1435): Trigonometric tables

REGIOMONTANUS (JOHANN MÜLLER) (1470): Trigonometry, astronomy

First printed arithmetic (1478)

LUCA PACIOLI (1494): Sūma (including double-entry bookkeeping), "divine proportion"

SCIPIONE DEL FERRO and ANTONIO FIOR (1506): Cubic equation

TARTAGLIA (NICOLO of Brescia) (1535): Cubic equation

GIROLAMO CARDANO (1545): Algebra

FERRARI (1545): Biquadratic equation

RHETICUS (1550): Tables of natural trigonometric functions
NICOLAUS COPERNICUS (1530): System of the universe
SIMON STEVIN (1590): Systematic explanation decimal fractions, compound interest table, statics, hydrostatics
ROBERT RECORDE (1542): Arithmetic, geometry, algebra **[162]**
VIETA (1580): Algebra, trigonometry, notation
First mathematical book published in the New World (1556)

II. History of Mathematics after 1600

A. The Seventeenth Century

JOHN NAPIER (1614): Logarithms, trigonometry, computing rods
HENRY BRIGGS (1624): Logarithms to the base 10
JOHANN KEPLER (1610): Laws for motion of planets, volumes, maxima and minima, star polyhedra, principle of continuity in geometry, logarithms
GALILEO GALILEI (1600): Falling bodies, projectiles, cycloid, momentum
THOMAS HARRIOT (1600): Algebra, symbolism
WILLIAM OUGHTRED (1630): *Clavis Mathematicae*, algebra, slide rule, notations, first table of natural logarithms
GIRARD DESARGUES (1640): Projective geometry, treatise on singing
ÉTIENNE PASCAL (1635): Limaçon
BLAISE PASCAL (1650): Conics, computing machine, cycloid, theory of probabilities
GEORGE MOHR (1672): Constructions with compasses only
RENÉ DESCARTES (1637): Analytic geometry, equations, $F+V=E+2$, folium, ovals
PIERRE DE FERMAT (1635): Theory of numbers, maxima and minima, theory of probabilities, analytic geometry
BONAVENTURA CAVALIERI (1635): Indivisibles
JAMES GREGORY (1670): Binomial theorem, "Taylor's" theorem, expansion of functions into series, integrations
JOHN MACHIN (1706): π in terms of inverse tangents **[201]**
CHRISTIAAN HUYGENS (1670): Circle quadrature, theory of probabilities, evolutes, pendulum clocks, catenary
ISAAC BARROW (1670): Editor of works of EUCLID, APOLLONIUS, THEODOSIUS; geometry, differentiation, optics **[210]**
JOHN WALLIS (1650): Algebra, imaginary numbers, length of curves, π as infinite product
ISAAC NEWTON (1680): Fluxions, dynamics, hydrostatics, hydrodynamics, gravitation, cubic curves, series, numerical equations, imaginary roots, challenges
GOTTFRIED WILHELM LEIBNIZ (1682): Newton controversy, calculus, determinants, polynomial expansions, notations
JAMES I BERNOULLI (1690): Isochronous curves, Euler's spiral, logarithmic spiral, calculus of probability
EDMOND HALLEY (1690): **[211]**; GILLES PERSONE DE ROBERVAL (1640): **[197]**; EVANGELISTA TORRICELLI (1640): **[197]**

B. The Eighteenth Century

ANTOINE PARENT (1700): ALEXIS CLAIRAUT (1731), and JAKOB HERMANN (1732–33): Analytic geometry of three dimensions
GIROLAMO SACCHERI (1733): Noneuclidean geometry prelude
ABRAHAM DEMOIVRE (1720): Actuarial mathematics, theory of probabilities, trigonometry with complex quantities, $m! \approx (2\pi m)^{\frac{1}{2}} e^{-m} m^m$
BROOK TAYLOR (1720): Expansion in series, finite differences, vibrating string
COLIN MACLAURIN (1740): Fluxions, organic description of curves
LEONARD EULER (1750): Publications, notations, $e^{i\pi}+1=0$, Euler's line, biquadratic

equation, indicatrix, vibrating string, calculus of variations, Euler's spiral, beta and gamma functions, Euler's constant

CASPAR WESSEL (1797): Imaginary quantities; JEAN CHARLES BORDA (1799); JEAN CHARLES CALLET (1795), and JOSEPH JERÔME FRANÇOIS DE LA LANDE (1775): Mathematical tables; JEAN ÉTIENNE MONTUCLA (1775): Histories of mathematics **[246]**

JOSEPH LOUIS LAGRANGE (1780): Calculus of variations, differential equations, mechanics, calculus foundations, theory of numbers, numerical equations

JOHANN HEINRICH LAMBERT (1770): *Freye Perspective*, parabola property, π, e^x, tan x, irrational, hyperbolic functions, map projection

WILLIAM WALLACE (1798): Geometry **[251]**

GASPARD MONGE (1795): Descriptive geometry, differential geometry

C. The Nineteenth Century and Later

PIERRE SIMON LAPLACE (1805): Astronomy, celestial mechanics, probabilities, differential equations

ADRIEN MARIE LEGENDRE (1805): Theory of numbers, elliptic functions and integrals, exercises in integral calculus, geometry

JEAN ROBERT ARGAND (1806): Imaginary quantities **[246]**

JOSEPH FOURIER (1812–1822): Series expansions, theory of equations, physics **[246]**

JEAN BAPTISTE DELAMBRE (1820): Histories of astronomy, trigonometry **[246]**

Noneuclidean geometry: BOLYAI and LOBACHEVSKY (1825)

KARL FRIEDRICH GAUSS (1820): Polygon constructions, theory of numbers, differential geometry, algebraic equations, etc.

Calculating prodigies: ZACHARIAS DASE (1855)

Projective and modern geometry: VICTOR PONCELET (1830) and JAKOB STEINER (1840); duality, CHARLES J. BRIANCHON (1806)

Trisection of an angle; duplication of cube proved impossible (1837)

Inversion: STEINER (1824), QUETELET (1825), linkages

Algebraic solution of the general quintic equation impossible: NIELS HENRIK ABEL (1824)

Approximating and locating roots of numerical equations: NEWTON (1669); RUFFINI (1804), HORNER (1819), FOURIER (1820), STURM (1829), CAUCHY (1831). See also **[246, 285]**

Convergence of series: NEWTON, GAUSS, CAUCHY

JOSEPH LIOUVILLE (1850): Transcendental numbers **[253]**

CHARLES HERMITE (1882): Transcendence of e **[254]**

Geometry of the triangle: LEMOINE and BROCARD (1873–75, and later)

Geometrography: LEMOINE (1888 f.)

Vector analysis: GIBBS (1881–84), HEAVISIDE (1891)

Quaternions: W. R. HAMILTON (1853), Ausdehnungslehre: H. G. GRASSMANN (1844, 1862) **[292]**

Determinants: LEIBNIZ, LAGRANGE, CAUCHY, JACOBI

Mathematical Tables: U. S. Nat. Res. Council Comm. on Math. Tables and Other Aids to Computation; U. S. Nat. Bureau of Standards Comm. on Math. Tables; Br. Assoc. Adv. Sci. [now Royal Soc.] Comm. Math. Tables, J. W. L. GLAISHER and CAYLEY (1873–1883, 1911); HENDERSON (1926), THOMPSON (1924–1948), ANDOYER (1915–1918), LEHMER (1909, 1914), DE MORGAN (1842–1861), DUFFIELD (1896), VEGA (1794). Also **[293–302a]**

Surveys of Pure and Applied Higher Mathematics: E. W. BROWN (1923), F. KLEIN (1926–27), R. COURANT (1926), R. S. WOODWARD (1900), BELL (1945), STRUIK (1948), Amer. Math. Soc. Semicentennial Addresses and History (1938), SMITH & GINSBURG (1934), CAJORI (1890), BEATTY (1939), PRASAD, BELL (1937), COOLIDGE (1940), KÖTTER (1901), LORIA (1931), *Teubners Sammlung, Grundlehren, Ergebnisse, Mémorials, Cambridge Tracts*. ISAAC BARROW. See also **[303–312]**

HISTORY OF MATHEMATICS BEFORE THE SEVENTEENTH CENTURY

A. BABYLONIAN AND EGYPTIAN MATHEMATICS AND ASTRONOMY 3100 B.C. TO 1 B.C. [13–26]

The period covered by this first lecture on the history of mathematics will be about 4700 years; for we shall begin by noting that in one of the great museums at Oxford is a royal mace [36] of 3100 B.C. on which there is a record of 120,000 prisoners, of 400,000 captive oxen, and of 1,422,000 captive goats. These numbers, written in Egyptian hieroglyphs, show that already in this ancient time not only was the decimal system of numeration, but also a method for writing very large numbers, thoroughly established. It is interesting to speculate on how many thousand years earlier must have been *the beginnings* when great communities, of a millennium later, with a highly developed social order, calling for the frequent use of mathematics of taxation, barter, interest, and of large numbers, were replaced by small groups of primitive people for whose simple needs such number words as "one," "two," and "heap" would wholly suffice [**4, 19**].

Any general primitive notions of this kind probably long antedated the execution of the recently discovered finest art-work of primitive man, reputed to be from 25,000 to 50,000 years old [**18**]. It is a sculpture of a white rhinoceros with a swarm of attendant tick-birds, hammered into a slab of basaltic rock, and showing extraordinary power of form, line, and perspective—elements intimately allied to things mathematical.

But there were other people besides the Egyptians who contributed notably to early mathematics. I refer to the non-Semitic Sumerians who lived just north of the Persian Gulf and south of the Semitic Akkadians, and who for many centuries prior to 2500 B.C. were generally predominant in Babylonia, but were absorbed into a larger political group by about 2000 B.C. One of the greatest of the Sumerian inventions was the adoption of cuneiform script; notable engineering works of the Babylonians, by means of which marshes were drained and the overflow of rivers regulated by canals, went back to Sumerian times, as also a considerable part of their religion, their law, and their mathematical notation [**13, 17, 19**].

An extraordinary number of tablets show that the Sumerian merchant of about 2000 B.C. was familiar with such things as weights and measures, bills, receipts, notes, and accounts. Long before coins were in use (7th century B.C.) it was common custom to pay interest for the loans of produce, or of a certain weight of a precious metal. Tablets indicate that the rate of interest varied from 20 to 30 per cent, the higher rate being charged for produce.

Sumerian or Babylonian arithmetic was essentially sexagesimal, although there are influences of a decimal system. Hence the general numeral might be written in the form $a_n \cdot 60^n + a_{n-1} \cdot 60^{n-1} + \cdots + a_1 \cdot 60 + a_0 \cdot 60^0 + b_{-1} \cdot 60^{-1} + \cdots + b_{-m} \cdot 60^{-m}$ or $a_n, a_{n-1}, \cdots a_1, a_0, b_{-1}, \cdots b_{-m}$. In particular 31, 6, 15 might

equal 111975. The symbol for unity was also the symbol for any one of the numbers 60^n where n is a positive or negative integer. Thus 31, 6, 15 might mean not only 111975, but also 6718500 or 1866 1/4 or 31 1/10 1/240, according as the 15 is taken as multiplied by 1, 60, or 1/60, or 1/3600. The uncertainty in this regard introduces certain difficulties in interpreting Babylonian mathematical texts. Another uncertainty was introduced through the fact that where a zero enters may be determined only by the context; so that, for example, 11, 7 or 11, , 7 might stand for 39607. But a blank space does not always mean zero in Old Babylonian tablets. So far as is at present known, no special symbol for zero was used by the Babylonians before about 400 B.C., and this symbol was used to indicate a zero both in the interior and at the end of the number.

In the field of geometry the Babylonian of 2000 to 1600 B.C. used the following results in concrete cases, from which we have to infer that they were familiar with the general rules:

1. The area of a rectangle is the product of the lengths of two adjacent sides.
2. The area of a right triangle is equal to one-half the product of the lengths of the sides about the right angle.
3. The sides about corresponding angles of two similar right triangles are proportional.
4. The area of a trapezoid with one side perpendicular to the parallel sides is one-half the product of the length of this perpendicular and the sum of the lengths of the parallel sides.
5. The perpendicular from the vertex of an isosceles triangle on the base bisects the base. The area of the triangle is the product of the lengths of the altitude and half the base. Indeed the Babylonians may have thought of this result for the area of a triangle other than right or isosceles, since such a triangle may be regarded as made up of adjacent or overlapping right triangles; but there is no known example of this use of the formula.
6. The "Pythagorean" theorem; for example, for triangles with sides corresponding to the numbers 3, 4, 5; 5, 12, 13; 8, 15, 17; 20, 21, 29; and many more; see after no. 11, below.
7. The angle in a semi-circle is a right angle **[55]**.
8. The length of the diameter of a circle is one-third of its circumference ($\pi=3$). The area of a circle is 1/12 of the square of its circumference (correct for $\pi=3$).
9. The volume of a rectangular parallelopiped is the product of the lengths of its three dimensions, and the volume of a right prism with a trapezoidal base is equal to the area of the base multiplied by the altitude of the prism.
10. The volume of a right circular cylinder is the area of its base multiplied by its altitude.
11. The volume of the frustum of a cone, or of a square pyramid, is equal to its altitude multiplied by one-half the sum of the areas of its bases. It has been conjectured that the Babylonians had also the equivalent of an exact

formula for the volume in the case of a square pyramid, namely

$$V = h\left[\left(\frac{a+b}{2}\right)^2 + \frac{1}{3}\left(\frac{a-b}{2}\right)^2\right],$$

where a and b are the lengths of the sides of the square bases [**20**]. This was known to HERON of Alexandria 1700 years later, and reduces to the extraordinary formula apparently known to the Egyptians.

The most remarkable recent discovery in connection with Babylonian mathematics was that made by Professor NEUGEBAUER, in tablet PLIMPTON 322, dated -1900 to -1600, at Columbia University, which contains a table of "Pythagorean" numbers [16]. Let l denote the longer, s the shorter side of a right triangle, and h its hypotenuse. The values of h and s are given in two columns of our text but the third column gives not l but h^2/l^2 with successive entries decreasing almost linearly while there is great variation in the other columns. He discusses the contents of this tablet at great length and is convincing in suggesting that the following Euclidean relations (*Elements*, book 10, prob. 28, lemma) were known more than a thousand years before Pythagoras.

$$l = 2pq, \qquad s = p^2 - q^2, \qquad h = p^2 + q^2,$$

p and q being relatively prime, and $p > q$. Also that the following relation (if $\bar{q} = 1/q$ and $\bar{p} = 1/p$) was not only known:

$$h/l = \tfrac{1}{2}(p\bar{q} + q\bar{p}),$$

but was the basis, with known tables of reciprocals, for the discovery of the successive entries of the tablet. In 13 of the 15 triangles the sides are relatively prime. The lengths of the sides of the largest triangle are:

650 700, 649 909, 1 080 541.

Was EUCLID indebted to Babylonians for his tenth book lemma? As yet no cuneiform mathematical tablets date from the period 1300 to 300 B.C. so that the origins of later Babylonian results are mostly unknown.

Numerous problems involving portions of a right triangle cut off by lines parallel to a side lead to systems of simultaneous equations, even as many as ten equations in ten unknowns; and also to the solution of quadratic equations. Moreover the Babylonian of 1800 B.C. evidently knew our formula for the solution of a quadratic equation with the positive sign before the radical. Many problems could be cited to prove this [**14, 16, 17, 21**]; let us consider one of them on a tablet in Strassburg dating from about 1800 B.C.

"An area [A] (consisting of) the sum of two squares [x^2+y^2] (is) 16, 40 [=1000]." The side of one square [y] (is) $\frac{2}{3}\left[\frac{\alpha}{\beta}\right]$ of the side of the other square [x], diminished by 10[d]. "What are the sides of the squares [x, and y]?"

Hence

$$x^2 + y^2 = A, \qquad y = \frac{\alpha}{\beta}\, x - d.$$

If $\zeta = x/\beta$, we get a quadratic equation:

$$(\alpha^2 + \beta^2)\zeta^2 - 2d\alpha\zeta = A - d^2,$$

the solution being

$$\zeta = \frac{1}{\alpha^2 + \beta^2}\, [d\alpha \pm \sqrt{\{d^2\alpha^2 + (\alpha^2 + \beta^2)(A - d^2)\}}].$$

Bearing in mind that, in the particular case, $A = 1000$, $d = 10$, $\alpha = 40$, $\beta = 60$, let us now compare with this the working given in the text, with our numbers substituted for the Babylonian, of which a few samples are given.

"You proceed thus:	
Square 10: this gives [1, 40] 100; subtract [1, 40] 100 from [16, 40] 1000: this gives [15, 0]900.	$A - d^2 = 900.$
Square [1, 0]60: this gives [1, 0, 0] 3600; 40^2 is [25, 40] 1600: 3600+1600=5200.	$\alpha^2 + \beta^2 = 5200.$
Multiply 5200 by 900: this gives 4680000.	$(\alpha^2 + \beta^2)(A - d^2) = 4680000.$
Multiply 40 by 10: this gives 400.	$\alpha d = 400.$
Square 400: this gives 160000.	$\alpha^2 d^2 = 160000.$
Add 160000 to 4680000: this gives 4840000.	$\alpha^2 d^2 + (\alpha^2 + \beta^2)(A - d^2) = 4840000.$
The square root of this is 2200.	$\sqrt{\{\alpha^2 d^2 + (\alpha^2 + \beta^2)(A - d^2)\}} = 2200.$
Add 400 already found: this gives 2600.	$d\alpha + \sqrt{\{\alpha^2 d^2 + (\alpha^2 + \beta^2)(A - d^2)\}} = 2600.$
What part of 5200 gives 2600? Answer: one-half (30 in text).	$\dfrac{d\alpha + \sqrt{\{\alpha^2 d^2 + (\alpha^2 + \beta^2)(A - d^2)\}}}{\alpha^2 + \beta^2}\ [= \zeta] = \frac{1}{2}.$
$\frac{1}{2}$ multiplied by 60 gives 30 as [side of] greater square.	$\frac{1}{2}\beta = \zeta\beta = x = 30.$
Multiply $\frac{1}{2}$ by 40: this gives 20.	$\frac{1}{2}\alpha = \zeta\alpha = 20.$
Subtract 10 from 20 and this gives 10 as [side of] lesser square."	$\zeta\alpha - d = y = 10.$

Surely such work "is wonderful in itself; it is equally extraordinary that these developments in arithmetic and algebra should have remained, for the most of 1800 years at all events, unknown to, or at least without (so far as we can judge) any traceable effect upon, the Greek pioneers in the same subject" [**22**].

Another interesting method of solution of quadratic equations may be noted. On a Louvre tablet of about 300 B.C. are four problems [**23**] concerning rectangles of unit area but with the sum of adjacent sides varying $x+y=a$, $xy=1$. The successive steps of the solution are equivalent to substitution in the formula

$\frac{1}{2}(x - y) = \pm \{[\frac{1}{2}(x + y)]^2 - xy\}^{1/2}$, or

$[\frac{1}{2}(x - y)]^2 = [\frac{1}{2}(x + y)]^2 - xy$ (EUCLID'S *Elements*, II, 5 and *Data*, prop. 85).

In a Yale text of about 1700 B.C. this same method of solution is applied [24] to the equations $x-y=7$, $xy=1$, 0. By this method also the solution of somewhat more complicated rectangle problems on Strassburg and Louvre tablets, [25] of about 1800 B.C., would follow at once.

Our present knowledge of Babylonian achievements in mathematics is mainly due to the extraordinary discoveries of OTTO NEUGEBAUER, a research professor and chairman of the History of Mathematics Department of Brown University. (Dr. SACHS is also a member of this Department.) He discovered not only most of the results set forth above, but also many other things; in particular that the Babylonians discussed problems involving cubic [26] and biquadratic equations. A tablet was discovered which gives not only the squares and cubes of all integers from 1 to 30, but also the results for the sum n^3+n^2 for this same range. Pure cubic equations with integral solutions could be solved by means of the table of cubes reversed, and an example of this kind is quoted. The use of tables of cubes was not previously surmised. Another problem, of about 1800 B.C., seems to call for the solution of simultaneous equations, $xyz+xy=1\frac{1}{6}$, $y=\alpha x$, $z=\mu x$, $(\alpha=2/3,\ \mu=12)$, which lead to $(\mu x)^3+(\mu x)^2=252$. The solution of this equation may be found from the n^3+n^2 table. Two problems lead to such equations. Another problem, of the same period, seems to lead to the general equation $\mu x^3+(1-\mu b)x^2-bx+a=0$, being derived from $xyz+xy=a$, $z=\mu x$, $x+y=b (a=7/6,\ b=5/6,\ \mu=12)$. But in the tablet it is stated that $\dfrac{xyz+xy}{\mu b^3}=\dfrac{x}{b}\cdot\dfrac{y}{b}\cdot\dfrac{z+1}{\mu b}=\dfrac{3}{5}\cdot\dfrac{2}{5}\cdot\dfrac{7}{10}$ whence it is inferred that $\dfrac{x}{b}=\dfrac{3}{5}$, $\dfrac{y}{b}=\dfrac{2}{5}$, $\dfrac{z+1}{\mu b}=\dfrac{7}{10}\cdot$ Thus, this general cubic equation is not solved by reduction to the "normal form," $n^3+n^2=c$, to which, however, NEUGEBAUER believes that the Babylonians were quite capable of reducing the general cubic equation, although as yet he has no evidence that they actually did do it. In connection with the n^3+n^2 table NEUGEBAUER noted also [27] that they may well have known the equivalent of the relation $\sum_{i=1}^{n} i^3=(\sum_{i=1}^{n} i)^2$ for various values of n.

Tablets at Yale University [16] containing hundreds of problems, without solutions, arranged in systematic order, are of great interest. Possibly these may date from 1600 B.C. In several cases of simultaneous equations for solution and leading to biquadratic equations, we find here a number of extraordinary examples of negative numbers in right-hand members. For instance, solve the equations

$$xy = 600 \quad \text{and} \quad 150(x-y)-(x+y)^2=-1000.$$

Babylonian astronomical texts of the third century B.C. [28] make explicit use of the rules, $+\times+=-\times-=+$;
$+\times-=-\times+=-$.

On a Berlin tablet closely related to Yale tablets, the equations (also leading

to a biquadratic equation)

$$xy = 600 \quad \text{and} \quad x^2 + \frac{1}{13}\left\{\frac{1}{19}\left[(x+y)^2 - 600\right] + 3y^2\right\} = 1000,$$

are especially interesting because of the "irregular" numbers in the fractions 1/13 and 1/19. All problems which have so far been found to call for division by such numbers as 7, 11, and 13, are arranged in such a way that the divisors will disappear in the course of the work.

Two other illustrations from the Yale tablets may be cited. A general cubic equation comes up in the discussion of volumes of frustums of a pyramid, as the result of eliminating z from equations of the type, $z(x^2+y^2)=A$, $z=(ay+b)$, $x=c$. An equation of the sixth degree (equivalent to a quadratic in x^3) results from the solution of equations of the form $xy=b$, $a_1x^2/y+a_2y^2/x+a_3=0$.

A sexagesimal number, n, is regular if its reciprocal, $\bar{n}$, is a finite sexagesimal expression, and irregular if it is not. The necessary and sufficient condition for n to be regular is that $n=2^\alpha 3^\beta 5^\gamma$, where α, β, γ are each a positive integer or zero. With the exception of a Yale tablet, tables of reciprocals contain only reciprocals of regular numbers. For an irregular number like 7 it might be thought that such approximations as 7/48 or 13/90 might be found, but this is not the case. On the unique tablet to which we have just referred are such approximations as the following:

$$\overline{59} = ;1,1,1 \qquad \overline{61} = ;0,59,0,59 \quad \text{and} \quad \overline{78} = ;0,56,9,13,50.$$

The Babylonian method of finding the reciprocal of any regular number, however complicated, is now well known [**29**]. For such a number c the basic relation is $\bar{c}=\bar{a}(\overline{1+b\bar{a}})$, where a and b are two numbers such that $c=a+b$, and a is a number whose reciprocal may be found in a standard table.

In Babylonian mathematics tables were constantly used, and in particular, tables of reciprocals, reducing division to multiplication, but these tables rarely go beyond two sexagesimal places (3600). There is, however, a Louvre text of about 300 B.C. with scores of larger entries and including one of an n, as a seven-place number (corresponding to eleven decimal places), leading to $\bar{n}$, a seventeen-place number (corresponding to twenty-nine decimal places). Such tables were necessary in the Babylonian astronomical calculations [**30**] of the time. Thus there was mathematical development to meet astronomical needs.

On another Louvre tablet about the time of ARCHIMEDES, NEUGEBAUER found two other suggestive problems [**32**]. One states that

$$1 + 2 + 2^2 + \cdots + 2^9 = 2^9 + 2^9 - 1.$$

Is this another way of writing $2\cdot 2^9 - 1 = (2^{10}-1)/(2-1)$, indicating knowledge of EUCLID's, or our own, formula for the sum of such a geometric series?

On the same tablet it is stated that

$$1\cdot 1 + 2\cdot 2 + 3\cdot 3 + \cdots + 10\cdot 10 = (1\cdot \tfrac{1}{3} + 10\cdot \tfrac{2}{3})\cdot 55 = 385.$$

Now $\sum_{i=1}^{n} i^2 = (1\cdot\frac{1}{3}+n\cdot\frac{2}{3})\cdot\sum_{i=1}^{n} i = \frac{1}{6}n(n+1)(2n+1)$, if we set $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$, a result known to the PYTHAGOREANS. This sum-of-squares formula was practically equivalent to one known to ARCHIMEDES. Did contemporary Babylonians also use it?

Istanbul tablets seem originally to have had tables of c^n, for $n=1$ to 10, for $c=9$, $c=16$, $c=100$, and $c=225$. An application for such a table would be in solving exponential equations of the type $a^x=b$. Such equations arose in work of the Babylonian, even in the case where x was not integral, as one may check by a Louvre tablet of about 1700 B.C. One problem here is to find how long it would take for a certain sum of money to double itself at 20% interest. The problem is, then, to find x, the number of years, in the equation $(1;12)^x=2$. $1;12^3<2<1;12^4$, hence $x>3$ but <4. Linear interpolation gives

$$\frac{1;12^4 - 2}{1;12^4 - 1;12^3} = 0;12,46,40 \text{ years};$$

or 2;33,20 months. That is, $x=4$ years$-$2;33,20 months.

Since the final result thus found is exactly what is given in the tablet, the method used in reaching it, may well have been that employed by the Babylonians. Both in the Berlin Museum and in Yale University are tablets with other problems in compound interest.

Babylonian approximations to the square roots of non-square numbers are of interest; for example, 1 5/12 for $\sqrt{2}$, and 17/24 for $1/\sqrt{2}$; here, and in finding $\sqrt{1700}$, an approximation formula equivalent to one employed by HERON of Alexandria [97] seems to have been used. In finding 2β as approximation to $\sqrt{2\frac{1}{2}}$, the use of the equivalent of a DIOPHANTINE equation is suggested. But the most remarkable approximation is that found in a Yale tablet of about -1600 for the diagonal of the side of a unit square $\sqrt{2}\approx 1;24,51,10$ which is 1.414213 instead of 1.414214. In explaining the possible derivation of this result NEUGEBAUER & SACHS present a strong case for the Babylonians having followed a procedure consisting of alternating approximation of $\sqrt{2}$ by arithmetic and harmonic means of approximations previously found [33].

The brief suggestions which we have given of discussions of purely numerical problems of very varied types, could be greatly magnified, and force upon us the conclusion that Babylonians during 2000 years before the Christian era laid the foundations of real algebra. Even where the foundation is apparently geometric the essence is usually strongly algebraic, illustrated by the fact that frequently operations occur which do not admit of a geometric interpretation, like the addition of areas and lengths, or multiplication of areas. The predominant problem consists in the determination of unknown quantities subject to given conditions. Of course a certain number of geometric relations were well known. Numerical calculations are everywhere carried out with the greatest facility and skill.

The criterion for scientific mathematics must be the existence of the concept of proof. Egyptian mathematics contains only two (rather trivial) general

rules (RHIND papyrus [43, 44], problems 61, 66), and the much more highly developed Babylonian mathematics displays substantial illustrations of general techniques for proving its procedures [34]. But the remarkable mathematical astronomy which flourished during the last three centuries preceding the Christian era, and had as its goal the computation of ephemerides for the moon and planets, was distinctly scientific [27, 28].

Extension of our knowledge in connection with Babylonian mathematics and astronomy is likely long to continue. Let us now consider achievements of the Egyptians in pure and applied mathematics [35], where recent research has contributed little that is new.

We have referred to the mace record [36] of 3100 B.C. The erection of the great pyramid at Gizeh about 2900 B.C. must have involved many mathematical problems. Here was a huge structure covering more than 13 acres which, with its marvelous connecting roadway to the Nile, took 100,000 workmen 30 years to build. Over 2,000,000 blocks of stone, averaging $2\frac{1}{2}$ tons in weight, and fitted together with great exactness were brought from sandstone quarries on the opposite side of the Nile. For the roofs of chambers granite blocks 27 feet long and 4 feet thick, weighing 54 tons each, were transported from the quarry more than 600 miles away, and placed in their position over 200 feet above the ground level [37].

But the problems of mechanics and engineering involved in handling even the large stone blocks of the great pyramid were slight as compared with those dealt with by the Egyptians in quarrying and setting up some of their huge obelisks of pink granite. The largest existing obelisk, quarried about 1500 B.C., was no less than 105 feet long, nearly 10 feet square at the larger end, and about 430 tons in weight. It was set up in front of the Temple of the Sun at Thebes [38].

A certain amount of mathematics was used by the Ancient Egyptians in connection with their devising of timepieces, namely: (a) the sun-dial or gnomon; and (b) the clepsydra or water-clock [39]. The same may be said concerning ship-building and navigation [40] and engineering involved in the building of notable aqueducts and tunnels for providing water supplies [41].

Most of our knowledge of Egyptian mathematics is derived from two mathematical papyri, the one written about 1850 B.C., usually called the Moscow papyrus [42], containing 25 problems; and the other dating from about 1650 B.C., commonly called the RHIND mathematical papyrus [43, 44], with 85 problems. All of the 110 problems are numerical and many of them are excessively simple. Those in the RHIND papyrus are preceded by a table giving the equivalents in unit fractions of 2 divided by all odd numbers from 5 to 101; for with the exception of $\frac{2}{3}$ the Egyptian had no notation for a fraction with a numerator other than unity. Two divided by 7 is expressed as the sum of $\frac{1}{4}$ and 1/28; 2 divided by 97 is expressed as the sum of 1/56, 1/679, 1/776. Both of these results, as well as $\frac{2}{3}$ expressed as the sum of $\frac{1}{2}$ and $\frac{1}{6}$, are used in a single problem later. The table was therefore a reference list for use in solving problems.

The Egyptian carried through the multiplication of two numbers by successive multiplication of one of the numbers by twos or tens or by $\frac{2}{3}$, or by the division of the number by twos or tens. A sort of transposition was also a common operation. For example, if it had been found that $\frac{1}{5}$ of 105 was 21 the Egyptian might at once write down 1/21 of 105 is 5. Problems in division are reduced to those of multiplication. It is rather extraordinary that in order to get *one*-third of a number, the Egyptian first found *two*-thirds of the number and then took one-half of the result. This is illustrated in more than a dozen problems of the RHIND papyrus.

Nearly a score of 110 problems are such as we would now solve by algebra with equations of the first degree in one unknown quantity. For example: "A quantity, its $\frac{2}{3}$, its $\frac{1}{2}$, and its $\frac{1}{7}$, added together, becomes 33. What is the quantity?" The method of solution used is generally that of trial, or false position, and in more than one case it is obvious that the idea of proportion was clearly understood.

Another score of the problems deal with such questions as the strength of bread and of different kinds of beer, the derivation of beer of great alcoholic strength from two others, and the exchange of beer for bread. An example of this type of problem is the following: "Given that 13 hekat of upper Egyptian grain is made into 18 des of besha date-substitute beer, and that 1 des of this makes $2\frac{1}{6}$ des of barley beer, what is the strength of the barley beer?"

The feed for geese, cranes, ducks, quails, doves, and also for bulls and common cattle is discussed in other problems. The following illustrates a rule-of-three problem: "A sandal maker works for 15 days receiving wages every 5 days. If he does the work in 10 days after what periods should he be paid?" Of problems in arithmetic progressions the following may be mentioned: "Divide 100 loaves among 5 men in such a way that the share received shall be in arithmetic progression and that one-seventh of the sum of the largest three shares shall be equal to the sum of the smallest two." Remember that these are problems of 1650 to 1850 B.C.

Twenty-six of the 110 problems are geometric [**45**], and both volumes and areas are discussed. The area of a circle is repeatedly taken as the square of 8/9 of the diameter; this leads us to the remarkable value $256/81 = 3.1605 \cdots$ for π, much better than the somewhat earlier Babylonian value 3. The volume of a right circular cylindric granary is taken as equal to the area of its base multiplied by the number of units in its height. The most recent discussion seems to make it clear that the Egyptian knew that in any triangle its area is equal to one-half the product of its base and altitude [**45**]. The cotangents of the angles which the faces of pyramids make with their bases are discussed. A numerical problem appears to prove the extraordinary fact that the Egyptian knew our formula for the volume of a frustum of a square pyramid $V = (h/3)(a^2 + ab + b^2)$, where a and b are the lengths of the sides of the square bases and h is the number of height units of the frustum. The editor of the Moscow papyrus, which was first completely published in 1930 [**42**], believed that yet another problem gave

the correct result for the area of a hemisphere, if we take the value of π to which reference was just made. But the late Professor PEET, a prominent English Egyptologist, argued that there was no proper ground for this conclusion, since the correct translation led to something quite different [**45**].

There is no document to prove that the Egyptian knew even a particular case of the PYTHAGOREAN theorem [**43**].

Our main sources of information concerning Egyptian mathematics consist of two papyri of much the same type, but all additional fragments which we possess match the same picture, which is paralleled by economic documents in which occur precisely those problems and methods which we find in the mathematical papyri. Furthermore, the Egyptian mathematical texts find their continuation in Greek papyri (on to the eighth century of our era), which again show the same pattern. It is therefore safe to say that Egyptian mathematics never rose above a very primitive level, and did not provide the most essential tools for astronomical computation of real importance [**30**].

Thus we conclude our consideration of the mathematics of the Babylonians and Egyptians in our first period. Before 600 B.C. there was no other mathematics than theirs worth considering.

B. GREEK MATHEMATICS 600 B.C. TO 600 A.D. [46–51]

Greek history began with the second millennium B.C. but in connection with the history of mathematics, of sculpture, of architecture, of art, of philosophy, of literature, of thought, the semi-millennium commencing about 600 B.C. is of the greatest importance. The Greeks were not confined to the Greek peninsula, as in modern times. They occupied Macedonia and Thrace, the islands of the Aegean, the northern and western seaboards of Asia Minor, Southern Italy and Sicily. Scattered settlements were also to be found as far apart as the mouth of the Rhone, the north of Africa, and the eastern end of the Black Sea. Such was the location of the people who were presently to set such marvelous everlasting beacon lights of Freedom, Truth, Reason, Beauty, Excellence, Fellowship between man and man, which were to inspire, guide, and sustain through the millenniums to follow.

What were the special aptitudes which the Greeks possessed for science? An answer by SIR THOMAS HEATH [**48**], long a leading authority on Greek mathematics, is given. But into this quotation two correcting interpolations are introduced.

"They had, first, a love of knowledge for its own sake, amounting, as Butcher says, to an instinct and a passion; secondly, a love of truth and a determination to see things as they are; thirdly, a remarkable capacity for accurate observation." [In "capacity for accurate observation" there seems to be no difference between Greek and other cultures. The representation of animals by Egyptians and Assyrians did not find their equal in Greece; because of their exactness they are constantly used by zoologists as source material. The astronomical observations of the Babylonians are certainly not of less importance

than those of the Greeks, and also the fundamental instruments such as gnomon, sundial, water-clock. Ancient Egyptian anatomy was of the highest order.] "Fourthly, while eagerly assimilating information from all quarters, from Egypt and Babylon in particular, they had an unerring instinct for taking what was worth having and rejecting the rest. As one writer has said, 'it remains their everlasting glory that they discovered and made use of the serious scientific elements in the confused and complex mass of exact observations and superstitious ideas which constitutes the priestly wisdom of the East, and threw all the fantastic rubbish on one side.'" ["Rejecting the rest" and "Superstitious ideas . . . rubbish on one side." These stand in contradiction to many historical facts. All forms of oriental culture found an entry into Greece (see, for example, R. REITZENSTEIN, *Die hellenistischen Mysterienreligionen nach ihren Grundgedanken und Wirkungen*, third ed., Leipzig, 1927.) Astrology in its absurd forms was first a product of Hellenism.] "Fifthly, they possessed a speculative genius unrivaled in the world's history.

"It was this unique combination of gifts which qualified the Greeks to lead the world in all the intellectual pursuits that make life worth living.

"Last, but not less important, the Greeks possessed the advantage over the Egyptians and Babylonians of having no priesthood which could monopolise learning as a preserve of its own, with the inevitable result of sterilising it by keeping it bound up with religious dogmas and prescribed and narrow routine."

Some of these attributes were possibly more in evidence in their great achievements in the fields of medicine, biology, and natural science, than in those of mathematics and astronomy, which now concern us for a few moments. Since hours would be necessary for any adequate description of their wonderful achievements in these fields, we must confine ourselves largely to references to a few names and results.

Greek theoretical geometry and astronomy began with **Thales** [**51**] of Miletus, on the West coast of Asia Minor, in the first half of the sixth century B.C. No wonder he was declared one of the Seven Wise Men since, apart from being a mathematician and astronomer, he was also a statesman, engineer, man of business, and philosopher. To THALES later tradition attributes the following results in elementary geometry, all of them in EUCLID'S *Elements:*

1. A circle is bisected by any diameter (I, def. 17);
2. The angles at the base of an isosceles triangle are equal (I, 5);
3. If two straight lines cut one another, the vertically opposite angles are respectively equal (I, 15);
4. If two triangles have two angles and one side in each respectively equal the triangles are equal in all respects (I, 26);
5. The angle in a semi-circle is a right angle (which we have seen, was already recognized by the Babylonians [**55**] some 1400 years earlier; III, 31).

Of practical problems he showed how to determine the distance of a ship from the shore and found the height of a pyramid by means of the shadows cast on

the ground at the same moment by the pyramid and a stick; that moment was chosen when the length of the stick and its shadow were equal. There is no evidence that Thales predicted a solar eclipse which took place in 585 B.C. (See A. PANNEKOEK, "The origin of the Saros," K. Akad. van Wetens., Amsterdam, *Proc.*, 1918, v. 20, p. 955.) It may be of interest to remark that THALES was the first known individual with whom definite mathematical discoveries were associated.

About half a century after THALES came **PYTHAGORAS** [52]. Under his inspiration geometry was first pursued as a study for its own sake. A man of great ability and a most interesting and magnetic mystic, he finally settled at Crotona on the southeastern coast of Italy. Here among the young men of well-to-do families he established a secret society or brotherhood most of whose mathematical discoveries were pooled. According to HEATH [22], it is to the PYTHAGOREANS, that is about 500–350 B.C., that the following geometric results are due:

1. The properties of parallels and their application to prove that the sum of the angles of a triangle is equal to two right angles. From this were deduced the familiar results concerning the sums of (*a*) exterior and (*b*) interior angles of a polygon.
2. The transformation of areas of rectilinear figures, and the sums and differences of such areas, into equivalent areas of different shapes. To this end they invented the powerful method of application of areas, the main constituent of the geometric algebra by which they effected the geometric equivalent of addition, subtraction, division, extraction of the square root, and the complete solution of the general quadratic equation $x^2 \pm px \pm q = 0$, so far as it has real roots.

At a considerably later day APOLLONIUS of Perga named the conic sections, parabola, ellipse, and hyperbola, because he had shown that these curves were respectively defined by the application of an area, the application of an area falling short, the application of an area exceeding; that is,

$$y^2 = px, \quad y^2 = px - \frac{p}{d}x^2, \quad \text{and} \quad y^2 = px + \frac{p}{d}x^2.$$

3. The PYTHAGOREANS had a theory of proportion pretty fully developed, though it was only applicable to commensurable magnitudes, being presumably a numerical theory. They knew properties of similar figures such as similar rectilinear figures being in the duplicate ratio of corresponding sides.
4. They had discovered, or were aware of the existence of, at least three of the regular solids—the tetrahedron, cube, and dodecahedron.
5. They discovered the existence of the incommensurable in at least one case, that of the diagonal of a square in relation to its side, and they devised a method of obtaining closer and closer approximations to the value of $\sqrt{2}$ in the form of numerical fractions, x/y whose elements

are the successive solutions of the indeterminate equations $x^2-2y^2=\pm 1$; $[y=2, 5, 12, 29, \cdots; x=3, 7, 17, 41, \cdots]$.

Arithmetic in the sense of the theory of numbers began in discussions connected with problems of the PYTHAGOREANS. With the Greeks, Arithmetic dealing with absolute numbers, or numbers in the abstract, was distinguished from Logistic [50], the science dealing with ordinary arithmetical operations, and certain problems of elementary algebra.

Two numbers are called amicable or friendly if each equals the sum of the aliquot divisors of the other. PYTHAGORAS gave the first pair 220, 284 [56]. The second and third pairs were given in the seventeenth century by FERMAT and DESCARTES respectively, and the next 61 pairs were obtained by Euler in the eighteenth century. PYTHAGOREANS discussed also various forms of figured numbers—triangular, square, pentagonal, *etc.*, the number of dots in a figure corresponding to the number. Thus $1+2+3+\cdots+n=\frac{1}{2}n(n+1)$ is any triangular number; $1+3+5+7+\cdots+(2n-1)=n^2$ is any square number and so on. To them is also attributed the formula $m^2+[\frac{1}{2}(m^2-1)]^2=[\frac{1}{2}(m^2+1)]^2$, where m was any odd number. But their most wonderful discovery about numbers was in showing the dependence of musical intervals upon numerical ratios. They found that for strings of the same tension, the different lengths of strings would be in the ratio of 2 to 1 for the octave, 3 to 2 for the fifth, and 4 to 3 for the fourth. They went on to form a diatonic scale and thus initiated a study which was to be extensively elaborated by later Greeks [57]. The PYTHAGOREANS showed interest in astronomy, associating musical notes with the heavenly bodies (the highest with the fixed stars, the next highest with Saturn, the lowest with the Moon) and also various numbers [58]. The planets were thought as moving in independent circles. There is no ground for thinking that they regarded the earth as spherical [59].

Thus geometry, arithmetic, music, and astronomy came to be grouped together by the PYTHAGOREANS as fundamental liberal arts for study and to form what was, in the middle ages, called the *quadrivium*. The word "mathematics" is derived from a Greek word μάθημα meaning simply a "subject of instruction"; but by the time of ARISTOTLE [60] (−340) the term was definitely restricted to subjects of the *quadrivium*.

PLATO [61, 62] regarded mathematics in its four branches, arithmetic, geometry, stereometry, and astronomy, as the first essential in the training of philosophers and of those who should rule his ideal State; "let no one destitute of geometry enter my doors," said the inscription over the door of his school near Athens. He emphasized that mathematics is of value for the training of the mind, and that by comparison, its practical value is of no account. It is not known that PLATO made any mathematical discovery. THEAETETUS, pupil of SOCRATES and friend of PLATO, one of whose dialogues, *Theaetetus*, is a commemorative tribute, advanced the theory of irrationals and was probably the first to construct all five of the regular solids [62] theoretically, and to in-

vestigate fully their relations to one another, and to the circumscribed sphere as elaborated in EUCLID'S *Elements*, XIII; indeed THEAETETUS was the discoverer of the octahedron and icosahedron. In the fifth century B.C. flourished **ZENO** of Elea [**63**] who also does not seem to have been a mathematician, yet his paradoxes of motion had a profound influence on the course which geometry thereafter took.

The PYTHAGOREAN difficulties in connection with geometric proportion and incommensurables were completely solved in the fourth century B.C. by a pupil of ARCHYTAS and PLATO named **EUDOXUS** [**64**], born at Cnidos on the west coast of Asia Minor. He was an original genius second only to ARCHIMEDES. His masterly setting forth of incommensurables, as in the fifth book of EUCLID'S *Elements*, is practically identical with the modern formulation of DEDEKIND. EUDOXUS discovered also the so-called "method of exhaustion" by means of which he gave the first rigorous proof of the results for the volume of a cone, and of a pyramid, and probably showed that: (1) the areas of two circles are to one another as the squares of their diameters; and (2) the volumes of spheres are to one another as the cubes of the lengths of their respective diameters.

It seems certain that **MENAECHMUS**, a pupil of EUDOXUS, was the discoverer of the conic sections—parabola, ellipse, hyperbola, which were originally thought of as sections perpendicular to generators of right-angled, acute-angled, and obtuse-angled cones; see NEUGEBAUER, 1948, [**86**]. He showed that by means of the intersection of two parabolas, or of a parabola and a rectangular hyperbola so obtained, we could find two mean proportionals between two lines of lengths a and b. This implies the recognition of a geometric relation, in the case of the parabola, equivalent to our ordinary Cartesian equation, and of a similar relation for a rectangular hyperbola referred to its asymptotes, as axes.

This discovery of MENAECHMUS was of particular interest because it gave a new solution of the problem of the duplication of the cube, one of the three famous problems which had been formulated in the fifth century B.C. These problems are [**65–69**].

(*a*) To find a line which shall be the edge of a cube whose volume is double that of a given cube, the problem of the duplication of the cube.
(*b*) To trisect any given angle.
(*c*) To find a line which shall be the side of a square whose area shall be exactly equal to that of a given circle, the problem of squaring the circle [**67**].

All of these problems were solved by the Greeks within a century, but more than 22 centuries were to pass before it was finally proved that no one of them could be solved with ruler and compasses alone. As early as the fifth century B.C., by means of a curve called the quadratrix [**69**], the problem of the trisection of an angle was solved by **HIPPIAS** of Elis, and **HIPPOCRATES** of Chios had shown that the problem of duplicating the cube could be reduced to that of finding two mean proportionals between such lengths as a and $2a$. Such mean

proportionals were found by **Archytas**, a Pythagorean, a friend of Plato, a statesman and philosopher, in a marvellous construction by means of the intersection of a right cone, a cylinder, and an anchor ring with inner diameter zero. **Dinostratus**, a brother of Menaechmus, showed that the quadratrix of Hippias could also be used to solve the problem of squaring the circle. In the third century this problem was also solved, in effect, by a spiral invented by Archimedes [69]. In the third century a quartic curve, the conchoid of **Nicomedes** [69], was used to solve the problems of trisecting an angle, and duplicating a cube, and the cissoid of **Diocles** [69], a cubic curve, was also employed for solving the latter problem.

We come now to the consideration of the golden period of Greek mathematics and of the greatest mathematical school of ancient times. This was at the magnificently endowed university of Alexandria which had been founded by Alexander the Great in 332 B.C. [70]. The university was opened about 300 B.C. and within the first 40 years of its existence over 600,000 rolls had been collected in its great library. **Euclid** [71] was a professor of mathematics in the university. Practically nothing is known about his life except that he was the author of at least ten treatises [72] of which approximately complete texts of five are available. These include three on applied mathematics, namely on phaenomena [75], on optics [76], and on music [77]. But by far the most famous one is his treatise, in 13 books, called the Elements [73, 74]. More than a thousand editions have appeared since the first one printed in 1482, and for 1800 years before that, manuscript copies dominated all teaching of geometry. Though a large portion of the subject matter had been investigated by predecessors the whole arrangement was due to the great genius of Euclid who supplied innumerable details. It is quite impossible to give in a few sentences any adequate idea of the contents of the 465 propositions in this monumental work. Practically all of the geometric material of American school texts in plane and solid geometry is contained in parts of six of the books (1, 3, 4, 6, 11 and 12) of the Elements. Most books are prefaced by definitions, but before the first book are certain postulates the fifth of which is the famous one which differentiates Euclidean from noneuclidean geometry [78]. The statement is as follows: "Let it be postulated that, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines if produced indefinitely, meet on that side on which are the angles less than two right angles."

In Book II of the Elements are a number of propositions giving geometric proofs of algebraic identities, such as $(a+b)^2=a^2+2ab+b^2$, and in Books II and VI are the propositions giving, among many other things, geometric solutions of quadratic equations [98], which were treated algebraically by the Babylonians. In Book V is expounded the remarkable Eudoxus theory of proportion, alike applicable to incommensurable and commensurable magnitudes of every kind. There are many masterly Euclidean developments. Book VI applies to plane geometry the general theory of proportion set out in Book V. The 102 proposi-

tions in Books VII, VIII, IX deal with questions in the theory of numbers. Special reference may be made to three propositions in Book IX. It is elegantly proved in the 20th that there are an infinite number of prime numbers; in the 35th there is a beautiful geometric derivation of what is practically equivalent to our algebraic formula for the sum of the first n terms of a geometric progression; and in the 36th it is proved that if $S_n = 2^n - 1$ is prime, then $P = 2^{n-1}S_n$ is a perfect number [**79**], that is, a number which is equal to the sum of its divisors smaller than itself. The first four perfect numbers are 6, 28, 496, and 8128; only 12 perfect numbers are known, and they are all of the EUCLIDEAN type.

Book X contains 115 propositions on incommensurables, and is perhaps the most remarkable and most finished of all the books of the Elements. Books XI, XII, XIII deal with geometry of three dimensions. In propositions 16 and 17 of the last book all the details are carried out for the actual construction of an icosahedron, and of a dodecahedron, inscribed in a sphere.

The logical form of the presentation of the propositions in the Elements is especially notable. There is first of all the enunciation of the proposition; secondly, the statement of the precise data; thirdly, the statement of what we are required to do with reference to the precise data mentioned; fourthly, the construction, the addition when necessary of more lines to the figure; fifthly, the proof; and sixthly, the conclusion, stating what has actually been done, which generally follows the wording of the original enunciation.

In the period between EUCLID and ARCHIMEDES came **ARISTARCHUS** of Samos [**58**], whose great achievement lies in the fact that he was the first to assert that the earth and the other planets (Venus, Mercury, Mars, Jupiter, and Saturn) revolved about the sun, thus anticipating COPERNICUS by seventeen centuries.

ARCHIMEDES [**80–82**] was born at Syracuse, in Sicily, about 287 B.C. and was killed in the sack of that city by the Romans in 212. He studied with successors of EUCLID at Alexandria and it was probably in Egypt that he invented the water screw, known by his name, for drawing water to irrigate fields. On returning to Syracuse he devoted his time wholly to mathematical research. Up to the time of NEWTON at least he was the greatest mathematical genius that the world had seen. The following summary of his work is mainly due to HEATH [**22**].

In geometry his works consisted chiefly of original researches in the quadrature of plane curvilinear figures and in the quadrature of curved surfaces and the cubature of solids bounded by such surfaces. By methods equivalent to integration [**82**], ARCHIMEDES found the area of a parabolic segment, the area of a spiral, the surface and volume of a sphere and a segment of a sphere, and the volumes of any segments of the solids of revolution of the second degree.

In arithmetic he calculated approximations to the value of π and in the course of these calculations showed that he could find approximate values for the square roots of large or small non-square numbers; he found $3\frac{1}{7} > \pi > 3\frac{10}{71}$, and also, apparently, $3.141697 \cdots > \pi > 3.141495 \cdots$, the mean of which is 3.141596. ARCHIMEDES also invented a system of arithmetic nomenclature by

which he could express in language enormously large numbers, in fact all numbers up to that which we would write as 1 followed by 80,000 million million ciphers.

In mechanics he laid down certain postulates and, on the basis of these postulates, established certain fundamental theorems on magnitudes balancing about a point and on centers of gravity **[82]**, going so far as to find the center of gravity of any segment of a parabola, a semi-circle, a cone, a hemisphere, a segment of a sphere, and a right segment of a paraboloid of revolution. As we learn from his *Method* **[81]**, discovered as recently as 1906, ARCHIMEDES made most ingenious use of mechanics as an aid for suggesting probable results in geometry; the first theorem which he found in this way was in connection with the quadrature of the parabola.

He invented the whole science of hydrostatics, which he carried so far as to give a complete investigation of the positions of rest and stability of a right segment of a paraboloid of revolution floating in a fluid with the base either upwards or downwards, but so that the base is entirely above or entirely below the surface of the fluid. HEATH then sums up as follows:

"The treatises themselves are, without exception, models of mathematical exposition; the gradual unfolding of the plan of attack, the masterly ordering of the propositions, the stern elimination of everything not immediately relevant, the perfect finish of the whole, combine to produce a deep impression, almost a feeling of awe in the mind of the reader."

In accordance with the desire of ARCHIMEDES the figure of a sphere inscribed in a cylinder was engraved on his tomb since he seemed to regard the discovery that the volume of the sphere was two-thirds that of the circumscribed cylinder as his greatest achievement. CICERO found the tomb with this inscription.

Other results due to ARCHIMEDES were the following: (a) our familiar formula for the area of a triangle in terms of its sides; (b) the construction of 14 inscribable semi-regular polyhedra **[83]**; (c) the solution of a three-term cubic equation, the term of the first degree lacking **[80]**, by means of the intersection of a parabola and a rectangular hyperbola; this occurs in his work on the sphere and cylinder.

The tradition that ARCHIMEDES destroyed Roman ships, by concentrating sun's rays by means of a series of mirrors, was proved by experiment to be entirely feasible **[84]**. And finally, ARCHIMEDES was much occupied with astronomy and wrote a book on the construction of a sphere so as to imitate the motion in the heavens of the sun, the moon, and the five planets. When CICERO was quaestor in Sicily in 75 B.C., he actually saw the contrivance and gives a description of it.

Almost contemporary with ARCHIMEDES was his friend **ERATOSTHENES**, who taught in Alexandria and was librarian at the University. To him ARCHIMEDES dedicated his treatise on *Method*. The most famous scientific achievement of ERATOSTHENES was his apparently very accurate determination of the polar circumference of the earth. This was done by observing that when the sun was

in the zenith at Syene, it was exactly 7° 12′ south of the zenith at Alexandria, known to be 5000 stadia distant. According to a recent interpretation [**85**], an ERATOSTHENES' device, called a sieve, for determining prime numbers, is of considerable interest.

The third great name in connection with the Alexandrian School was **APOLLONIUS** of Perga [86] who was about 25 years younger than ARCHIMEDES. He was called by his contemporaries the "Great Geometer," because of his extraordinary treatise on Conics, and he was also mentioned as a famous astronomer. He studied at Alexandria. Only seven of the eight books of his Conics have been preserved. The first three seem to correspond to a treatise which EUCLID wrote, but books V, VI, VII contain the discoveries which APOLLONIUS himself had made. To him are due the terms parabola, ellipse, hyperbola, for reasons which we have already explained, and his lines of reference are any diameter and a tangent at its extremity, the cases of these when at right angles being considered only as special cases. Book V treats of various questions concerning normals to conics, and from certain propositions we can readily deduce the CARTESIAN equations to the evolutes of the three conics. Book VII deals mainly with various results in connection with conjugate diameters. There are nearly 400 propositions. The suggestions that I have given are sufficient to indicate that in this work of APOLLONIUS there is far more than is contained in any of our American textbooks on analytical geometry, so far as conic sections are concerned. Foci of central conics are constructed, and that the sum or difference of focal radii of points of such curves is constant, is proved. There is no reference to the focus of the parabola. It seems likely that the focus-directrix property of all three conics was given in EUCLID's Conics, or well known in his time, although it is not explicitly mentioned before PAPPUS, hundreds of years later [**86a**].

Among a number of other works of APOLLONIUS we shall refer to only three. In his Plane Loci the following familiar results were given: (*a*) If A, B be fixed points, P any other point in the plane, and AP and BP are in a given ratio, the locus of P is a straight line or a circle (circle locus given earlier by ARISTOTLE) according as the given ratio is or is not one of equality; (*b*) If A, B, C, $\cdots$ be any number of fixed points and $\alpha, \beta, \gamma, \cdots$ any constants, the locus of point P, such that $\alpha \cdot AP^2 + \beta \cdot BP^2 + \gamma \cdot CP^2 + \cdots =$ a constant, is a circle.

The second work, on Contacts or Tangencies, has not come down to us, but its principal problem is to describe a circle tangent to three given circles, which is usually known as the problem of APOLLONIUS. Many celebrated mathematicians, such as VIETA, EULER and NEWTON, have worked on this problem, and what is known of the work as a whole has made it the basis of numerous restorations and discussions [**87**]. The analogous form for spheres was treated synthetically by FERMAT. From the Plane Loci, and the account of Tangencies by PAPPUS, it has been deduced that APOLLONIUS made the following result, in effect, the basis of his solution of the general tangencies problem: The six centers of similitude of three coplanar circles lie by threes on four straight lines [**88**].

The third work, on Vergings, is also lost, but some of its problems and solu-

tions have been preserved. One of these problems is: Between the side of a given rhombus and its adjacent side produced, insert a straight line of given length and verging to the opposite corner. HUYGENS gave many solutions and NEWTON used the particular case of a square in illustration of discussion [**89**]. The work of APOLLONIUS in Astronomy was also of importance [**89**].

We have already seen the marvellous development of geometry by the Greeks from the beginning up to difficult problems of the integral calculus, and all of this within the short space of 350 years.

Possibly contemporary with scientific activity among the Greeks of the fourth and fifth centuries B.C. are two notable Babylonian astronomers [**90**], **NABU-RIANU** and **CIDENAS** or **KIDINNU**, who had at their disposal an extraordinary number of Babylonian observations of eclipses for more than three centuries. The next name of mathematical interest among the Greeks is that of their most eminent astronomer, **HIPPARCHUS** [**91**], who flourished about 140 B.C. From the great mass of data available from about 750 B.C., it was almost inevitable that he should be the first to note the precession of the equinoxes [**92**]; but it was not till the time of NEWTON that the explanation of this precession made it a matter of real importance. In masterly fashion HIPPARCHUS improved on ARISTARCHUS's calculations of the sizes and distances of the sun and moon, and developed a systematic theory with regard to them. He also compiled a catalogue of 850 stars [**93**], stating their places and apparent sizes. Trigonometry as a science seems to have begun with him, and also the introduction into Greece of the division of the circle [**94**] into 360°.

About two centuries after HIPPARCHUS we come to **HERON** [**95**] of Alexandria whose date, until comparatively recently, has been a matter of uncertainty, even to the extent of 400 years [**96**]. He was an almost encyclopedic writer on mathematical and physical subjects, and aimed at practical utility rather than theoretical completeness. In his *Pneumatica* are many mechanical devices such as a siphon, a fire engine, a device whereby temple doors are opened by fire on an altar, an altar organ blown by the agency of hand labor or by a windmill, and a jet of steam supporting a sphere [**314**].

He gave an elegant geometric proof of the formula for the area of a triangle in terms of the sides, now attributed to ARCHIMEDES. His method of approximating to the square root of non-square numbers [**97**] seems to have been used by early Babylonians. His formula for the volume of the frustum of a square pyramid can be readily reduced to the one used 2000 years earlier in the Moscow papyrus [**42**].

Quadratic equations are solved in a manner very similar to that of the Babylonians 1900 years before [**98**]. HERON showed how he obtained the cube root of a non-cube number, 100. He found also the volume of the five regular solids, and of an anchor ring (as given by DIONYSODORUS, a contemporary of APOLLONIUS of Perga). His works on surveying, and the *Dioptra* are of interest, and it is by means of the latter that his date has been narrowed down.

Modern research seems to indicate that **DIOPHANTUS** [**99**] of Alexandria,

one of the great mathematicians of Greek civilization, should be assigned to the first century of our era [100], rather than to the third century. He was the first to make systematic use of symbols in algebraic work, a sign for unknown, a sign for minus, and signs for the various powers, that is, square, cube, and so on. Of his great work called Arithmetic only 6 of its thirteen books survive and they are mostly taken up with problems in indeterminate analysis of the second degree. Hence the term Diophantine Analysis [101]. The answers are always in positive rational numbers. The collection is extraordinarily varied and the devices resorted to are highly ingenious. In Book V, 9, 11, not only does he find particular solutions of $x^2-26y^2=1$ and $x^2-30y^2=1$, but also shows (VI, 15, lemma) that they have an infinite number of solutions. DIOPHANTUS may have discussed the equation $x^2-Ay^2=1$ more fully in the part of the Arithmetic which has been lost. See later under BHĀSKARA. It has been surmised that the last seven books of the Arithmetic may have been filled with material between the most difficult of DIOPHANTUS' problems and the famous cattle problem of ARCHIMEDES.

A possible contemporary of HERON was **MENELAUS** of Alexandria [102], who wrote a variety of treatises among which his *Sphaerica* [103] contained for the first time the conception and definition of a spherical triangle; here also are the MENELAUS theorem for the sphere, and deductions from it, furnishing the equivalent of formulae in spherical trigonometry.

Much of our knowledge of the achievements of HIPPARCHUS is derived from a work of **CLAUDIUS PTOLEMY** [104] of Alexandria, who flourished in the second century of our era. This great work, of extraordinary compactness and elegance, was called the *Almagest*, which overshadowed all previous works of the kind. In the trigonometry of the early part is derived a table of chords equivalent to a table of $\sin A$ for each $15'$ of the quadrant, the values being expressed in parts, minutes and seconds.

In the first book of the *Almagest* is PTOLEMY'S theorem regarding a quadrilateral inscribed in a circle, and from this $\sin^2 A+\cos^2 A=1$, and the well-known formulae for $\sin (A\pm B)$ and $\cos (A\pm B)$, among others, are readily derived [105]. In the fourth book occurs a solution of the so-called "Problem of SNELL" (1617), or of POTHENOT 1692 (publ. 1730): to determine the point from which pairs of three given points are seen under given angles [105].

In this same work results of much of PTOLEMY'S observational work are contained in his catalogue of 1028 stars, with the latitude and longitude of each [106]. But his outstanding achievement was to develop a lunar theory and a planetary theory, close to KEPLER'S, with planetary movements in eccentric orbits almost elliptical.

PTOLEMY used various projections including stereographic. The essentials of his argument in an attempted proof of EUCLID'S parallel postulate have been preserved. It is recorded that he wrote also a work on dimension in which he attempted to prove that the possible number of dimensions is only three.

PTOLEMY wrote also a remarkable geographical treatise with maps, various

later printed editions of which are treasured by libraries [**106a**]. His work on Optics is especially interesting because it contains the first attempt at a theory of refraction [**106b**]. He also wrote a notable treatise on music [**106c**].

And finally we come to **PAPPUS** [**107**] of Alexandria who lived at the end of the third century. His great work, entitled *Mathematical Collection,* [**108**] covers the whole range of Greek geometry and was intended "to be read with the original works (where extant) rather than independently. Where, however, the history of a subject is given, *e.g.*, that of the duplication of the cube, or the finding of the two mean proportionals, the solutions themselves are reproduced, presumably because they were not readily accessible but had to be collected from scattered sources. Even when some accessible classic is being described, the opportunity is taken to give alternative methods or to make improvements in proofs, extensions and so on. Without pretending to great originality, the whole work shows, on the part of the author, a complete grasp over the subjects treated, independence of judgment, and mastery of technique; the style is concise and clear; in short PAPPUS stands out as an accomplished and versatile mathematician and a worthy representative of the classical Greek geometry" [**22**]. Among the original contributions of PAPPUS are—(*a*) a generalization of the "Pythagorean" theorem to any triangle with certain parallelograms on its sides; (*b*) generalization of the four-line locus to five- or six- or *n*-line loci [**86**]; (*c*) proof of the invariance of the cross-ratio under a projective transformation; and (*d*) a result often attributed to GULDIN [**109**], a Swiss mathematician of the seventeenth century: The volume of a solid of revolution is equal to the area of the generating figure, multiplied by the circumference described by the center of gravity of the figure. PAPPUS was also the first to give results possibly known earlier: (*e*) the construction of a conic through five points [**86**]; and (*f*) the presentation of the focus-directrix property of the three conics [**86a**].

The fifth book is mainly devoted to the subject of isoperimetry. There is a very interesting passage concerning bees, their orderliness, and the hexagonal form of their cells exactly filling up space about a point [**110**]. It is later shown that (*a*) a circle is greater than any regular polygon of equal contour; (*b*) a sphere is greater than any of the five regular solids with equal area. The sixth book is mostly astronomical [**111**], dealing with treatises studied as an introduction to the great *Almagest* of PTOLEMY. There are, however, propositions of mathematical interest including some related to EUCLID's *Optics* [**76**]. The seventh book, *On the Treasury of Analysis,* is historically the most important of all the books, since it gives an account of a collection of treatises which, after the Elements of EUCLID, provided the body of doctrine necessary for the professional mathematician to know if he was to be regarded as fully equipped for the solution of problems arising in geometry. Of the 12 works listed seven are by APOLLONIUS and three by EUCLID.

THEON [**111a**] of Alexandria, a fourth century author of an edition of Euclid's Elements, and a commentator on works of EUCLID and PTOLEMY, is a name of importance in the history of mathematics. In the fifth century, **PRO-**

CLUS, more of a philosopher than a mathematician, placed the historian under great obligation by his commentary on the first book of EUCLID's elements, which is one of our main sources of information on the history of elementary geometry. [**313**].

In the early part of the sixth century JOHN PHILOPONUS of Alexandria wrote the earliest extant Greek treatise on the Astrolabe [**111b**], which played an important role in the history of astronomy and of a mathematical projection.

Alexandria was razed by the Arabs in the seventh century. Contrary to the imaginings of some writers, no library of importance was then destroyed, or even in existence there [**112**].

For the period of history which we have so far been considering, it is to be noted that, in the case of the Babylonians and Egyptians, the actual source material written two to four thousand years ago is available for study. On the other hand, in the case of the Greeks practically not a single source manuscript is dated earlier than a thousand years after the writer flourished. In spite of this great handicap, scholars feel that some reliable complete or partial original texts of works of such writers as EUCLID, ARCHIMEDES, and APOLLONIUS are now available [**313**].

C. HINDU [113–116], ARABIC, PERSIAN MATHEMATICS 600 to 1200

With PAPPUS creative Greek mathematicians came to an end, and very soon all the great traditions of Greek learning had died out; then followed nearly a thousand years in which slight additions to the sum of mathematical knowledge were made in Europe. Significant contributions came from India and the Arabs.

In this period the Arabian Empire stretched from India to Spain, Bagdad and Cordova being the centers for the reigning caliphs [**117**]. In extraordinary fashion Arabs at Bagdad developed profound interest in translated Greek mathematical and astronomical works, and also in similar Hindu material. The ninth and tenth centuries may be regarded as the golden age of Arabian mathematicians [**118**], to whom the world owes a great debt for preserving, and thus making possible the transmission to Europe, of classics in Greek mathematics otherwise lost. A vast field for further investigation in this connection awaits the attention of scholars.

The formula for the area of an inscribed quadrilateral, similar in form to that for a triangle, ascribed to ARCHIMEDES, was first given in the early part of the seventh century by an outstanding Hindu mathematician, named **BRAHMAGUPTA** [**120**], but it was not recognized as true for a cyclic figure only. If a, b, c, and d are the lengths of the sides of the quadrilateral, $s=\frac{1}{2}(a+b+c+d)$, and m and n are the lengths of the diagonals, the area is equal to $\sqrt{[(s-a)(s-b)(s-c)(s-d)]}$, $m=\sqrt{[(ab+cd)(ac+bd)/(ad+bc)]}$, and $n=\sqrt{[(ac+bd)(ad+bc)/(ab+cd)]}$. These formulae were considered with a view to determining quadrilaterals whose sides, diagonals, and areas were all rational quantities. In particular BRAHMAGUPTA gave the rule: If $a^2+b^2=c^2$ and $\alpha^2+\beta^2=\gamma^2$, then the

quadrilateral $(a\gamma, c\beta, b\gamma, c\alpha)$ is cyclic, the area is rational, and its rational diagonals are at right angles. BHĀSKARA pointed out that similarly rational is such an inscribed quadrilateral, if a pair of adjacent sides are interchanged; he also indicated how to determine the length of the third diagonal, no longer orthogonal to the other diagonal of the new quadrilateral. BRAHMAGUPTA also noted, in terms of any three rational numbers, the formulae for the lengths of sides of an oblique triangle whose altitudes and areas are rational [121].

We have already seen that the PYTHAGOREANS were led to solutions of the equations $x^2-2y^2=\pm 1$, in getting approximations to $\sqrt{2}$. BRAHMAGUPTA and BHĀSKARA [120, 122] (of the twelfth century) gave the method for finding remarkable particular solutions of the equations $x^2-Ay^2=1$, $A=8$, 61, 67, and 92; and the latter found general solutions [123]. BRAHMAGUPTA's rules lead, in effect, to the general solution of $ax+by=c$, a, b, c integers, and a and b relatively prime, as $x=\pm(cq-bt)$, $y=\pm(-cp+at)$, where t is zero or any integer, and p/q is (in modern phraseology) the second last convergent of a/b, expressed as a continued fraction. These are the most important developments in Hindu mathematics of this period.

Let us now outline the origin of our common numerals [124]. So far as we know the largest number of our numeral forms were first used in India [125]. The 1, 4, and 6 are found in inscriptions of the third century B.C.; the 2, 4, 6, 7, and 9 appear in another inscription about a century later; and the 2, 3, 4, 5, 6, 7, and 9 in caves of the first and second centuries of our era, all in forms which have a considerable resemblance to our own.

The first definite external reference to Hindu numerals is in a note by a bishop who lived in Mesopotamia about 650 A.D. Early in the ninth century the numerals became known to Arabic scholars. Indeed, one of the most prominent of these, MOHAMMED IBN MÛSÂ AL-KHOWÂRIZMÎ [126] (that is, of Khowârizmî, now Khiva, in Russian Turkestan) worked at Bagdad, and wrote a treatise on Hindu numeration and arithmetic, which became known to Europeans through a twelfth century Latin translation. The earliest undoubted occurrence of zero in India [127] was in an inscription of 876, in connection with the numbers 50 and 270. But it was used much earlier, indeed by HYPSICLES (−180) and later Greek astronomers, from whom Hindus probably borrowed the sign. A somewhat similar sign is also in astronomical records of the Mayas of Central America [128], which may date back to the beginning of our era; they had a vigesimal system with the principle of local value. About −400 the Babylonians possessed both the principle of relative local value, and a zero symbol which was used systematically in astronomical texts and computations. In Europe the complete system of numerals with the zero was derived from the Arabs in the twelfth century.

We have referred to the astronomer and mathematician AL-KHOWÂRIZMÎ (the first of several mathematicians often referred to by their place of birth). He wrote a work on algebra with the title "ḥisāb alǧabr walmuḳābalah," which has been translated "the science of reduction and cancellation." The Arabic

word for reduction, "alḡabr," thus became our word alḡebra. The English word algorism is simply a corruption of AL-KHOWÂRIZMÎ, as in the Latinized title form *Algoritmi di numero indorum*, of AL-KHOWÂRIZMÎ's arithmetic. The algebra contained sections on geometry, quadratic equations solved geometrically [**129**], and problems of inheritance. But most of AL-KHOWÂRIZMÎ's work was in astronomy, and among 100 of his tables in this field are tables of sines and cotangents [**130**], the former doubtless indicating Hindu influence. For already in the fifth century Greek trigonometry operating with chords was transformed by the Hindus into consideration of half-chords or sines. Thus we there have a table [**131**] of 3438 sin A, where the radius $=3438$.

From the time of the PYTHAGOREANS, constructions of elementary geometry were imagined as carried through with a ruler and compasses with variable opening. A prominent Arabian mathematician of the tenth century, **ABÛ'L WEFÂ** [**132**], lectured on geometric constructions with a ruler and compasses with a fixed opening [**133**]. It was not until centuries later that it was recognized that there was no real limitation here, since it was shown by PONCELET and later by STEINER, in the early part of the nineteenth century, that if a single circle and its center are once drawn in a plane, every construction with ruler and compasses can be carried through with ruler alone.

ABÛ'L WEFÂ contributed notably to the development of trigonometry. Already in his time all six of the trigonometric functions were in use, and he employed our formulae for versed sine, for the tangent and cotangent in terms of sine and cosine, and for the sin of an angle in terms of the sine and cosine of the half angle. PTOLEMY's table of sines for each $\frac{1}{2}°$ was extended by him to each $\frac{1}{4}°$, and he made a similar table for tangents. He found the value of sin 30′ correct to the equivalent of eight places of decimals [**134**], but his great contributions were in spherical trigonometry where he first used our formulae for a right spherical triangle and also our law of sines formula for an oblique spherical triangle.

In concluding the period under consideration we should make brief reference to a remarkable work of the Persian mathematician, astronomer, poet, **OMAR KHAYYÂM** (that is, OMAR the Tentmaker) [**135**], known to the western world as the author of the Rubaiyat [**136**]. He wrote a treatise on algebra in which his geometric treatment of systematically classified cubic equations is central [**137**]. He obtains a root as the abscissa of a point of intersection of a conic and a circle, or of two conics. Various forms of cubic equations are considered; negative roots are rejected, and not all positive roots are discovered. While we have already noted the solution of cubic equations by MENAECHMUS and ARCHIMEDES it is to be remarked that here the point of view is different; the problem is: How can we solve cubic equations with numerical coefficients? Another notable achievement of OMAR was his correction of the calendar by the introduction of cycles of 33 years. This calendar was more accurate than the one we use today.

In order to round out our survey of Islamic contributions to the develop-

ment of trigonometry we append to the treatment of this period some remarks on contributions of two men of later date. The first is **NASÎR ED-DÎN AL-TÛSÎ** [**138**], exceptionally able astronomer, mathematician and politician at Bagdad in the thirteenth century. Displaying remarkable scientific thoroughness, and power to integrate and extend earlier discoveries, he wrote the first complete plane and spherical trigonometry independent of any astronomical application [**139**]. In his discussion the theorem of MENELAUS is basic. All six trigonometric functions are used, and necessary formulae for solving right, and all cases of oblique, spherical triangles are given. Such was the status of trigonometry at the close of the thirteenth century. A recently discovered fourteenth century Arabic commentary by NASÎR ED-DÎN suggests that it contains the first part of OMAR KHAYYÂM's discussion of difficulties of EUCLID [**140**]. This part treats material similar to introductory propositions in the noneuclidean geometry of SACCHERI, hundreds of years later.

The second supplementary name is that of **ULUGH BEG** [**141**], a fifteenth century Persian prince and astronomer, who compiled extraordinary tables of sines and tangents for every minute and correct to the equivalent of eight or ten places of decimals [**142**]. In calculating the sine table he was led approximately to solve the cubic equation of the angle-trisection problem [**143**]: given sin 3°, to find sin 1°.

D. EUROPEAN MATHEMATICS 1200 TO 1600

During the period 500 to 1200 the student went to the teacher in the monastery and heard his lectures. But in the thirteenth century universities commenced to spring up at such places as Bologna, Padua, Naples, Paris, Oxford and Cambridge. Scribes making copies of treatises were thus kept busily employed by the universities. By the middle of the fifteenth century, however, their products were being sold as books are today. But such methods of disseminating knowledge were crude when compared with that of the distribution of the printed work. The publication of these with movable type commenced about 1450. More than two hundred mathematical works were printed, in Italy alone, before 1500; but this number was increased to 1527 in the next century.

During three and a half of the four centuries now under consideration Italy made the chief contributions to mathematics, and by far the most outstanding mathematician was one who flourished at the beginning of this period, **LEONARDO** of Pisa [**144**], often called FIBONACCI, that is, son of BONACCIO. During early life he travelled extensively about the Mediterranean, visiting Egypt, Syria, Greece, Sicily, and southern France, and knowledge thus gleaned regarding arithmetic systems used by merchants of different countries was the basis of a notable work, entitled *Liber Abaci* [**145**], which he wrote in 1202. This is a storehouse from which for centuries authors got material for works on arithmetic and algebra. The Hindu-Arabic system of numerals was here strongly advocated and illustrated, and the work did much to introduce it into Europe. LEONARDO discussed problems in arithmetic processes, barter, alligation, false

position, square and cube roots. It is in this work that we find the problem: "7 old women went to Rome; each woman had 7 mules; each mule carried 7 sacks; each sack contained 7 loaves; and with each loaf were 7 knives; each knife was put up in 7 sheaths. What is the sum total of all named?" It was this problem which gave the clue to the interpretation of a problem in the RHIND papyrus 2800 years earlier, "In each of 7 houses are 7 cats, each cat kills 7 mice, each mouse would have eaten 7 ears of spelt, and each ear of spelt will produce 7 hekat of grain; how much grain is thereby saved?" The modern conundrum starting out: "As I was going up to St. Ives I met a man with 7 wives, each wife had 7 sacks," etc., further illustrates the perpetuation of this number succession through the centuries. An interesting chapter on similar perpetuation of other problems has been written [**146**].

In *Liber Abaci* is also the following: How many pairs of rabbits can be produced from a single pair in a year if it is supposed: (*a*) that every month each pair begets a new pair which from the second month on becomes productive; and (*b*) that deaths do not occur? In this way we are led to the famous recurrent FIBONACCI series: $1, 1, 2, 3, 5, 8, 13, 21, \cdots$, in which each term after the second is the sum of the two preceding [**147**].

Among many other things the work contains also a proof of the well-known algebraic identity expressing the product of the sums of two squares as the sum of two squares: $(a^2+b^2)(c^2+d^2)=(ac+bd)^2+(ad-bc)^2=(ac-bd)^2+(ad+bc)^2$.

LEONARDO wrote two other important works: *Liber Quadratorum* [**148**] in 1220 and *Practica Geometriae* in 1225. The first of these is a brilliantly written, original, and able work on indeterminate analysis, stamping the author as the outstanding mathematician in the field from the time of DIOPHANTUS to the time of FERMAT over 400 years later. The great *Practica Geometriae* brings together a vast amount of material in geometry and trigonometry, and it would seem as if some works of the ancients now lost had been still available to LEONARDO. In particular this seems to have been true of EUCLID's work on the Division of Figures [**72**].

LEONARDO's great reputation led to a sort of mathematical tournament when JOHN of Palermo presented three problems which LEONARDO solved. The second of these was to find the solution of the equation $x^3+2x^2+10x=20$, the value for which, given without discussion in his *Flos*, correct to 10 places of decimals, has excited much wonder. Various surmises have been made as to his method of arriving at this result evidently based on Arabic methods [**149**].

Since we have already referred to NASÎR ED-DÎN [**138**] in the latter part of the thirteenth century, and ULUGH BEG [**141**] in the fifteenth century, we shall skip over the period of about 250 otherwise barren years and come to a German named **JOHANN MÜLLER**, born near Königsberg, Lower Franconia, and as a result known in the history of mathematics as **REGIOMONTANUS** [**150**]. He was the most influential and best-known German mathematician of the fifteenth century. We shall simply note that his work on trigonometry, written about 1464, but posthumously published in 1533, was the first European systematic

exposition of plane and spherical trigonometry; the only functions here introduced are sines and cosines [151]. The work had great influence in establishing trigonometry as a science independent of astronomy. The period of REGIOMONTANUS is also that of the Italian, **LUCA PACIOLI**, as well as that in which two mathematical works were printed, namely the first dated arithmetic [152] (Treviso, 1478), an anonymous commercial work, and the first edition of EUCLID'S *Elements*, a Latin translation by CAMPANUS (Venice, Ratdolt, 1482) [153]. PACIOLI [154] published two notable works, the first, usually referred to as *Sūma* (Venice, 1494), a great mathematical compilation drawn almost wholly from a variety of sources, and containing the first account of double-entry bookkeeping [155]; the second, *De divina proportione* (Venice, 1509), of special interest from the geometrical point of view, since the "divine proportion" here discussed was called "golden section" in the nineteenth century.

In the sixteenth century the chief Italian achievement was the solution of equations of the third and fourth degrees. The facts with regard to the general solution of the cubic equation seem to be as follows:

SCIPIONE DEL FERRO, a professor of mathematics at the University of Bologna, solved the equation $x^3+mx=n$ in 1515, imparting the result to his pupil **ANTONIO FIOR**, without publication. About 1535 **NICOLO** of Brescia who stammered badly because of an injury received as a child, and was therefore called **TARTAGLIA**, the stammerer, discovered the solution of the cubic $x^3+px^2=n$ as well as that of FERRO's form. In a public contest provoked by FIOR who had been suspicious of TARTAGLIA's achievements, TARTAGLIA triumphed completely [156]. Under a pledge of secrecy TARTAGLIA confided his method of solution to **GIROLAMO CARDANO** [157], a genius of great ability and the foremost Italian mathematician of his time, who taught mathematics and practised medicine in Milan. By far the most notable of the numerous works which CARDANO wrote was his *Ars Magna* published at Nürnberg, Germany, in 1545. This was the first great Latin treatise devoted solely to algebra. It was here that TARTAGLIA's general solution of the cubic equation was first published without TARTAGLIA's consent. Here, too, appeared the first solution of the general biquadratic equation [158]; this had been found by **FERRARI**, a pupil of CARDANO.

Already at this time the astronomers were feeling the need of trigonometric tables. After twelve years of incessant labor with computers, two works of extraordinary merit, and of value even to the present day, were finally prepared by **RHETICUS** or GEORG JOACHIM of Rhaetia, but not published until (1596, 1613) after his death (1576). One of the works was a ten-place table, with differences, of all six of the trigonometric functions, at interval 10″; the other was a fifteen-place table of natural sines, to every ten seconds of arc, with first, second, and third differences [159]. RHETICUS was the first: (i) to define trigonometric functions in connection with the ratios of sides of a right triangle; (ii) to employ the semiquadrantal arrangement of trigonometric tables. His table of secants was also the first table of the kind. RHETICUS was the leading mathematical astronomer in Teutonic countries in the sixteenth century. From 1539–

1542 he was a disciple of the Polish astronomer **Copernicus** [**160**] who after thirty-six years of labor had completed his great work *De Revolutionibus Orbium Coelestium* [**161**] whose publication in 1543 was due to importunities of Rheticus, like the *Principia* of Newton because of Halley. The Copernican Theory thus formulated had a great effect on thought of the time. We have seen that it had been put forward by **Aristarchus** about 1800 years earlier. A very influential sixteenth century English mathematician, **Robert Recorde** [**162**], was one of the first to bring the Copernican system to the attention of English readers.

The most influential of all the mathematicians in the Netherlands in the sixteenth century was **Simon Stevin** [**163**], famed in his time for his contributions to statics and hydrostatics. But he published the first compound interest tables [**164**] (1582), and in a tiny publication of 1585 he was the first to give a systematic explanation of decimal fractions. Here also he advocated the decimal division of the degree [**164**].

And finally we come to the greatest of all French mathematicians of the sixteenth century, **François Vieta** [**165**], a lawyer and Royal Commissioner who devoted most of his leisure to mathematics. His collected mathematical works form a considerable volume. He contributed extensively to the development of algebra and trigonometry [**166**]. He was among the first to employ letters to represent numbers in algebra, using vowels for the unknowns, and consonants for the knowns. He found the formula for $\cos n\phi$ in terms of $\cos \phi$ for any natural number n and wrote it down for values of n up to $n=9$; made an advance towards proving that a polynomial of the nth degree is made up of n linear factors; showed how to increase, decrease, multiply, or divide the roots of the equation $f(x)=0$, by k; gave the earliest evaluation of π as an infinite product [**167**]; applied algebra to geometry in such a way as to lay a foundation for analytic trigonometry; indicated powers more simply than his predecessors had done, using *A quadratum* or *A quad* for the square of the unknown, *A cubum* or *A cub* for its cube, *A quad quad* for its fourth power, and so on; and showed clearly the relation between the problems of the trisection of an angle and the solution of a cubic equation [**65, 143**].

In concluding our notes on mathematical developments during the 4700 years ending with the sixteenth century, we may draw attention to the fact that the first work on mathematics printed in the New World appeared at Mexico City in 1556, within 64 years of the discovery of America. It was the *Sumario Compendioso* by **Juan Diez** [**168**], and contained (*a*) tables intended to assist merchants in buying gold and silver, (*b*) an arithmetic suited to needs of apprentices in counting-houses, and (*c*) a few pages of algebraic problems, chiefly relating to quadratic equations.

HISTORY OF MATHEMATICS AFTER THE SIXTEENTH CENTURY

A. THE SEVENTEENTH CENTURY

The seventeenth century is especially outstanding in the history of mathematics. It saw FERMAT lay foundations for modern number theory, DESCARTES formulate algebraic geometry, PASCAL and DESARGUES open new fields for pure geometry, KEPLER discover laws of heavenly bodies, GALILEO GALILEI initiate experimental science, HUYGENS make notable contributions in the theory of probability and other fields, NEWTON create new worlds with calculus, curves, and physical observations, LEIBNIZ contribute notably in applications of the calculus and in mathematical notations, and NAPIER reveal a new method of computation. One cannot help being struck by the fact that in this century, with a single exception, all of these creators in mathematics were to be found in the north, where supremacy reigned to the end of the nineteenth century, in practically all fields.

While NAPIER, KEPLER, and GALILEO each lived many years in the sixteenth century, practically all of their important results to which we shall refer were obtained or announced in the seventeenth. It may be well to consider first the contributions of these men.

JOHN NAPIER [169] was a Scot, spent most of his life at Merchiston Castle near Edinburgh, and took an active part in the political and religious controversies of the day. As BALL expresses it [1]: "The business of his life was to show that the Pope was Antichrist, but his favorite amusement was the study of mathematics and science." In connection with the history of mathematics he is usually thought of only as the great inventor of logarithms; but the "rule of circular parts" (a mnemonic for readily reproducing the formulae used in solving right spherical triangles), the four formulae (known as "NAPIER's analogies") for solving general spherical triangles, and the calculating rods [170], called "NAPIER's Rods," for multiplying, dividing and taking of square roots, were also products of his genius. NAPIER seems to have had the idea of logarithms in mind as early as 1594, but it was not until 1614 that he published his researches in a "description of the admirable canon of logarithms" [171]. Besides explaining his logarithms, he gives a table of the logarithms of natural sines from 0 to 90° for each minute. The base of these logarithms is $q^{1/d}$, where $q=(1-10^{-7})$, $d=(1+\frac{1}{2}\cdot 10^{-7})10^{-7}$. Hence $q^{1/d}\approx e^{-1}(1-\frac{1}{3}\cdot 10^{-14})\approx e^{-1}$. Thus it is wholly incorrect to refer to NAPERIAN logarithms as natural logarithms [172].

While, from the time of ARCHIMEDES on to the time of NAPIER, there were numerous instances of the recognition of such a relation as $a^m\cdot a^n=a^{m+n}$, the law of exponents was in no sense a common idea, and it was not till later that the notion of a logarithm as an exponent became general. In NAPIER's system the logarithm of 10^7 (not unity) was zero, although NAPIER recognized that a system with the logarithm of unity taken as zero was more desirable. It was effected by his friend, the English mathematician, **HENRY BRIGGS**, whose tables

of the logarithms of numbers from 1 to 20,000, and from 90,000 to 100,000, to 14 places [173], were published in 1624, seven years after NAPIER's death. The final part of a great new table to 20 places of decimals has recently appeared in England [174] in partial (though belated) celebration of the tercentenary of the discovery of logarithms. In Italy NAPIER's wonderful discovery was taken up enthusiastically by **BONAVENTURA CAVALIERI**, a pupil of GALILEO and professor of mathematics at the University of Bologna for the last 18 years of his life [175]; we shall presently have something to say about his work. In Germany KEPLER's zeal and reputation soon brought them into vogue there.

Let us now consider some of the contributions made by the second man of our group **JOHANN KEPLER** [176], mathematician, mystic, one of the founders of modern astronomy and an exceptional genius, outstanding as a brilliant calculator and patient investigator. In 1601 at Prague he became the assistant of the remarkable Danish astronomical observer **TYGE BRAHE** [176], who died a little later. The vast body of accurate observations of the planets which Kepler thus inherited led him in 1609 and 1619 to formulate the following laws:

I. All the planets move round the sun in elliptic orbits with the sun at one focus;

II. The line joining the planet to the sun sweeps out equal areas in equal intervals of time;

III. For all planets the square of the time of one complete revolution is proportional to the cube of the mean distance from the sun.

Purely as a thrilling intellectual experience, without any imaginable practical application, APOLLONIUS of Perga, and other Greeks, developed a marvellous body of knowledge with regard to conic sections. Then suddenly, 1800 years later, to a KEPLER this knowledge had most illuminating practical applications.

KEPLER's contributions to the infinitesimal calculus included finding volumes of surfaces of revolutions of conics about lines (93 different solids including the torus); deriving, in effect, the value of $\int \sin t dt$, and other integrals; and in solving maximum-minimum problems concerning cylinders, cones, and wine barrels. Of the four known star polyhedra KEPLER discovered two, the other two being found centuries later (1809) by LOUIS POINSOT. Star polygons and problems of filling a plane and space by regular figures were studied by KEPLER. He discovered how to determine whether a conic is a hyperbola, ellipse, parabola, or circle when given a vertex, the axis through it, and an arbitrary tangent with its point of contact. In one of his works, which consists rather in the enunciation of certain general principles, illustrated by a few cases, than in a systematic exposition, he laid down what has been called a Principle of Continuity, and gives as an example the statement that a parabola is at once the limiting case of an ellipse and of a hyperbola. He illustrated the same doctrine by reference to foci of conics; he explained also that parallel lines should be regarded as meeting at infinity. KEPLER contributed to the simplification of computations and published a volume of logarithms (1624–1625).

The last member of our group, **GALILEO GALILEI** [177], one of the most inter-

esting figures in the history of science, was the founder of the science of dynamics. His work in astronomy we shall not consider. A native of Pisa, he is said to have carried on experiments from the leaning tower there and to have arrived at the law that the distance of descent of a falling body was proportional to the square of the time, in accordance with the well-known law that $s=\frac{1}{2}gt^2$. GALILEO was the first to realize that, neglecting air resistance, the path of a projectile is a parabola. He speculated interestingly on laws involving momentum. He suggested that arches of bridges should be built in the form of cycloids [**178**], and by weighing pieces of paper he is believed to have found that the area of a cycloid is less than three times that of its generating circle. From 1609 on, he constructed telescopes.

Another mathematician who lived many years in the sixteenth century, although his outstanding publication appeared in the seventeenth century, was the Englishman **THOMAS HARRIOT** [**179**], who is of special interest to Americans since he was sent in 1585 to survey and map Virginia, now North Carolina. He died in 1621; one of his works published in 1631, *Artis Analyticae Praxis*, deals with algebraic equations of the first, second, third, and fourth degrees. It is more analytic than any algebra that preceded it, and marks an advance both in symbolism and notation, though negative and imaginary roots are rejected. It was potent in bringing analytic methods into general use. Harriot was the first to use the signs $>$ and $<$ to represent is greater than and is less than. When he denoted the unknown quantity by a, he represented a^2 by aa, a^3 by aaa, and so on. This is a distinct improvement on Vieta's notation. In unpublished manuscripts, attributed to HARRIOT, there is a table of binomial coefficients worked out in the form of a PASCAL triangle [**180**]. It was more than thirty years after HARRIOT's death that PASCAL is known to have used this triangle. MORLEY stated that the manuscripts contained a well-formed analytical geometry, but D. E. SMITH found that the manuscript in question was not in HARRIOT's handwriting.

HARRIOT's posthumous work appeared in the same year, 1631, as the first edition of the *Clavis Mathematicae of* **WILLIAM OUGHTRED** [**181**], a work on arithmetic and algebra. After NAPIER'S *Descriptio* in 1614, this was the most influential mathematical publication in Great Britain, in the first half of the seventeenth century. It was one of the few books that contributed to laying the foundations of the mathematical knowledge of NEWTON as he was starting on his career. OUGHTRED exceptionally emphasized the use of mathematical symbols and, although he introduced more than a hundred of them, only three have come down to modern times. These are: our cross-sign for multiplication, the four-dot sign in proportion (::), and our sign for difference between ($\sim$). An OUGHTRED publication of 1618 contains [**181a**] (1) the first use of the sign $\times$; (2) the first abbreviations, or symbols, for the sine, tangent, cosine, and cotangent; (3) the first invention of the radix method of calculating logarithms; (4) the first table of natural logarithms. About 1622 OUGHTRED invented the slide rule [**182**], and his priority in the invention is unassailable. OUGHTRED was by

profession a minister of the gospel but his avocation was the teaching of mathematics and the writing of mathematical books. Among his pupils were JOHN WALLIS, and CHRISTOPHER, afterward SIR CHRISTOPHER, WREN. He was born in the last third of the sixteenth century and lived to be over ninety.

We may appropriately consider next the contributions of a member of the French group, **GIRARD DESARGUES**, born in the last decade of the sixteenth century. By profession an engineer and architect, his gratuitous lectures in Paris, when he was in his thirties, made a great impression on his contemporaries. Among his books was a treatise on how to teach children to sing well [**183**]. But it was his remarkable treatise on conics which stamps him as the most original contributor to pure geometry in the seventeenth century. DESARGUES commences with a statement of the doctrine of continuity as laid down by KEPLER and develops fundamental theorems on involution, homology, poles and polars, and perspective [**184**], such as arise to-day in our courses on synthetic projective geometry. This subject was not developed further, to any extent, until the early part of the nineteenth century.

When a mathematician hears the name PASCAL he usually thinks of another Frenchman, **BLAISE PASCAL** [**185**], author of the famous *Provincial Letters*, and not of his father **ÉTIENNE** whose name is connected with the quartic curve called PASCAL's limaçon [**186**]. BLAISE showed phenomenal ability in mathematics at an early age, and DESCARTES could not at first credit that the manuscript on conics, which BLAISE wrote at the age of 16, had been written by him and not by his father. It was here that the famous PASCAL's theorem occurs: If a hexagon be inscribed in a conic the points of intersection of opposite sides will be collinear [**187**]. In 1642, at the age of 19, he invented a computing machine which he later improved [**188**]. This, and one constructed by LEIBNIZ about 1694, were the first of their kind. PASCAL's last work was on the cycloid. In correspondence with Fermat he laid down the principles of the theory of probability [**189**].

In connection with pure geometry in the seventeenth century it should now be noted that a work published at Amsterdam in 1672 by a Dane named **GEORG MOHR** [**190**] contained the elegant basis of a proof that all constructions with ruler and compasses could be carried through with compasses alone. Up to the recent republication of this work, it was generally thought that this theorem was first proved 125 years later by a capable Italian mathematician **LORENZO MASCHERONI**, and the term "MASCHERONI constructions" (which should much more correctly be "MOHR constructions") is current in mathematical literature.

The development of analytic geometry and of the infinitesimal calculus diverted attention from pure geometry. Let us now consider the work of the Frenchman, **RENÉ DESCARTES** [**191**], born in a family of wealth and culture, who greatly enriched the world by his writings in many fields. We are chiefly interested in his *La Geometrie* [**192**], a section of his large epoch-making work published at Amsterdam in 1637. The work has the title, ***Discours de la Methode Pour bien conduire sa raison, & chercher la verité dans les sciences. Plus La Dioptrique. Les Meteores. Et La Geometrie. Qui sont des essais de cette Methode.*** In the ADAM and TANNERY edition of *Œuvres de Descartes*, the whole fills 515 pages.

Of these *La Geometrie* occupies pp. 367–485, 511–514. The essence of plane analytic geometry is the study of loci by means of their equations. This was an important part of the investigation of conic sections by the Greeks. Even in the time of MENAECHMUS, the reduction of the problem of the duplication of the cube to finding two mean proportionals between two quantities was connected with parabolas and hyperbolas, and the equivalent in geometric form of their equations, as we now use them. APOLLONIUS of Perga established geometric relations equivalent to the equations of conics referred to a tangent and a diameter through its point of contact as axes. In the Greek scheme of variables a line segment corresponded to a first degree variable; the area of a rectangle, to the product of two such variables; the volume of a rectangular parallelopiped, to the product of three variables. Hence, after linear, plane, and solid magnitudes, the Greeks could go no further. On this DESCARTES made a tremendous advance by arithmetizing his geometry, and using excellent algebraic symbolism. The expression x^2 did not suggest to him an area but the fourth term in the proportion $1:x=x:x^2$. He laid off x on a certain line and then a length y at a fixed angle and sought points whose x's and y's satisfied certain conditions. He applied his method to the famous three- and four-line locus, and showed not only that this was a conic, but also that generalized forms of the problem, like that of PAPPUS, gave loci of higher degree. He showed that in the case of each of several curves mechanically constructed we could translate the mechanical process into algebraic language and so find an equation for the curve. The quartic ovals that bear his name [**193**] occur in this work. In the third book DESCARTES suggests that his method of geometric determination of a root may be extended to any degree. Here also he made use of what we now call "DESCARTES's rule of signs," distinguished between algebraic and transcendental curves, introduced the system of indices now in use, as a^4, and set the fashion of denoting variables by x, y, z, and constants by a, b, c.

DESCARTES was the first one to formulate the theorem, commonly attributed to EULER, on the relation between the number of faces, edges, and vertices of a convex polyhedron, $F+V=E+2$. He did not completely imagine the so-called folium of DESCARTES [**193**], $x^3+y^3=3axy$.

Another great French mathematical genius in the seventeenth century was **PIERRE DE FERMAT** [**194**], who was counselor for the local parliament at Toulouse, and who, except during 1641–53, devoted most of his leisure to mathematics after he was thirty years of age. He was the founder of the modern theory of numbers and a master in the field of DIOPHANTINE analysis [**195**]. Many theorems, of which he left no proof, have later been proved to be correct. Since it appears that practically no positive statement which he made has been shown to be incorrect, extraordinary celebrity has been developed in connection with the following theorem of which FERMAT stated that he had a proof: No integral values of x, y, z, can be found to satisfy the equation $x^n+y^n=z^n$ if n is an integer greater than 2. For, even to the present day, no mathematician has been able either to prove or to disprove the statement.

The numbers $F_n=2^{2^n}+1$ are often called FERMAT numbers because FERMAT

thought that F_n might possibly be a formula for giving an infinity of primes; F_n is prime for $n=0, 1, 2, 3, 4$, but it has since been shown that it was not prime for 13 other values of n. The numbers F_n became famous through the fact that about 1800 GAUSS discovered that when F_n is prime, it represents the number of sides of a regular polygon which can be constructed with ruler and compasses [196].

FERMAT was in possession of the general idea of finding maxima and minima; he also obtained the subtangents and areas of a number of curves, certain quadratures, and the center of mass of a paraboloid of revolution.

FERMAT gave an analytic treatment of the straight line and conic sections, and he and PASCAL were the founders of the theory of probability, but they published nothing on this subject.

In reviewing NAPIER'S work we found **CAVALIERI** enthusiastically advocating the use of logarithms. CAVALIERI'S chief contribution to mathematics was his principle of indivisibles discussed somewhat by ARISTOTLE, and also receiving the attention of GALILEO, TORRICELLI, and ROBERVAL [197]. It occupies a place intermediate between the method of exhaustion of the Greeks, and the calculus methods of NEWTON and LEIBNIZ. Each indivisible is capable of generating the next higher continuum by motion; a moving point generates a line, a moving line a plane, a moving plane a solid. Though lacking in scientific foundation it was a sort of integral calculus which yielded solutions of difficult problems. A theorem called CAVALIERI'S theorem [198] has recently come into use in America in the teaching of elementary geometry, as a means of unifying the ideas that lie at the basis of the mensuration of solids. The substance of this theorem is as follows: If two solids have equivalent bases, and if sections parallel to the bases and equally distant from them in the two solids are also equivalent, then the solids are equivalent.

On turning once more to the north the most eminent mathematician in Scotland in the seventeenth century after NAPIER was **JAMES GREGORY** [199], for a few years professor of mathematics at the Universities of St. Andrews and Edinburgh. In 1671 he was an independent discoverer of the binomial theorem in its most general form, of the interpolation formula (usually referred to as GREGORY-NEWTON), and of expansions including those for $\tan x$, $\sec^{-1} x$, and $\tan^{-1} x$; for this last

$$\tan^{-1} x = x - \tfrac{1}{3}x^2 + \tfrac{1}{5}x^3 - \cdots ,$$

from which when $x=1$, $\pi/4$ may be expressed (not noted by GREGORY) as a series (known to LEIBNIZ in 1674). He evaluated

$$u = \int_0^\alpha \sec \theta d\theta = \ln (\sec \alpha - \tan \alpha) = \ln \tan (45^\circ + \tfrac{1}{2}\alpha),$$

thus settling an Edward Wright problem [200], discovered (1671–72) TAYLOR'S (1735) theorem, and, inspired by FERMAT, solved DIOPHANTINE problems.

From GREGORY's series for $\tan^{-1} x$, JOHN MACHIN, an eighteenth century professor of astronomy [201], derived the relation (1706)

$$\frac{1}{4}\pi = 4 \tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239},$$

which was used to calculate π to 707 decimal places (1874). In 1947 (with another formula) it was shown by another English computer, who calculated π to 808 decimal places, that the earlier extended computation was correct to only 526 decimal places.

On the continent we find for our story much of interest in the work of CHRISTIAAN HUYGENS, eminent as physicist, astronomer, and mathematician [202], the greatest scientist that Holland has ever produced. Most of his discoveries were made with the aid of ancient Greek geometry, for which, like his friend NEWTON, he always showed partiality; but at times he called to his aid ideas suggested by DESCARTES, CAVALIERI, and FERMAT. NEWTON and HUYGENS were kindred spirits and always had the greatest admiration for each other.

ARCHIMEDES secured his approximation to the value of π correct to two places of decimals by considering inscribed and circumscribed polygons of $96 = 3.2^5$ sides. Using the same method VIETA calculated the value correctly to nine places of decimals by means of polygons of $393\ 216 = 3.2^{17}$ sides, and Ludolf van Ceulen to 20 places by means of polygons of $32\ 212\ 254\ 720 = 60.2^{29}$ sides; after spending most of his life in calculation he used this same method to determine the result to 35 places of decimals. As a result of a series of geometric propositions given by HUYGENS in his remarkable work on the magnitude of a circle, the ARCHIMEDES' value may be obtained from an inscribed hexagon, and a value correct to 9 places from a regular polygon of only sixty sides [203].

HUYGENS wrote also the first formal treatise on probability, which was the best until superseded by more elaborate works of JAMES I BERNOULLI and DEMOIVRE. To the theory of curves he added the important theory of evolutes and showed in particular that the evolute of a cycloid is an equal cycloid, which led to his construction of the isochronal pendulum. Stimulated by the astronomer's need, he invented the pendulum to regulate the movement of clocks. HUYGENS showed also that the cycloid has the property that when the accelerating force is gravity a particle moving along a vertical curve will arrive in the same time at a given point wherever the initial point is taken on the curve. HUYGENS solved the problem of determining the equation of the catenary curve formed by a perfectly flexible chain suspended by its ends. He found also many other results concerning curves, some of them generalizations of particular cases considered by FERMAT, and dealt with many interesting problems of pure geometry.

In England of the seventeenth century the work of HARRIOT and of OUGHTRED has been considered; but a much abler mathematician, one of the most

original of his day was **John Wallis** [204], for 54 years professor of geometry at Oxford University. He was a man of great erudition, and wrote works in a dozen different fields. His great work on *Algebra* (not without a blemish) contains the first record of an effort to represent an imaginary number graphically; while different from the method now used, it is complete and consistent in itself. In discussing fourth dimension he refers to it as a "Monster in Nature, and less possible than a *Chimaera* or *Centaure*." He used Cavalieri's method of indivisibles to find the areas of curves $y=x^m$ when m is a positive integer or a fraction. He failed to get the approximate quadrature of the circle $y=(1-x^2)^{1/2}$ by series, since he did not know the binomial theorem for a fractional power; but by an original method of interpolation he expressed π in the form of an infinite product,

$$\frac{4}{\pi}=1\times\frac{9}{8}\times\frac{25}{24}\times\frac{49}{48}\times\frac{81}{80}\cdots;$$

thus bringing us to the beginning of the second period in the history of the problem of squaring the circle [67], the first having ended with discussions of polygon perimeters.

But the greatest scientist of any century was **Isaac Newton** [205–208], who was born December 25, 1642, o.s., or January 5, 1643, n.s., and died in 1727, aged 84. Already at Cambridge University, at the age of 22, in the year that he took his B.A. degree, he had invented the method of fluxions [209] or calculus, and discovered the binomial theorem [209]. In the following year he made applications of the calculus to tangents, curvature, concavity and convexity, maxima and minima, and other things, some of which were new and of importance. It was about this time also that he made interesting discoveries in optics and formulated, but did not announce, the law of universal gravitation. Referring to these years later in life Newton wrote, "in those days I was in the prime of my age for invention, and minded Mathematicks and philosophy more than at any time since."

His former teacher **Isaac Barrow** [210], fully recognizing the transcendent ability of Newton, resigned his chair (although only thirty-nine years old) to make way for Newton, then only twenty-six years of age, to be appointed in his place as professor of mathematics at Cambridge. At the age of forty-seven Newton represented the University in Parliament and by the time he was fifty his scientific creations were practically ended; but in later years he received many honors and a number of his earlier writings or lectures were published.

His greatest work was indisputably the "Mathematical Principles of Natural Philosophy" usually known by a single word of the Latin title, *Principia* [211], published in 1687. This work (1) treats of the motion of particles or bodies in free space either in known orbits, or under the action of known forces or under mutual attractions; (2) treats of motion in a resisting medium, and of hydrostatics and hydrodynamics with special applications to waves, tides, and acoustics; and (3) applies theorems obtained to the chief phenomena of the solar system and determines the masses and distances of the planets. The motions of

the moon, the various inequalities therein, and the theory of tides are worked out in detail. The great principle underlying the whole work is that of universal gravitation characterized as "indisputably and incomparably the greatest scientific discovery ever made." Possibly because he regarded the presentation as more convincing for students of his time, NEWTON achieved the stupendous task of translating his arguments into the language of Greek geometry even though many of his results were first worked out by methods of the calculus. In the *Principia* are a great many results concerning well-known curves, such as epi- and hypocycloids, and many beautiful results in geometry, such as the following: (1) If a point P moving along a given straight line is joined to two fixed points O and O', and if lines OQ and $O'Q$ make fixed angles with OP, $O'P$, then the locus of Q is a conic. (2) The locus of the centers of conics tangent to the sides of a quadrilateral is the line ("NEWTON's line") [**212**] through the centers of its diagonals (lemma 25, cor. 3).

NEWTON studied curves of the third order [**213**] with a view, apparently, to illustrating the application of analytic geometry to curves other than conics. He enumerated 72 of the possible 78 forms which a cubic may assume, in his classification; and in the course of the work he states the remarkable theorem that just as the shadow of a circle cast by a luminous point on a plane gives rise to all the conics, so the shadow of the curves represented by the equation $y^2=ax^3+bx^2+cx+d$ gives rise to all cubics. This remained an unsolved puzzle till a proof was discovered in 1731.

In another monograph on the quadrature and rectification of curves by means of infinite series, NEWTON indicates at one point the importance of determining whether the series are convergent—an observation far in advance of his time—but there is no record that he knew a general test for the purpose; and in fact it was not until GAUSS and CAUCHY took up the question, early in the 19th century, that the necessity for such limitation was commonly recognized. In this monograph arcsin x is expanded in a series, and the process of reversion of series is employed.

In his *Universal Arithmetic*, containing the substance of lectures [**214**], DESCARTES's rule of signs is extended to give limits to the number of imaginary roots of an equation, and a method is given for finding the approximate values of the roots of a numerical equation [**215**], a method which applies equally to algebraic or transcendental equations. He enunciated also the theorem, known by his name, for finding the sum of the nth powers of the roots of an equation, and laid the foundation of the theory of symmetric functions of the roots of an equation.

In connection with various challenges among mathematicians, NEWTON was never beaten; in one from LEIBNIZ he solved the problem of finding the orthogonal trajectories of a family of curves. One of the most interesting eulogies on NEWTON's work is that attributed to LEIBNIZ who, upon being asked by the Queen of Prussia what he thought of NEWTON, answered: "Taking mathematicians from the beginning of the world to the time when NEWTON lived, what he had done was much the better half" [**216**].

LEIBNIZ [217], the only pure mathematician of the first class produced by Germany during the seventeenth century, was equally notable in the field of philosophy. The last years of his life were embittered by a controversy as to whether he had discovered the calculus independently of NEWTON's previous investigations, or had derived the fundamental idea from NEWTON. There is now no doubt that LEIBNIZ developed his calculus quite independently, and that he and NEWTON are each entitled to credit for their respective discoveries. The two lines of approach were radically different, although the respective theories accomplished results that were practically identical. LEIBNIZ was original in much that he did. He made the CAVALIERI method scientific. He introduced the idea of fractional differentiation [218]. He laid the foundation for the theory of envelopes and defined the osculating circle, showing its importance in the study of curves.

The history of determinants [219] begins with LEIBNIZ in connection with the eliminant of linear equations. He gave a generalization of NEWTON's binomial expansion rule in his expansion of any polynomial $s+x+y+z+\cdots$ to any arbitrary power r. We have already referred to the calculating machine [220] he constructed.

LEIBNIZ made important contributions to the notation of mathematics [221]. Not only is our notation of the differential and integral calculus due to him but also our signs, in geometry, for similar ($\sim$), and for equal and similar, or congruent, ($\simeq$).

And finally, among the names of mathematicians of the last half of this century we shall briefly note results found by JAMES I BERNOULLI of Switzerland, a member of a remarkable family in which 8 members distinguished themselves in mathematics [222]. He was professor of mathematics at the University of Basel, and was interested in the fields of astronomy, mathematics, and physics. He was the first to solve the problem of isochronous curves proposed by LEIBNIZ, that is, to find the curve along which a body falls with uniform vertical velocity; in his solution of this problem (1690) we meet with the word integral for the first time. HUYGENS also solved this problem and found it to be the semicubical parabola [69] $x^3=ay^2$.

He worked on the problem: To find the curvature a lamina should have in order to be straightened out horizontally by a weight suspended at the end. He noted the relation $Rs=a^2$, where R is the radius of curvature at a distance s along the lamina from the point of suspension, and gave a construction for points of the curve. But it was EULER in the next century, who first really visualized the curve as a double spiral [223] with asymptotic points, later called the clothoïde [69]; the equation of the spiral is:

$$x = \int_0^s \sin\left(\tfrac{1}{2}t^2/a^2\right)dt, \qquad y = \int_0^s \cos\left(\tfrac{1}{2}t^2/a^2\right)dt.$$

This is the spiral which comes up in connection with FRESNEL's discoveries of

the diffraction of light, and in railway easement curves [**224**]. It has a number of interesting properties.

BERNOULLI did much work on the logarithmic spiral [**225**] and found for example that (1) the evolute of a logarithmic spiral is another equal logarithmic spiral, (2) the pedal of a logarithmic spiral with respect to its pole is an equal logarithmic spiral; (3) the caustic by reflection and refraction of a logarithmic spiral for rays emanating from the pole as a luminous point is an equal spiral. Such perpetual renascence delighted BERNOULLI and it was, in accordance with his directions, recorded on his tomb with the inscription: *Eadem mutata resurgo* —"I arise the same though changed."

BERNOULLI wrote also a posthumously published (1713) work entitled *Ars Conjectandi* which established principles of the calculus of probability [**226**] in more elaborate form than HUYGENS had done. It is here that we meet certain numbers, known in a very extensive later literature as "BERNOULLI numbers." In 1913 the Academy of Sciences at Leningrad celebrated the bicentenary of the law of large numbers, here published.

Thus closes our sketch of the seventeenth century during which such great advances were made in fields of analytic geometry, calculus and its applications, mechanics and dynamics, geometry, theory of probabilities and the theory of numbers.

B. THE EIGHTEENTH CENTURY

The early part of this century saw the beginnings of Analytic Geometry of Three Dimensions [**227**]. In 1679 PHILIPPE DE LA HIRE solved a simple problem; but by 1700 ANTOINE PARENT had gone much farther and before 1750 ALEXIS CLAIRAUT and JAKOB HERMANN were discussing problems in analytic geometry of curves and surfaces.

The first half of the century saw also the chief publication of one of the forerunners of noneuclidean Geometry, **GIROLAMO SACCHERI** [**140, 228**], Jesuit teacher of theology, logic, metaphysics, and mathematics, at Turin, Pavia, and Milan.

The principal mathematicians flourishing in this century were, DEMOIVRE, TAYLOR, MACLAURIN, EULER, D'ALEMBERT, LAGRANGE, LAMBERT and MONGE.

Since **ABRAHAM DEMOIVRE** [**229**], although born in France, spent 66 years of his life in England, he is properly considered a member of the English school. He was an intimate friend of NEWTON. Apart from numerous memoirs he published a work called *Annuities upon Lives*, of which there were seven editions, a work of importance in the history of actuarial mathematics. His very valuable *Doctrine of Chances* passed through three editions and contained much new material connected with the theory of probability [**230**]; indeed, he ranks with LAPLACE in making the most important contributions to the subject up to the end of the eighteenth century. He gave the first treatment of the probability integral, and essentially of the normal curve. He formulated and used a theorem improperly called STIRLING's theorem [**231**].

In his *Miscellanea Analytica* [230], 1730, which brought about his election as a Fellow in the Berlin Academy, there is important material in connection with recurrent series, analytic trigonometry involving imaginaries, and problems of chance, and analysis is applied to problems of astronomy. From one part of this work it is clear that DEMOIVRE was thoroughly familiar with the formula now connected with his name [232]:

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx,$$

where n is an integer. But the result is not explicitly stated in any of DEMOIVRE's writings; a form equivalent to this was, however, given in 1748 by EULER [232]. In this work DEMOIVRE studied also the factors of $x^{2n}-2x^n \cos n\theta +1 [=y]$. If O is the center of a unit circle circumscribing a regular polygon of n sides $A_1A_2 \cdots A_n$, if $OP=x$, and if $\angle POA_1=\theta$, then $\overline{PA}_1^2\cdot\overline{PA}_2^2\cdot\overline{PA}_3^2 \cdots \overline{PA}_n^2=y$, called "DEMOIVRE's property of the circle."

The name of the Englishman **BROOK TAYLOR** [233] is familiar to every student of the calculus, through the theorem of TAYLOR's expansion of $f(x+h)$, given in his *Methodus Incrementorum directa et inversa.* (This is the first treatise dealing with finite differences; it was published in 1715.) The convergency of the expansion is not considered.

In this work TAYLOR solved the following problem which he believed to be new: "To find the number of vibrations that a string will make in a certain time, having given its length, its weight, and the weight that stretches it." In discussing the form of the vibrating string, his suppositions regarding initial conditions, including that it vibrated as a whole, led to a differential equation whose integral gave a sine curve. Thus started a discussion which was to culminate a century later in the work of Fourier [234].

We shall next consider **COLIN MACLAURIN** [235], a Scot, and one of the ablest mathematicians of the eighteenth century. His name is associated with TAYLOR in the expansion of $f(x+h)$, when $h=0$. This was given in his *Treatise of Fluxions* in 1742, but the same thing appeared twenty-five years earlier, in a publication by STIRLING. MACLAURIN's *Treatise* is especially valuable for the solutions it contains of numerous problems in geometry, statics, the theory of attractions, and astronomy. He followed NEWTON in using geometric methods. By avoiding the use of analysis and of the infinitesimal calculus he helped to keep alive methods not abandoned in England till about 1820. In 1740 MACLAURIN shared with EULER and D. BERNOULLI a prize of the French Academy of Sciences for his memoir on the tides, winning from LAGRANGE the highest praise [236].

When only 21 years of age MACLAURIN wrote a work called *Geometria Organica* which was inspired by NEWTON's work on curves of the third order, and by the organic description of a conic which we quoted from NEWTON's *Principia*. He attempted to generalize this organic description of a conic so as to obtain curves of all possible degrees by a mechanical description. He discusses also various pedal curves [237]. In this way he was led to such familiar curves as the cissoid, strophoid, cardioid, limaçon, and lemniscate, and many

others [237] which have since become notable in connection with the study of special curves.

We have already more than once referred to LEONARD EULER [238], the great Swiss mathematical genius, born in 1707 at Basel, the home of the BERNOULLIS. The project of erecting an Academy at Petersburg, which had been formed by PETER THE GREAT, led to EULER being invited in 1727 to the chair of mathematics there. After fourteen years he accepted the invitation of FREDERICK THE GREAT [238a] to go to Berlin to direct the Prussian Academy which was then rising in fame; he continued there for twenty-five years, until 1766, when he returned once more to Petersburg, where he remained till his death in 1783 at the age of 76.

Few writers in mathematics ever contributed so extensively or so fruitfully. In a sense he was the creator of modern mathematical expression. He revised almost all the branches of mathematics then known, filling up details, adding proofs and new results, and arranging the whole in a consistent form. More than 700 memoirs and papers came from his hand, besides many treatises, half of them being published after he was totally blind. He wrote on philosophy, astronomy, physics, geography and agriculture; and on mathematics in the following fields, among others: arithmetic, theory of numbers, Diophantine equations, algebraic equations, series, continued fractions, calculus of probabilities [239], analysis, definite and indefinite integrals, elliptic functions and integrals [240], differential equations, calculus of variations, elementary geometry, analytic geometry, differential geometry, mathematical recreations, music [241], mechanics, ballistics, and the motion of ships.

Certain of our ordinary notations [242] originated with EULER,

$f(x)$ for function symbol,
e for $2.71828\cdots$,
s for half the sum of the sides of a triangle,
a, b, c for the sides of a triangle ABC,
$\sum$ for summation sign,
i for $\sqrt{(-1)}$.

From the formula $e^{ix}=\cos x+i \sin x$, for $x=\pi$, $e^{i\pi}+1=0$. This relation was to be the basis of a proof that the problem of squaring the circle was impossible [243]. It should be noted that thirty years before EULER published his result ROGER COTES gave the equivalent in words of the relation:

$$ix = \log(\cos x + i \sin x).$$

EULER showed that $\frac{1}{2}\pi = -i \log i$, and that $(\log i)/i$, and i^i had an infinite number of values [244]. He gave the formula,

$$\pi = 20 \tan^{-1}\frac{1}{7} + 8 \tan^{-1}\frac{3}{79},$$

and from it calculated π to 20 places of decimals.

EULER showed that the center of the circumscribed circle of a triangle, its center of gravity, and the point of meeting of the altitudes are collinear [245]; hence the name *Euler's Line*. It was later shown that the center of the nine-point circle is also on this line.

EULER gave a new method of solving the general equation of the fourth degree. He introduced the idea of an indicatrix for studying the variations of the radii of curvature of normal sections of a surface at a point. He worked on the problem of a vibrating string [241] initiated by BROOK TAYLOR, and later carried on by Lagrange, preparatory to the important work of **FOURIER** [246] in the early part of the 19th century. It was in his work on the calculus of variations that Euler considered the problem of JAMES I BERNOULLI for a coiled lamina [238], to which reference has been made. Beta and gamma [247] functions originated with EULER, as well as the EULER constant [248],

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right) = 0.5772156649\cdots.$$

When EULER left Berlin in 1766, he was succeeded by **JOSEPH LOUIS LAGRANGE** [249], an Italian by birth, a German by adoption and a Frenchman by choice, one of the great mathematicians of all time. He was then 30 years old. In sending the invitation FREDERICK THE GREAT wrote that the greatest king in Europe wanted the greatest mathematician of Europe at his court. LAGRANGE remained there till the death of FREDERICK 20 years later. Many of his great memoirs, as well as the composition of his monumental work on *Analytic Mechanics* (containing his well-known general equations of motion of a dynamical system), date from this period. During the remaining 25 years of his life he was in Paris, a professor at the newly established École Normale Supérieure, and École Polytechnique. These two great schools occupy a most important place in the history of mathematics. Most of the greatest mathematicians of France have been professors in them, and practically all have there been trained. Towards the close of the eighteenth century appeared two more of LAGRANGE's great works, the *Resolution of Numerical Equations of Every Degree*, and the *Theory of Analytic Functions Containing the Principles of the Differential Calculus*. In the first of these he attempted a proof (see later, under GAUSS), that every algebraic equation has a root. His work in the field of Calculus of Variations was important, and his name is linked with that of EULER in founding the subject. His contributions in the field of Differential Equations were notable. Like DIOPHANTUS and FERMAT, he had a special genius for the theory of numbers. As a whole his work had a very profound influence on later mathematical research.

Let us now turn back to Germany to consider contributions made by **JOHANN HEINRICH LAMBERT** [250], who enjoyed association with EULER and LAGRANGE in Berlin. He was a many-sided scholar and wrote on a great variety of subjects.

His *Freye Perspective* of 1774 contained numerous geometric problems of interest [250].

In a monograph of 1761, on the determination of a comet's orbit, he gave incidentally the results that a circle about a triangle circumscribing a parabola goes through the focus; and hence if four tangents are given the focus is determined. From the rediscovery of this parabola property, nearly forty years later, **William Wallace** [251] deduced his theorem: The feet of the perpendiculars on the sides of a triangle, from any point on its circumscribed circle, are collinear.

In 1767 Lambert had proved with most minute rigor [252] that π, e^x, and $\tan x$ were irrational. (In 1737 Euler had shown that e and e^2 were irrational.) This result for π was the beginning of the third period in the history of the problem of the squaring of the circle [67]. In 1844 and 1851 a prominent French mathematician named Liouville [253] proved the existence of transcendental numbers, that is of numbers which are not roots of algebraic equations with integral coefficients. In 1873 a great French mathematician named **Hermite** [254] proved that e was transcendental. As a result, and by means of Euler's relation $e^{i\pi}+1=0$, it was shown, in 1882, that π was transcendental, and hence that the problem of squaring the circle was impossible of solution with ruler and compasses. This last result was found by Lindemann, a German.

To Lambert we owe the systematic development of the theory of hyperbolic functions, introduced by Vincenzo Riccati a few years earlier [255]. Lambert's name is also associated with important projections used in map making [256].

Three other French mathematicians of prominence, Monge, Laplace, and Legendre, were born about the middle of the eighteenth century, but the culminating period of the most important work of the two latter was in the nineteenth century. For the final name in the eighteenth century, we shall, then, consider that of **Gaspard Monge** [257], a professor of mathematics in the École Polytechnique of Paris. He was the inventor of descriptive geometry, and his treatise on this subject, which appeared first in 1794, went through a number of editions. There were five editions of his *Applications of Analysis to Geometry* which first appeared in 1795; it was an important early work on the differential geometry of surfaces.

Our survey has suggested that while the eighteenth century saw the further development of such subjects as trigonometry, analytic geometry, analysis, theory of numbers, differential equations, probabilities, and analytic mechanics, it saw also the creation of new subjects such as actuarial science, calculus of variations, finite differences, gamma and elliptic functions, descriptive geometry, and differential geometry [258].

C. THE NINETEENTH CENTURY AND LATER

Moritz Cantor's great history of mathematics to the end of the eighteenth century contained four large volumes [12] with an average of 988 pages to the

volume. It has been estimated that if the history of mathematics in the nineteenth century were written with the same detail it would require fourteen or fifteen more similarly sized volumes [259]. As indicated in our introductory remarks we shall, in conclusion, touch very briefly on only a very few topics in the enormous mass of available material.

We saw that LAPLACE and LEGENDRE were two great French mathematicians who were contemporaries of LAGRANGE, but whose principal work was published in the nineteenth century. Let us now consider them. **PIERRE SIMON LAPLACE** [260], born in obscurity, became titled through political activities. He participated in the organization of the École Normale Supérieure and École Polytechnique. His researches were carried on chiefly in the fields of astronomy, celestial mechanics, probabilities [261], calculus, differential equations, and geodesy. In 1796 he published a non-mathematical popular treatise on astronomy entitled, *Exposition of the system of the world.* It was followed by his monumental mathematical *Celestial mechanics*, embracing all the discoveries in this field, of NEWTON, EULER, LAGRANGE, and others who preceded him, as well as his own. This work stamped him at the time as an unrivalled master in the field. (Four volumes were translated into English with elaborate commentary by NATHANIEL BOWDITCH, the New England author of the *New American Practical Navigator* [262], which has passed through more than 65 editions since the first in 1799.) In 1812–1814 LAPLACE published both popular and mathematical treatises on probability. He was characteristically neglectful of crediting predecessors with their discoveries. It may well be, then, that LAPLACE knew that DEMOIVRE had treated the probability integral, as well as what is essentially the normal curve, long before him [231].

The notion of a potential function introduced into analysis by Lagrange was much used by LAPLACE who showed that if V is such a function it satisfies the partial differential equation:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

called LAPLACE'S equation.

NEWTON died in 1727, and LAPLACE in 1827. How vast the mathematical advances in that century!

ADRIEN MARIE LEGENDRE [263] is chiefly known in higher mathematics for important works on the theory of numbers (1798, third ed., 2 v., 1830), on elliptic functions and integrals [264] including great tables, the result of 30 years of meditation (3 v., 1825–1832), and on exercises in the integral calculus (3 v., 1811–1817). Most of these works deal with topics in higher mathematics the discussion of which is beyond the scope of this sketch. LEGENDRE (1752–1833) was the first one to publish the method of least squares (1805) [265]. LEGENDRE wrote also a work on the elements of geometry and trigonometry, first published in 1794, which has gone through many editions [266]. This work

is based on EUCLID's Elements, but rearrangements and simplifications are introduced without lessening the rigor of the ancient methods of treatment. In successive editions LEGENDRE tried to prove the fifth postulate of EUCLID, and every student of the foundations of geometry takes account of results thus found [266]. The work contains also a proof that both π and π^2 are irrational, a proof of the first result having been already given by LAMBERT [252].

A geometry which is built up without assuming EUCLID's fifth postulate as true is said to be noneuclidean. Such a geometry was worked out almost simultaneously by a Russian and a Hungarian. The Russian, **NICOLAI LOBACHEVSKY** [267], studied at the University of Kazan and became a professor of mathematics there in 1814 at the age of twenty-one. In 1826 he made known, through his lectures, his conception of such a geometry which should not depend upon the Euclidean postulate of parallel lines. These ideas were published in 1829 and later. The Hungarian, **JÁNOS BOLYAI** [268], whose results were worked out in 1825–26, published them in 1832. The results thus presented were, later in the century, developed into a body of theory of high importance in mathematics [269].

The fundamental ideas of noneuclidean geometry were also at this time familiar to **CARL FRIEDRICH GAUSS** [270], who spent most of his life at Brunswick and Göttingen, Germany. As a child of three he detected an error in his father's calculation of the wages of an employee; he became the greatest mathematician that Germany, and one of the greatest mathematicians that the world, has ever known. He was born in 1777 and was led to take up the study of mathematics by his discovery in 1796 that it was possible to construct a regular polygon of 17 sides by means of a ruler and compasses. He was later able to show that when the FERMAT number $F_n = 2^{2^n}+1$ is prime, it represents the number of sides of a regular polygon constructible with ruler and compasses [271]. He did not prove, however, what is a fact, namely that these are the *only* polygons, the number of whose sides is prime, which are so constructible. Only four such polygons are known at present. This discussion occupied a very small part of GAUSS's greatest single work, *Disquisitiones Arithmeticae*, which deals with matters of fundamental importance in the theory of numbers. His memoirs of 1825–27 on general investigations of curved surfaces, fundamental in differential geometry, were translated into English, edited with a bibliography showing resulting developments, and published by Princeton University (1902).

GAUSS worked and created in many other fields, analytic functions of a complex variable, astronomy [272], geodesy, electromagnetism, least squares, and almost every leading branch of mathematics. He gave the first wholly satisfactory proof that every algebraic equation of degree m has exactly m roots [270] (which GAUSS calls the Fundamental Theorem of algebra), in his doctoral dissertation written when he was twenty years of age. He here shows defects in the earlier consideration of this question by **D'ALEMBERT** [273] (1746), EULER (1749), FONCENEX (1759), and LAGRANGE (1772). GAUSS gave three other proofs of this theorem in 1816 and 1850. He died in 1855 aged 78.

GAUSS was a notable instance of a youthful arithmetical prodigy who later achieved greatness in mathematics. The most extraordinary mental calculating prodigy who ever lived was **ZACHARIAS DASE** [274] of Hamburg who died in 1861, aged 37. He multiplied two 8-figure numbers in 54 seconds, two 20-figure numbers in 6 minutes, two 40-figure numbers in 40 minutes and two 100-figure numbers in $8\frac{3}{4}$ hours, and found the square root of a 100-figure number in 52 minutes. GAUSS tried to turn his abilities to some useful purpose, and as a result we have DASE's seven-place tables of the natural logarithms of numbers from 1 to 10,500 (1850) and his factor tables of all numbers in the seventh, eighth, and ninth (with assistance) millions (1862–1865).

In considering the mathematics of the seventeenth century we referred to the notable development of synthetic geometry by PASCAL and DESARGUES. There was practically no extension of their work until the early part of the nineteenth century when enormous strides were made by the Frenchman **VICTOR PONCELET** [275] and the Swiss **JAKOB STEINER** [276], who spent most of his life in Berlin. It was PONCELET who first indicated that every construction with ruler and compasses is also possible: (1) with ruler alone, if a single circle and its center are given in the plane; and (2) with an angle of given opening, α, alone [277]. Very early in the nineteenth century the principle of duality was formulated, and **BRIANCHON** [278], a student in the École Polytechnique, enunciated the dual of PASCAL's theorem, namely, The three diagonals of a hexagon circumscribing a conic are concurrent.

We have referred to the proof of the impossibility of the problem of squaring the circle. The other two famous problems of the ancients lead to cubic equations. In 1837 these problems also were shown to be impossible of solution with ruler and compasses, when it was proved [279] that the corresponding cubic equations were irreducible in Euclidean domain.

The transformation of figures in the plane by circular inversion is of importance in both pure and applied mathematics. This seems to have been first conceived [279a] by STEINER in 1824, and by the Belgian astronomer and statistician ADOLPHE QUETELET in the following year. German, Italian and Irish expounders of the method soon followed.

An outstanding geometrical problem in the latter part of the nineteenth century was to discover a linkage for drawing a straight line. A beautiful arrangement of seven jointed rods was finally devised in 1864 by a Frenchman named **PEAUCELLIER** [280]. The construction of this was based upon the principle that the inverse of a circle through the center of inversion is a straight line. The same principle was used later in a new straight-line linkage containing only five links. Many linkages were found [281] for constructing special curves, such as conics, cardioids, lemniscates, and cissoids. It has been proved that a linkage is possible for tracing every algebraic curve, but that it is not possible to construct a linkage for tracing any transcendental curve. Linkages have also been used for solving algebraic equations.

We have seen that the solutions of the general equations of the third and

fourth degrees were found in the sixteenth century by TARTAGLIA and FERRARI. Nearly three hundred years were to elapse before it was shown that a similar solution by radicals of the general equation of the fifth degree was not possible. This was proved in 1799 by an Italian physician **PAOLO RUFFINI** [282], and in a pamphlet published in 1824 by the Norwegian genius **NIELS HENRIK ABEL** [264]. We have already referred [264] to ABEL'S priority in connection with elliptic functions.

Of methods for approximating to the real roots of numerical equations we noted that of NEWTON by successive substitutions in derived equations. This dates from 1669 although it was first published in WALLIS'S *Algebra* in 1685. It is the method which to this day has held undisputed sway in France. The so-called "HORNER'S method" in a paper read by **WILLIAM G. HORNER** [283] before the Royal Society of London in 1819, and published in the same year, was really given fifteen years earlier by the Italian PAOLO RUFFINI. It soon became widely used in England, and later in the United States, and to a less degree in Germany, Austria, and Italy. A valuable work on numerical equations by FOURIER was published in 1831, after his death [246]; but for nearly twenty-five years before this he had taught his students at the École Polytechnique the theorem now associated with his name, concerning the limit to the number of real roots in any given interval. In 1829 **CHARLES STURM** [284] stated a theorem for determining the exact number of roots in any interval; this was merely a by-product of his extensive investigations of linear difference equations of the second order, although it was suggested by a close study of the manuscript of FOURIER'S work of 1831. Considerable work has been done also in deriving the roots of numerical equations by means of infinite series [246].

The problem of determining the imaginary roots of numerical equations has been considered by many writers. LAGRANGE'S work contained a valid method. STURM'S theorem gave a method for determining the number of such roots, but not their location. This was revealed by a general theorem [285] of a great French mathematician **AUGUSTIN LOUIS CAUCHY** [286] (1789–1857), giving the number of roots, real or complex, which lie within a given contour (1831).

A few comments in connection with the convergency of infinite series, so necessary for consideration by every mathematician, are now appropriate. We noted that this question was discussed by NEWTON in a particular case. But the first important and strictly rigorous investigation of infinite series (compare LAMBERT [252]) was made by GAUSS in 1812 in connection with hypergeometric series. In 1821, however, CAUCHY gave a discussion and tests of general application, such as are now constantly in use. A paper of ABEL on power series, published in 1826, is also of importance in this connection [264]. The convergence of trigonometric series was also naturally considered about this time. We have seen that problems of TAYLOR and EULER in connection with a vibrating string called for trigonometric series. So also in other connections such series arose in works of BERNOULLIS, LEGENDRE, and LAPLACE. But FOURIER [246] was the first to assert and to attempt to prove (1812–1822) that any function, even

though for different values of the argument it is expressed by different analytical formulae, can be developed in such a series. Since the real importance of trigonometric series was thus shown by FOURIER for the first time, it was natural that his name should thereafter be associated with them [287].

In the nineteenth and early twentieth centuries considerable work was done in connection with two topics, of elementary geometry, namely: Geometrography, and, especially, the Geometry of the Triangle. Geometrography is a system for measuring "the coefficient of simplicity" and the "coefficient of exactitude" of constructions with ruler and compasses. It was initiated in 1888 and kept alive for a score of years by publications of **ÉMILE LEMOINE**, and many others [288].

We have already referred to EULER'S Line, MALFATTI'S Problem, Nine-Point Circle, WALLACE'S Line—results in the Geometry of the Triangle, which was so extensively developed, especially after the impetus given by publications of LEMOINE and BROCARD 1873–1875. Thus a large new vocabulary was evolved [289].

Vector analysis as employed today originated mainly with two mathematical physicists, **J. W. GIBBS** [290], an American, 1881–1884, and **OLIVER HEAVISIDE** [291], an Englishman, 1891. Quaternions, an algebra of vector magnitudes, was developed earlier, about 1853, by that Irish genius, **W. R. HAMILTON** [292], who also contributed notably in the fields of dynamics and optics.

We have already noted that the subject of determinants originated with LEIBNIZ. LAPLACE, LAGRANGE, GAUSS, CAUCHY, JACOBI, and many others contributed to the development of the subject before 1841, as indicated in the first volume of the late Sir THOMAS MUIR'S monumental work [219]. Four other volumes of a similar size were necessary for setting forth the history of the subject during the next eighty years.

For several centuries Mathematical Tables have notably contributed to advances in science, and in latest years the development of tabular material and of computing machines has been enormous. During the past decade an international Committee on Mathematical Tables and Other Aids to Computation, of the Division of Mathematical and Physical Sciences of the United States National Research Council, has been especially active, and for six years has published not only a quarterly periodical [293] which, throughout the world, disseminates information dealing with tabular material old and new, as well as with other computational aids, but also a Report on tables in the Theory of Numbers [294]. Moreover in this same decade the United States National Bureau of Standards has been directing a computing group [295] (at one time numbering 350, working in two shifts, 15 hours a day, for five days a week, with 169 machines) which has already published more than a score of valuable volumes of tables.

But from 1871–1948 the British Association for Advancement of Science has had a Mathematical Tables Committee, constantly arranging for the

calculation and publication of new tables, or reprints of old ones, mainly in the field of BESSEL functions. Thus, apart from tables published in annual *Reports* (1877–1929), 10 separate volumes (one in two editions) have been published (1900, 1931–1946). The Committee's first publication (1873) was a valuable annotated catalogue by **J. W. L. GLAISHER** of existing mathematical tables [**296**]. Its first published volume of tables, by CUNNINGHAM [**302a**], was followed by three volumes of factor tables by J. GLAISHER [**302**]. Since June 1948 the B.A. Committee has been under the direction of the Royal Society.

Under other auspices the British also published (i), Cambridge, 1926, a detailed report on various published tables of *Logarithms of Numbers* [**297**], and (ii), London, 1946, an exceedingly valuable *Index of Mathematical Tables* [**298**] listing not only those great ones incidentally already mentioned in our survey, such as: tables of RHETICUS and PITISCUS, BRIGGS, DASE, and LEGENDRE,—some of them not yet replaced—but also a host of others published and unpublished.

The BRIGGS' (1624) tables of logarithms of numbers 1(1)20000, 90000(1)101000 to 14 places of decimal, and the 10 decimal tables of VLACQ have been replaced (1924–1949) by the magnificent 20D tables of A. J. THOMPSON [**174**]. The RHETICUS (1596) 10D tables of natural trigonometric functions, for interval 10″, were elaborated (1911–1916) in ANDOYER'S tables [**299**], 14–17D, and the superb factor tables and table of primes of D. N. LEHMER [**300**] (1909, 1914) render unnecessary the earlier works of BURCKHARDT [**301**] (1814–1817), J. GLAISHER [**302**] (1879–1883), DASE, (1862–1865), and others. The BRIGGS' 15D table of natural sines at interval 0°.01 (1633) was recalculated and published with central second differences (1949) by the National Bureau of Standards, but other tables of BRIGGS (1633) have not been replaced.

The mathematical tabular field is exceedingly extensive and is being constantly developed. Historical summaries in special fields serve a very useful purpose.

In conclusion we may indicate a few samples of references to historical accounts of developments of higher mathematics. E. W. BROWN [**303**] discussed the history of mathematics to the end of the second decade of the twentieth century. The lectures on the development of mathematics in the nineteenth century by KLEIN [**304**] are masterly. COURANT has discussed RIEMANN and mathematics of the last hundred years [**305**]. The development of applied mathematics in the nineteenth century was surveyed by R. S. WOODWARD [**306**]. The general sketches of BELL [**5**] and STRUIK [**6**] will also be found of interest. American activity 1888–1938 is indicated in the excellent *Semicentennial Addresses of the American Mathematical Society* [**307**] and in the biographies and bibliographies of all its presidents and secretaries published in its *History* [**308**]. Most of SMITH & GINSBURG'S history [**309**] is devoted to the nineteenth century mathematics in America, and a certain amount of reference to higher mathematics may be found in CAJORI'S *Teaching and History of Mathematics in*

the United States [**310**]. Then there is also the comparatively recent *A History of Science in Canada* [**311**]. Biographies of other nineteenth century mathematicians are to be found in works of BELL [**53**] and PRASAD [**254**] to which we have already made reference. Various histories of geometrical developments have been set forth by COOLIDGE [**245**], KÖTTER [**275**] and LORIA [**312**]. The 30 volumes of the American Mathematical Society's *Colloquium Lectures* and *Colloquium Publications* (1896–1948) are valued treatments of special modern topics of importance. Along with these may be noted the highly important 43 titles (1908–1930) of the *Teubners Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen*, and 54 volumes (1921–1947) of *Grundlehren der mathematischen Wissenschaften*, the briefer 25 *Ergebnisse der Mathematik und ihrer Grenzgebiete* (1932–1942), 105 *Mémorials des Sciences Mathématiques* (1925–1947), and the 39 *Cambridge Tracts in Mathematics and Mathematical Physics* (1905–1946).

We may conclude these lectures with recalling a passage more true today than when spoken by ISAAC BARROW in his oration of more than two and three-quarter centuries ago: *The Mathematics . . . is the fruitful Parent of . . . Arts, the unshaken Foundation of Sciences, and the Plentiful Fountain of Advantage to Human Affairs.*

LITERATURE LIST AND NOTES

Some of the more extensive general and original histories of mathematics in English are the following (LLN 1–5):

1. W. W. R. BALL, *A Short History of Mathematics*, fifth ed. London, 1912, xxiv, 522 pp. Although this work is much out of date and filled with erroneous statements, it is still the most readable of old-style histories in English.

2. F. CAJORI, *A History of Mathematics*, second ed., rev. and enl. New York, 1919. x, 514 pp. Includes the nineteenth and twentieth centuries.

3. F. CAJORI, *A History of Elementary Mathematics*, second ed., rev. and enl., New York, 1917 or 1921, viii, 324 pp. R. C. ARCHIBALD, "Florian Cajori (1859–1930)," *Isis*, vol. 17, 1932, pp. 384–407+portrait plate.

4. D. E. SMITH, *History of Mathematics*, corrected and rev. ed. Boston, v. 1, 1928, xxii, 596 pp.; v. 2, 1930, xxii, 725 pp.; there are changes in more than 100 places in v. 1 of the first ed. Many illustrations and portraits. The second vol. contains the topical history, and also many illustrations. W. D. REEVE, "David Eugene Smith," *Scripta Math.*, v. 11, 1945, pp. 209–212+2 portraits. Also anonymous biographical notes, pp. 364–369+4 portraits. Also "Bibliography of the critical, historical, and pedagogical writings of D. E. S.," *Osiris*, v. 1, 1916, pp. 9–78 and "The D. E. S. mathematical library of Columbia University" by BERTHA M. FRICK, pp. 79–84. Also L. G. SIMONS, "D. E. S.—In memoriam," Amer. Math. Soc., *Bull.*, v. 51, 1945, pp. 40–50.

5. E. T. BELL, *The Development of Mathematics*, second ed. enl. New York, 1945, xiv, 637 pp. A most readable and original modern exposition.

6. D. J. STRUIK, *A Concise History of Mathematics*. Volume I: *The Beginnings—The Beginnings in Western Europe*. Volume II: *The Seventeenth Century—The Nineteenth Century*. Paged continuously, New York, 1948, xxvii, 300 pp. More than 40 pages are devoted to portraits and other illustrations. Quotation: "We hope that . . . we have been able to give a fairly honest description of the main trends in the development of mathematics throughout the ages and of the social and cultural setting in which it took place."

7. O. E. NEUGEBAUER, "History of Mathematics," *Encyclopedia* (?), New York, P. Collier & Sons Corp., 1949?; 6000-word article.

7a. O. E. NEUGEBAUER, "Mathematical methods in ancient astronomy," Amer. Math. Soc., *Bull.*, v. 54, Nov. 1948, pp. 1013–1041.

7b. O. ORE, *Number Theory and its History*. New York, 1948, x, 370 pp. Attractively written v. in 15 chapters based on a course given for several years primarily for juniors and seniors at Yale University.

Among general histories of science, with most important mathematical sections, is the monumental and outstanding

8. G. SARTON, *Introduction to the History of Science. Volume I, from Homer to Omar Khayyam*, xii, 839 p.; *volume II, from Rabbi ben Ezra to Roger Bacon*, xxxviii, 1251 pp.; *volume III, Science and Learning in the Fourteenth Century*, xxxv, 1018, xiv, 1019–2155 pp. Washington, D. C., 1927, 1931, 1947–1948.

9. G. SARTON, *The Study of the History of Mathematics*. Cambridge, Mass., 1936. iv, 112 pp. Essay on the study of the history of mathematics, pp. 3–28; Note on the study of the history of modern mathematics, pp. 29–38; Bibliography, pp. 39–98.

10. J. W. N. SULLIVAN, *The History of Mathematics from the Fall of Greek Science to the Rise of the Conception of Mathematical Rigour*. London, 1925, 109 p. A very pleasantly written volume covering about 1000 years.

11. J. TROPFKE, *Geschichte der Elementar-Mathematik in systematischer Darstellung*. Berlin, 7 v., v. 1, third ed. 1930, viii, 222 pp.; v. 2, third ed., 1933, iv, 266 pp.; v. 3, third ed., 1937, iv, 239 pp.; v. 4, third ed., 1940, iv, 316 pp., edited after Tropfke's death by K. VOGEL; v. 5, second ed., 1923, iv, 185 pp.; v. 6, second ed., 1924, iv, 169 pp.; v. 7, second ed., 1924, iv, 128 pp.–Index v. A most admirable up-to-date history, rich in bibliographic references. J. E. HOFMANN, "Johannes Tropfke," *Deutsche Math.*, v. 6, 1941, pp. 114–118+1 plate (with list of publs.).

12. M. Cantor, *Vorlesungen über Geschichte der Mathematik.* Leipzig, v. 1, third ed., vi, 942 pp.; v. 2, second ed., 1900, xii, 943 pp.; v. 3, second ed., 1901, x, 923 pp.; v. 4, 1908, vi, 1113 pp. in cooperation with others. The great history of mathematics (to the close of the eighteenth century), which, even with the important corrections in the historical periodical, *Bibliotheca Mathematica*, while offering many things of great value, leaves much to be desired in accuracy, and is now largely out of date. F. Cajori, "Moritz Cantor, the historian of mathematics," Amer. Math. Soc., *Bull.*, v. 27, 1920, pp. 21–28. D. E. Smith, "Moritz Cantor," *Scripta Math.*, v. 1, pp. 204–207, 1932.

13. Students of Babylonian mathematics will find the following popular work of special interest: (a) *They Wrote on Clay. The Babylonian Tablets Speak Today.* By E. Chiera, ed. by G. G. Cameron. Chicago, 1939, xv, 234 pp. Some other references are: (b) R. C. Archibald, (i) "Babylonian Mathematics with special reference to recent discoveries," *Mathematics Teacher*, v. 29, 1936, pp. 209–219; (ii) "Babylonian Mathematics," *Isis*, v. 26, 1936, pp. 63–81. (c) F. Thureau-Dangin, *Esquisse d'une Histoire du Système Sexagesimal.* Paris, 1932, 81 pp. (d) English ed. of (c) by S. Gandz, rev. and enl. by the author, "Sketch of a history of the sexagesimal system," *Osiris*, v. 7, 1939, pp. 95–141. (e) V. G. Childe, "The oriental background of European science," *Modern Quarterly*, v. 1, 1938, pp. 105–120.

14. O. Neugebauer, *Mathematische Keilschrift-Texte*, 3 v. *Quellen u. Studien zur Geschichte d. Math.*, v. 3A, 1935–1937, xii, 516 pp.; iv, 64 pp.+69 plates; viii, 84 pp.+3 plates. This, and the works listed in [**15–17**], are fundamental among those dealing with Babylonian mathematical tablets. For a biographical sketch and portrait of Neugebauer, see *National Mathematics Mag.*, v. 11, pp. 14–16, 1936.

15. F. Thureau-Dangin, *Textes Mathématiques Babyloniens transcrits et traduits.* Leyden, 1938, xi, 243 pp.

16. O. Neugebauer & A. Sachs, *Mathematical Cuneiform Texts* . . . with a Chapter by A. Goetze (*Amer. Oriental Series*, v. 29). New Haven, Conn., 1945. x, 177 pp.+49 pages of plates. Only a very few of the 191 mathematical cuneiform tablets discussed in this volume are not in the United States. Among these tablets and those listed in the work of [**14**], 103 tablets at Yale University are elucidated. This is the most valuable lot of problem-texts in any existing collection. At the University of Pennsylvania are about 300 tablets (of which 100 have been published) but they, as well as 200 mathematical texts in the Istanbul Museum, are nearly all table-texts.

17. O. Neugebauer, *Vorlesungen über Geschichte der antiken mathematischen Wissenschaften*, v. 1, *Vorgriechische Mathematik.* Berlin, 1934, 224 p. Egyptian Mathematics is discussed only on pp. 110–165.

18. *Illustrated London News*, v. 173, 14 July 1928; and v. 175, 31 August 1929.

19. V. G. Childe, *What Happened in History.* (*Pelican Books*), New York, 1946, viii, 280 pp. An excellent history of civilization with numerous references to mathematics and astronomy. Bronze and Iron Ages are periods of early mathematics.

20. O. Neugebauer, "Das Pyramidenstumpf-Volumen in der vorgriechischen Mathematik," *Quellen u. Studien zur Geschichte d. Math.*, v. 2B, pp. 347–351, 1933; K. Vogel, "Eine Pyramidenstumpf-Aufgabe bei den Babyloniern," *Arch. Orientforschung*, v. 8, 1933, pp. 220–221. Compare [**37**].

21. K. Vogel, "Zur Berechnung der quadratischen Gleichungen," *Unterrichtsblätter f. Math. u. Naturw.*, v. 39, 1933, pp. 76–81. S. Gandz, "The origin and development of the quadratic equations in Babylonian, Greek, and early Arabic algebra," *Osiris*, v. 3, 1937, pp. 405–557. See also [**13**] and Tropfke [**11**], v. 3, pp. 50–56.

22. T. L. Heath, *A Manual of Greek Mathematics*, Oxford, 1931, xvi, 552 pp. "Ancient Babylonian mathematics," pp. 522–530. This work is an abridgment, brought up to date, of Heath's valuable, but somewhat out-dated, *History of Greek Mathematics*, 2 v., Oxford, 1921, xxviii, 1032 pp.

23. H. S. Schuster, "Quadratische Gleichungen der Seleukidenzeit aus Uruk," *Quellen u. Studien* . . . , v. 1B, pp. 194–200. O. Neugebauer, *Quellen u. Studien* . . . , v. 3B, pp. 247–248, 1936. K. Vogel, "Bemerkungen zu den quadratischen Gleichungen der babylonischen Mathematik," *Osiris*, v. 1, 1936, pp. 703–717. Tropfke [**11**], v. 3, pp. 54–56.

24. Neugebauer & Sachs [**16**], p. 130.

25. O. Neugebauer, *Quellen u. Studien . . .*, v. 1B, 1930, p. 124 and F. Thureau-Dangin, "Le prisme mathématique AO 8862," *Rev. d'Assyr.*, v. 29, 1932, pp. 1–10, 89–90. See also Thureau-Dangin [34], pp. 307–311.

26. See Neugebauer [17], pp. 193–197; Neugebauer [14], v. 3A, part 1, pp. 210–211, 215–216. O. Neugebauer, "Über die Lösung kubischer Gleichungen in Babylonien," Gesell. d. Wissen., Göttingen, *Math.-phys. Kl.*, *Nachr.*, 1933, pp. 316–321. K. Vogel, "Kubische Gleichungen bei den Babyloniern?" Bayer. Akad. d. Wissen., *Math.-naturw. Kl.*, *Sitzb.*, 1934, pp. 87–94.

27. *Quellen u. Studien . . .*, v. 2B, p. 304, 1932.

28. O. Neugebauer, *Quellen u. Studien . . .*, v. 4B, 1937, pp. 82–85, also paragraph 61.

29. A. J. Sachs, "Reciprocals of regular sexagesimal numbers," *Jn. of Cuneiform Studies*, v. 1, 1947, pp. 219–240.

30. O. Neugebauer, "The history of ancient astronomy; problems and methods," *Jn. of Near Eastern Studies*, v. 4, 1945, pp. 1–38; reprinted in enlarged form in Astron. Soc. of the Pacific, *Publs.*, v. 58, 1946, pp. 17–43, 104–142. This up-to-date monograph is of extraordinary interest and value for both the astronomer and the mathematician, not only on account of the clarity and breadth of its sweep, but also on account of the very extended list of references (170+) to the literature of the subject. Since this list is readily accessible, we shall not make specific reference to a number of items of the list as we might otherwise have done. A few supplementary titles are given in [31].

31. J. K. Fotheringham, "The indebtedness of Greek to Chaldean astronomy," *The Observatory*, v. 51, 1928, pp. 301–315; reprinted with comments by O. Neugebauer, *Quellen u. Studien . . .*, v. 2B, 1932, pp. 28–44. B. L. van der Waerden, "Egyptian 'Eternal Tables,' " Akad. v. Wetens., *Proceedings*, v. 50, 1947, pp. 536–547 and pp. 782–788, with special reference to Neugebauer's "Planetary Texts." Also, E. J. Knudtzon & O. Neugebauer, "Zwei astronomische Texte," Soc. R. d. Lettres de Lund, *Bull.*, 1946–1947, II, 1947, pp. 77–88+2 plates.

32. Neugebauer [14], v. 3A, part 1, pp. 96–103, and part 2, plate 1; also H. Waschow & O. Neugebauer, "Reihen in der babylonischen Mathematik," *Quellen u. Studien . . .*, v. 2B, pp. 298–303, 1932.

33. See Neugebauer & Sachs [16], p. 43; this very exact value for $\sqrt{2}$ is also found in Ptolemy's Almagest (Heiberg ed., v. 1, chap. 10, p. 35, l. 16). See also [13] (b) (i). O. Neugebauer, "Über die Approximation irrationaler Quadratwurzeln in der babylonischen Mathematik," *Arch. Orientforschung*, v. 7, 1931, pp. 90–99.

34. For a good example of such proof see Sachs [29], pp. 237–240. What underlies the formation of the Yale series texts and such solutions as are referred to in [23], are notably suggestive in this connection. Reference may also be given to an exceedingly interesting paper on algebra before 1000 A.D. F. Thureau-Dangin, "L'origine de l'algèbre," Acad. d. Inscriptions & Belles-Lettres, Paris, *Comptes Rendus*, *Bull.*, Jan.–June, 1940, pp. 292–318.

35. R. C. Archibald, "Mathematics before the Greeks," *Science*, n.s., v. 71, 1930, pp. 109–121, 342, and v. 72, 1930, p. 39. Both Babylonian and Egyptian mathematics are here surveyed. See also Bell [5], pp. 26–48; and O. Neugebauer, "Über vorgriechische Mathematik," Hamburg Univ., Math. Sem., *Abh.*, v. 7, pp. 107–124.

36. J. E. Quibell, Hierakonopolis. London, 1900, plate 26B.

37. N. F. Wheeler, "Pyramids and their purpose," *Antiquity*, Gloucester, Engl., v. 9, 1935, pp. 5–21, 161–189 (pyramid at Gizeh), 292–304 (pyramid mysticism and mystification.) Alleged mathematical relations in the pyramids of Egypt are totally unfounded. Herodotus (−450), *History*, English translation A.D. Godley, (Loeb Classical Library), London, v. 2, 1921 pp. 424–427. Wheeler ridicules Herodotus along with the "pyramidiots." J. H. Cole, *Determination of the Exact Size and Orientation of the Great Pyramid of Giza*. Survey of Egypt, *Paper* no. 29, Cairo, 1925, 9 pp.+1 plan. L. Borchardt, *Längen und Richtungen der vier Grundkanten der grossen Pyramide bei Gise*. Berlin, 1920, ii, 21 pp.+5 plates. L. Borchardt, *Einiges zur dritten Bauperiode der grossen Pyramide bei Gise . . . mit einer Bemerkung zur zweiten Bauperiode der dritten Pyramide* by Herbert Ricke. Berlin, 1932, 22 pp.+12 plates. An excellent recent work is Leslie Grinsell, *Egyptian Pyramids*. Gloucester, England, 1947, 195 pp.+1 plate.

38. R. ENGELBACH, *The Problem of the Obelisks from a Study of the Unfinished Obelisk at Aswan*, New York, 1923, 134 pp. S. CLARKE & R. ENGELBACH, *Ancient Egyptian Masonry*. London, 1930, xvi, 242 pp.

39. O. NEUGEBAUER, "Studies in ancient astronomy, VIII. The water clock in Babylonian astronomy," *Isis*, v. 37, 1947, pp. 37–43. R. W. SLOLEY, "Primitive methods of measuring time, with special reference to Egypt," *Jn. Egypt. Archeology*, v. 17, 1931, pp. 166–178+7 plates. A. POGO, "Egyptian water clocks," *Isis*, v. 25, 1936, pp. 403–425. L. BORCHARDT, *Die altägyptische Zeitmessung*, Berlin-Leipzig, 1920, x, 70 pp.+18 plates. F. THUREAU-DANGIN, "Clepsydre Babylonienne et Clepsydre Egyptienne," *Revue d'Assyriologie*, v. 30, 1933, pp. 51–52. See also ALBERT REHM, (a) "Horologium-Wasseruhren," PAULY's *Real-Encyclopädie d. class. Altertumswissen.* new ed. G. WISSOWA, v. 8, Stuttgart, 1913, cols. 2428–2433; (b) "Neue Beiträge zur Kenntnis der antiken Wasseruhren," Bayer. Akad. d. Wissen., *Phil. hist. Kl., 1920*, no. 17, 1921, 27 pp. with 5 illustrs. Also THALHEIM, article "κλεψύδρα," Pauly's *Real Encycl.* v. 11_1, 1921, cols. 807–809. The word clepsydra is derived from two Greek words meaning to steal water. The clepsydra was introduced into Greece in the time of Plato and later into Italy; see M. C. P. SCHMIDT, *Kulturhistorische Beiträge zur Kenntnis des Griech. u. Röm. Altertums. II: Die Entstehung der antiken Wasseruhr*. Leipzig, 1912, vi, 113 pp., 13 plates. A water clock of bronze inlaid with gold was presented by the King of Persia to CHARLEMAGNE in 807. Clepsydrae are operated even today by the Chinese, in Canton, for example; see D. J. MAGOWAN, "Modes of keeping time known among the Chinese," (a) *Chinese Repository*, Canton, v. 20, 1851, pp. 426–432; reprinted in (b) *Smithsonian Report 1891*, pp. 607–612. Also S. COULING, *The Encyclopaedia Sinica*, Shanghai, 1917. See also *Fir-Flower Tablets, Poems from the Chinese* by FLORENCE AYSCOUGH, *English Versions* by AMY LOWELL, Boston, 1921, pp. 11, 180–181. Perhaps the best historical article in English on the clepsydra is in F. J. BRITTEN, *Old Clocks and Watches & their Makers*, 6th ed., London, 1932, pp. 9–14, 7 illustrs. See also **[314]**.

40. C. BOREUX, *Études de Nautique Égyptienne. L'Art de la Navigation en Égypte jusqu'à la Fin de l'Ancien Empire*. Cairo, 1925, vii, 569 pp.+plates.

41. A. NEUBURGER, *The Technical Arts and Sciences of the Ancients*, transl. by H. L. BROSE. New York, 1930, xxii, 518 pp. H. P. & M. W. VOWLES, *The Quest for Power from Prehistoric Times to the Present Day*. London, 1931, xv, 354 pp., especially pp. 1–100. J. H. BREASTED, "The earliest civilization and its transition to Europe," *Scientific Mo.*, v. 10, 1920, pp. 87–105, 183–209, 249–255; fully illustrated.

42. *Mathematischer Papyrus des Staatlichen Museums der Schönen Künste in Moskau.* Edited with Commentary by W. W. STRUVE, with the use of a hieroglyphic transcription by B. A. TURAEV. *Quellen u. Studien . . .* , Berlin, v. 1A, 1930, xii, 198 pp.+10 folding plates.

43. *The Rhind Mathematical Papyrus, British Museum 10057 and 10058. Introduction, Transcription, Translation and Commentary* by T. E. PEET. London, 1923, iv, 136 pp.+23 plates.

44. *The Rhind Mathematical Papyrus, British Museum 10057 and 10058. Volume I: Free Translation and Commentary* by A. B. CHACE with the assistance of H. P. MANNING. *Bibliography of Egyptian Mathematics* by R. C. ARCHIBALD. *Volume II: Photographs, Transcription, Transliteration, Literal Translation* by A. B. CHACE, L. BULL & H. P. MANNING. *Bibliography of Egyptian and Babylonian Mathematics* (*Supplement*) by R. C. ARCHIBALD. *The Mathematical Leather Roll in the British Museum* by S. R. K. GLANVILLE. Oberlin, Ohio, 1927, x, 210 pp.; 1929, xvi, 126 plates, 20 pp. Almost every entry of the Bibliography has detailed annotations. With regard to the Leather Roll, see: S. R. K. GLANVILLE, "The math. leather roll in the British Museum," *Jn. Egypt. Arch.*, v. 13, 1927, pp. 232–238+4 plates. A. SCOTT & H. R. HALL, "Laboratory notes: Egyptian leather roll of the seventeenth century B.C.," *British Museum Quarterly*, v. 2, 1927, pp. 56–57+1 plate. O. NEUGEBAUER, "Zur ägyptischen Bruchrechnung," *Z. f. ägypt. Sprache*, v. 64, 1929, pp. 44–48; K. VOGEL, "Erweitert die Lederrolle unsere Kenntnis ägyptischer Mathematik?" *Archiv f. Gesch. d. Math. . . .* , v. 11, 1929, pp. 386–407.

45. B. GUNN & T. E. PEET, "Four geometrical problems from the Moscow mathematical papyrus," *Jn. Egypt. Arch.*, v. 15, 1929, pp. 167–185+1 plate. K. VOGEL, "The truncated pyramid in Egyptian mathematics," *Jn. Egypt. Arch.*, v. 16, 1930, pp. 242–249. T. E. PEET, "A problem in Egyptian geometry," *Jn. Egypt. Arch.*, v. 17, 1931, pp. 100–106+1 plate, and 154–160. O. NEUGE-

BAUER, "Die Geometrie der ägyptischen mathematischen Texte," *Quellen u. Studien . . .*, v. 1B, pp. 413–451. See also NEUGEBAUER [17], pp. 122–137. Compare [20].

46. BELL [5], "Firmly established Greece 600 B.C.–A.D. 300," pp. 49–84; SARTON [8], v. 1, pp. 65–413; HEATH [22].

46a. A work of reference of first importance and interest for the student is *Selections Illustrating the History of Greek Mathematics with an English Translation* by IVOR THOMAS, *in two volumes*, I: *From Thales to Euclid;* II: *From Aristarchus to Pappus.* (*Loeb Classical Library*), London, and Cambridge, Mass., 1939, xvi, 506 pp.; 1941, xii, 683 pp. Translations are given facing the Greek texts. There is an excellent subject index for the two volumes.

46b. The fundamental Bibliography of ancient mathematics during 1873–1930 has been published in 8 v. of *Jahresb. ü. d. Forts. d. kl. Altertumswissen.* The last vol. "Antike Mathematik," for the period 1906–1930, by CLEMENS TAER, filled v. 283, 1932, 144 pp. There are 498 entries, each with descriptive and critical notes, and the contents. The entries deal with mathematics before 600 A.D. The other 7 surveys are as follows: v. 1, 1873, "Jahresb. ü. d. exacten Wissen. bei den Griechen u. Römern," by B. LANGKAVEL, pp. 680–720; mathematics and astronomy, pp. 681–687. V. 11, 1879, "Jahresb. ü. Math. Astron. u. Mechanik im Alterthum für 1873–1877," by M. CURTZE, pp. 159–217. V. 23, 1882, by M. CURTZE, 1878–1879, pp. 217–242. V. 40, 1886, by M. CURTZE, 1879–1882, pp. 1–50 h. V. 108, 1902, by W. SCHMIDT, 1890–1901, pp. 59–128. V. 124, 1905, by J. L. HEIBERG, 1902–1905, pp. 129–143 (very brief). V. 129, 1907, by K. TITTEL, 1902–1905, pp. 113–219.

47. T. L. HEATH, "Mathematics and astronomy," in *The Legacy of Greece*, ed. by R. W. LIVINGSTONE, Oxford, 1922, pp. 97–136.

48. T. L. HEATH, "Greek mathematics and science," *Mathematical Gazette*, v. 10, 1921, pp. 289–301.

49. T. L. HEATH, "Greek geometry with special reference to infinitesimals," *Mathematical Gazette*, v. 11, pp. 249–259.

50. J. KLEIN, "Die griechische Logistik und die Entstehung der Algebra," *Quellen u. Studien . . .*, v. 3B, 1934, pp. 18–105; K. VOGEL, "Beiträge zur Griechischen Logistik," Bay. Akad. d. Wissen., *Math.-Naturw. Abt., Sitzungsb.*, 1936, pp. 357–472.

51. H. W. TURNBULL, *The Great Mathematicians*, London, 1929. Contents: THALES, PYTHAGORAS and the Pythagoreans, pp. 1–15; EUDOXUS and the Athenian School, pp. 16–29; Alexandria: EUCLID, ARCHIMEDES and APOLLONIUS, pp. 30–41: PAPPUS and DIOPHANTUS, pp. 42–53; NAPIER and KEPLER, pp. 54–69; DESCARTES and PASCAL, pp. 70–85; NEWTON, pp. 86–94; BERNOULLIS and EULER, pp. 95–100; MACLAURIN and LAGRANGE, pp. 101–107; GAUSS and HAMILTON, pp. 108–118.

52. W. W. R. BALL, "Pythagoras," *Mathematical Gazette*, v. 8, 1915, pp. 5–12. See also [51, 54] and Frank [57].

53. E. T. BELL, *Men of Mathematics*, New York, 1937, xxii, 593 pp. Extraordinarily readable and valuable volume. Contents: ZENO, EUDOXUS, ARCHIMEDES (pp. 19–34); DESCARTES (35–55); FERMAT (56–72); PASCAL (73–89); NEWTON (90–116); LEIBNIZ (117–130); the BERNOULLIS (131–138); EULER (139–152); LAGRANGE (153–171); LAPLACE (172–182); MONGE and FOURIER (183–205); PONCELET (206–217); GAUSS (218–269); CAUCHY (270–293); LOBATCHEVSKY (294–306); ABEL (307–326); JACOBI (327–339); HAMILTON (340–361); GALOIS (362–377); SYLVESTER and CAYLEY (378–405); WEIERSTRASS and SONJA KOWALEWSKI (406–432); BOOLE (433–447); HERMITE (448–465); KRONECKER (466–483); RIEMANN (484–509); KUMMER and DEDEKIND (510–525); POINCARÉ (526–554); CANTOR (555–579).

54. *Portraits of Eminent Mathematicians, with Brief Biographical Sketches*, by D. E. SMITH, I, II, New YORK, 1936–38; I: ARCHIMEDES, COPERNICUS, VIÈTE, GALILEO, NAPIER, DESCARTES, NEWTON, LEIBNIZ, LAGRANGE, GAUSS, LOBACHEVSKY, SYLVESTER; II: EUCLID, CARDAN, KEPLER, FERMAT, PASCAL, EULER, LAPLACE, CAUCHY, JACOBI, HAMILTON, CAYLEY, CHEBYSHEV, POINCARÉ. *Portraits of Famous Philosophers who were also Mathematicians, with Biographical Accounts* by C. J. KEYSER, New York, 1939: PYTHAGORAS, PLATO, ARISTOTLE, EPICURUS, ROGER BACON, DESCARTES, PASCAL, SPINOZA, LEIBNIZ, BERKELEY, KANT, PEIRCE. Of course there are no known portraits of ARCHIMEDES, ARISTOTLE, EUCLID, PLATO. These three portfolios were published by *Scripta Mathematica.*

55. O. NEUGEBAUER & W. W. STRUVE, "Über die Geometrie des Kreises in Babylonien," *Quellen u. Studien* . . . , v. 1B, 1929, pp. 81–92.

56. According to IAMBLICHUS (c. 325). In E. B. ESCOTT, "Amicable numbers," *Scripta Math.*, v. 12, 1946, pp. 61–72, 390 pairs of amicable numbers are listed. See also *Math. Tables and Other Aids to Computation*, v. 1, pp. 95–96; v. 2, p. 211. In *Scripta Mathematica*, v. 14, Mar. 1948, p. 77 are "43 new couples of amicable numbers" by the late PAUL POULET.

57. R. C. ARCHIBALD, "Mathematicians and music," *Amer. Math. Mo.*, v. 31, 1924, pp. 1–25. See also the admirable work, E. FRANK, *Plato und die sogenannten Pythagoreer, ein Kapitel aus der Geschichte des griechischen Geistes*. Halle, 1923, x, 400 pp.

58. T. L. HEATH, (a) *Aristarchus of Samos, the Ancient Copernicus. A History of Greek Astronomy to Aristarchus together with Aristarchus's Treatise on the Sizes and Distances of the Sun and Moon.* Oxford, 1913, viii, 425 pp. In this work Heath's account of Greek astronomy is mainly derived from researches of GOMPERZ. More recent investigation [27] renders unreliable these researches, as well as references to Pythagorean music. In this connection see FRANK [57], with many references to literature. (b) *Greek Astronomy*. London, 1932, viii, 192 pp. Translations of astronomical passages in many Greek authors with an Introduction; (c) *The Copernicus of Antiquity* (*Aristarchus of Samos*) (*Pioneers of Progress*). London, 1920, iv, 59 pp.

59. There is a valuable discussion of this in W. A. HEIDEL, *The Frame of the Ancient Greek Maps with a Discussion of the Discovery of the Sphericity of the Earth.* New York, 1937, x, 141 pp. The first unequivocal evidence of the earth as a sphere is found in PLATO's *Phaedo;* see also ARISTOTLE, *De Caelo*, I, §§5–12 and II, §§1, 4, 5, 6.

60. I. B. HART, *Makers of Science—Mathematics, Physics, Astronomy.* London, 1928, 320 pp., illustr.; ARISTOTLE pp. 19–32. KEYSER [54]. The first historian of mathematics, EUDEMUS, was a pupil of ARISTOTLE. He wrote, among other works, a History of Arithmetic, a History of Astronomy, and a History of Geometry.

61. HEATH [22], *History*, v. 1, pp. 284–315; BELL [5], p. 73; HEIDEL [59]; FRANK [57], the most valuable survey; H. CHERNISS, *The Riddle of the Early Academy.* Berkeley, Cal., 1945, viii, 103 pp.; SARTON [8], v. 1, p. 113 ff; W. A. HEIDEL, "The Pythagoreans and Greek mathematics," *Amer. Jn. Philology*, v. 61, 1940, pp. 1–33 (the writer seemed ignorant of FRANK [57]); J. BURNET, *Platonism*, Berkeley, 1928, "Plato's mathematics," pp. 96–112; F. SOLMSEN, "Plato's Einfluss auf die Bildung der mathematischen Methode," *Quellen u. Studien* . . . , v. 1B, pp. 93–107, 1929; S. DEMEL, "Platons Verhältnis zur Mathematik," *Forschung zur Geschichte der Philosophie und der Pädogogik*, Leipzig, v. 4, 1929, part 1, vi, 146 pp.; KEYSER [54].

62. Should we accept the suggestion of PROCLUS that the PYTHAGOREANS were the first to declare the four Elements (earth, fire, air, water) to be the material principles from which the universe was evolved? Certainly PLATO in his *Timaeus* was the first to associate these Elements with the Regular Solids: the earth arising from the cube, fire from the tetrahedron, air from the octahedron, and water from the icosahedron. The theory of the Elements is not related to the PYTHAGOREAN school but to DEMOCRITUS (–400). Such are important results in the following valuable book, clarifying the rôle of the PYTHAGOREANS in connection with mathematics, a role grossly exaggerated in antiquity: EVA SACHS, *Die Fünf Platonischen Körper zur Geschichte der Mathematik und der Elementenlehre Platons und der Pythagoreer* (*Philologische Untersuchungen*, v. 24). Berlin, 1917, x, 242 pp.

63. BELL [53]; H. D. P. LEE, *Zeno of Elea. A Text with Translations and Notes.* Cambridge Univ. Press, 1936, vi, 125 pp.; F. CAJORI, "The history of Zeno's arguments on motion: phases in the development of the theory of limits," *Amer. Math. Mo.*, v. 22, 1915, pp. 1–6, 39–47, 77–82, 109–115, 143–149, 179–186, 215–220, 253–258, 292–297; B. L. VAN DER WAERDEN, "Zenon und die Grundlagenkrise der griechischen Mathematik," *Math. Annalen*, v. 117, 1940, pp. 141–161.

64. BELL [5]; HEATH [49]; TURNBULL [51]; O. BECKER, "Eudoxus-Studien," *Quellen u. Studien* . . . , "I. Eine voreudoxische Proportionslehre und ihre Spuren bei Aristoteles und Euklid," "II. Warum haben die Griechen die Existenz der vierten Proportionale angenommen?," v. 2B, 1933, pp. 311–333, 369–387; "III. Spuren eines Stetigkeitsaxioms in der Art des Dedekind'schen zur Zeit des Eudoxus," v. 3B, 1936, pp. 236–244; "IV. Das Prinzip des ausgeschlossenen Dritten in der griechischen Mathematik"; "V. Die eudoxische Lehre von den Ideen und den Farben," v. 3B,

1936, pp. 370–410. HANS KÜNSSBERG, *Der Astronom, Mathematiker und Geograph Eudoxus von Knidos. I. Teil: Lebensbeschreibung des Eudoxus, Überblick über seine astronomische Lehre und geometrische Betrachtung der Hippopede* (Progr. Realschule, Dinkelsbühl, 1888, 59 pp.+folding plate). *II. Teil: Mathematik* (Progr. Realschule, Dinkelsbühl, 1890, 61 pp.+folding plate.) Eudoxus also did important work in geography and in explanations of planetary motion; see NEUGEBAUER [30], HEIDEL [59], pp. 95–102; and G. SCHIAPARELLI, *Scritti sulla Storia della Astronomia Antica*, Bologna, v. 2, 1926, pp. 2–112. In the outstanding work on the planetary theory by EUDOXUS he made an attempt to explain the peculiarities of a planetary movement known as retrogradation by the assumption of the superposition of the rotation of two concentric spheres around inclined axes and in opposite directions. In this way he reached a satisfactory explanation of the general type of planetary movement and thereby inaugurated in the history of astronomy a new period which was marked by attempts to explain the movements of the planetary system by mechanical models.

65. W. W. R. BALL, "Three classical geometrical problems," in his *Mathematical Recreations & Essays*, eleventh ed., rev. by H. S. M. COXETER, London, 1939, pp. 326–349. Also THOMAS [46a], see Index, under circle-squaring, cube-duplication, trisection of an angle. See also J. S. MACKAY, "The ancient methods for the duplication of the cube," Edinburgh Math. Soc., *Proc.*, v. 4, 1886, pp. 2–20; and H. DÖRRIE, *Triumph der Mathematik. Hundert berühmte Probleme aus zwei Jahrtausenden mathematischer Kultur*, Breslau, second ed., 1940, viii, 386 pp.

66. F. GOMES TEIXEIRA, *Sur les Problèmes Célèbres de la Géométrie Élémentaire non résolubles avec la Règle et le Compas*, Coimbra, 1915, also in his *Obras sobre Mathematica*, Coimbra, v. 7, 1915, pp. 285–412; F. ENRIQUES, ed., *Fragen der Elementargeometrie*, 2 Teil, second ed., Leipzig, 1923: Aufgaben dritten Grades: Verdoppelung des Würfels, Dreiteilung des Winkels," by A. CONTI, pp. 189–226; "Über die transzendenten Aufgaben, insbesondere über die Quadraturen des Kreises" by B. CALÒ, pp. 267–326. ORE [7b], "The classical construction problems," pp. 340–348.

67. E. W. HOBSON, *"Squaring the Circle," a History of the Problem*, Cambridge, 1913, viii, 58 pp.

68. F. KLEIN, *Famous Problems of Elementary Geometry, The Duplication of the Cube, the Trisection of an Angle, the Quadrature of the Circle*, transl. by W. W. BEMAN and D. E. SMITH. Second ed. rev. and enl. by R. C. ARCHIBALD, New York, 1930, xii, 92 pp.

69. R. C. ARCHIBALD, "Curves, Special," *Encyclopædia Britannica*, New York, fourteenth ed., v. 6, 1929, pp. 887–899.

70. All teachers of the history of Greek mathematics will want to refer their students to the excellent address of R. E. LANGER, "Alexandria—shrine of mathematics," *Amer. Math. Monthly*, v. 48, 1941, pp. 109–125.

71. TURNBULL [51]; SMITH [54]; HEATH [73], v. 1, pp. 1–240; A. DEMORGAN, article "Euclid" in W. SMITH, *Dictionary of Greek and Roman Biography*, v. 2, London, 1890, pp. 63–74; F. HULTSCH, "Eukleides," PAULY-WISSOWA, *Real-Encyclopädie*, v. 11, 1907, cols. 1003–1052.

72. EUCLID's ten treatises are 1. Elements, 2. Data, 3. Optics, 4. Phaenomena, 5. Music, 6. Pseudaria, 7. Surface-Loci, 8. Conics, 9. Porisms, 10. On Divisions of Figures. See HEATH [67], v. 1, pp. 7–18. Of 1–5 approximately complete texts have come down to us. In the case of 6–7 our fragmentary knowledge derived wholly from Greek sources makes conjecture as to their content rather futile. As to 10, PROCLUS alone among Greeks makes explanatory reference. But an Arabic ms. (translated into French) gives the enunciation of all of the propositions (36) and also the proofs of four of them. This and other documents were the basis of R. C. ARCHIBALD, *Euclid's Book on Division of Figures . . . with a Restoration based on Woepcke's Text and on the Practica Geometriae of Leonardo Pisano*. Cambridge, 1915, viii, 88 pp.

73. T. L. HEATH, *The Thirteen Books of Euclid's Elements translated from the text of Heiberg with Introduction and Commentary*, second ed. revised with additions. Cambridge, 3 vols., 1926, xii, 432 pp.+folding plate; vi, 436 pp., iv, 546 pp. This monumental edition of the *Elements* is also an encyclopedic history of elementary geometry. Who first laid down the criterion that the ruler and compasses shall be the only instruments used in elementary geometry, "plane" problems? In EUCLID's Elements these instruments are postulated as follows: 1. To draw a straight line from any point to any point; 2. To produce a finite straight line continuously in a straight line; 3. To describe a circle with any center and distance. See the doctoral dissertation, A. D. STEELE, "Über die Rolle

von Zirkel und Lineal in der griechischen Mathematik," *Quellen u. Studien . . .* , v. 3B, pp. 287–369, 1936. ORE [7b], "Euclid's algorism," pp. 41–49.

74. The first English edition of EUCLID's *Elements* was brought out at London in 1570, under the name of Sir HENRY BILLINGSLEY, later Sheriff and Lord Mayor of London. (For the small part which BILLINGSLEY may have had to do with the translation see A. DEMORGAN, *British Almanac and Companion for 1837*, pp. 38–39 of *Companion*). It contained "a very fruitful Praeface" by JOHN DEE, a man of great erudition; see W. F. SHENTON, "The first English Euclid," *Amer. Math. Monthly*, v. 35, 1928, pp. 505–512.

75. The *Phaenomena* consists of propositions in spherical geometry so far as required for observational astronomy. The earliest mathematical work which has come down to us entire is *On the Moving Sphere* by AUTOLYCUS of Pitane, an elder contemporary of EUCLID. Both works consider special circles on the heavenly sphere, the equator and parallel circles, the zodiac or ecliptic, and the horizon which here appears for the first time as a single word. But EUCLID's work seems to be based also on an unknown earlier work of more mathematical content. Of this work of AUTOLYCUS, and of another of his works, *On Risings and Settings*, as well as of the *Sphaerica* of the mathematician and astronomer, THEODOSIUS (bet. -180 and -25) of Bithynia, there is a handy German translation: *Autolykos Rotierende Kugel und Aufgang und Untergang der Gestirne. Theodosios von Tripolis* [sic] *Sphaerik, übersetzt und mit Anmerkungen versehen* (*Ostwald's Klassiker*, no. 232), Leipzig, 1931, viii, 180 pp. The work of THEODOSIUS, of the same general type as the other works mentioned above, is that of a laborious compiler offering practically nothing original. Such were the predecessors of the great work of MENELAUS.

76. The first English translation of EUCLID's *Optics*, by E. H. BURTON, was published in Amer. Optical Soc., *Jn.*, v. 35, 1945, pp. 357–372. EUCLID here proves the equivalent of the fact that $\tan \alpha/\tan \beta < \alpha/\beta$; and in another proposition it is assumed that the angles of incidence on a mirror and reflection therefrom are equal.

77. See ARCHIBALD [57], pp. 8–10, with various references to the literature.

78. HEATH [72], v. 1, pp. 202–220. L. R. LIEBER, *Non-Euclidean Geometry; or, Three Moons in Matheses*. Brooklyn, N. Y., 1931, 46 pp. D. E. SMITH, general ed., *A Source Book in Mathematics*. New York, 1929, xviii, 701 pp. +7 portrait plates. The 96 extracts are edited by many collaborators. R. BONOLA, *Non-Euclidean Geometry. A Critical and Historical Study of Its Development*. English translation by H. S. CARSLAW, Chicago, 1912, xii, 268 pp.

79. R. C. ARCHIBALD, (a) "Perfect numbers," *Amer. Math. Monthly*, v. 28, 1921, pp. 140–141; (b) "Mersenne's numbers," *Scripta Mathematica*, v. 3, pp. 112–119, 1935. There are 55 MERSENNE numbers which are defined by $M_n = 2^n - 1$, n prime, and less than 263. The largest known prime number (39 integers) is M_{127}, discovered in 1914. Regarding the M_n the following facts are now known: 12 (prime), 14 (composite and completely factored), 9 (two or more prime factors), 8 (only one prime factor), 12 (composite but no factor known). See *Math. Tables and Other aids to Computation*, v. 1, pp. 333, 404; v. 2, pp. 94, 341; and H. S. UHLER, Nat. Acad. Sci., *Proc.*, v. 34, 1948, pp. 102–103; and Amer. Math. Soc., *Bull.*, v. 54, 1948, pp. 378–380.

80. HARRIET H. SHOEN, "Archimedes. The reconstruction of a personality," *Scripta Math.*, v. 2, 1934, pp. 261–264, 342–347. T. L. HEATH, (a) *Archimedes* (*Pioneers of Progress*). London, 1920. (b) *The Works of Archimedes, Edited in Modern Notations with Introductory Chapters*. Cambridge, 1897. clxxxvi, 326 pp. The long and very valuable introduction to (b) gives a life of ARCHIMEDES, an account of (1) the relation of A. to his predecessors; (2) arithmetic and A.; (3) the problems known as "neuseis" (Vergings or Inclinations); (4) cubic equations; (5) anticipations by A. of the integral calculus. Then follow works of A.: (i) On the sphere and cylinder; (ii) Measurement of a circle; (iii) On conoids and spheroids; (iv) On spirals; (v) On the equilibrium of planes; (vi) The sand-reckoner; (vii) Quadrature of the parabola; (viii) On floating bodies; (ix) Book of lemmas; (x) The cattle problem. For a historical summary with regard to the cattle problem see R. C. ARCHIBALD, *Amer. Math. Monthly*, v. 25, 1918, pp. 411–414. In connection with (2) references may be given to C. MÜLLER, "Wie fand Archimedes die von ihm gegeben Näherungswerte von $\sqrt{3}$," *Quellen u. Studien . . .* , v. 2B, pp. 281–285, 1932, to [97]; and to G. A. GIBSON, "The treatment of arithmetic progressions by Archimedes," Edinburgh Math. Soc., *Proc.*, v. 16, 1897, pp. 2–12. Under the general heading of this note we refer also to BELL [53]; SMITH [54]; and to an extra-

ordinary ancient colored mosaic representing the manner of death of A. This mosaic came originally from the city of Herculaneum, destroyed by an eruption of Vesuvius in 79 A.D. A reproduction of the mosaic in colors is given in a publication of a Bonn archaeologist, F. WINTER, *Der Tod des Archimedes.* Berlin, 1924, 24 pp.+2 plates; see also F. CAJORI, *Science*, v. 61, 1925, pp. 415.

81. As a supplement to the edition of his *Works* of A., Heath published a translation from the Greek of *The Method of Archimedes Recently Discovered by Heiberg* . . . , Cambridge, 1912, 51 pp. An earlier English translation, from a German translation, was published in *The Monist*, April 1909, and reprinted as a pamphlet with an Introduction by D. E. SMITH, *Geometrical Solutions Derived from Mechanics*, Chicago, 109, ii, 28 pp. Discussion in the *Method* is also related to the topic treated in J. M. CHILD, "Archimedes' principle of the balance and some criticisms upon it," in C. SINGER, ed., *Studies in History and Method of Science.* Oxford, v. 2, 1922, pp. 490–520. A sort of "neusis" is used by Archimedes in his remarkable solution of the construction of a regular heptagon, found comparatively recently in an Arabic ms.; see *Die trigonometrischen Lehren des Persischen Astronomen Abu'l-Raihân Muh ibn Aḥmad* AL-BÎRÛNÎ, *dargestellt nach al-Qânûn al-Mas'ûdî von* CARL SCHOY. *Nach dem Tode des Verfassers herausgegeben von* JULIUS RUSKA *und* HEINRICH WIELEITNER. Hannover, 1927, xii, 108 pp. Also C. SHOY, "Graeco-Arabische Studien nach Handschriften der Vizeköniglichen Bibliothek zu Kairo," *Isis*, v. 8, 1926, pp. 21–40; J. TROPFKE, "Die Siebeneckabhandlung des Archimedes," *Osiris*, v. 1, 1936, pp. 636–651.

82. HEATH [49]; H. WIELEITNER, "Das Fortleben der Archimedischen Infinitesimalmethoden bis zum Beginn des 17. Jahrh., insbesondere über Schwerpunktbestimmungen," *Quellen u. Studien* . . . , v. 1B, pp. 201–220, 1930; W. STEIN, "Der Begriff des Schwerpunktes bei Archimedes," *Quellen u. Studien* . . . , v. 1B, pp. 221–224, 1930. When the idea of center of gravity was first conceived by the Greeks is unknown; in writings of Archimedes it appears to be already assumed as familiar.

83. HEATH [22], *History*, v. 2, pp. 98–101; M. BRÜCKNER, *Vieleck und Vielflache Theorie und Geschichte.* Leipzig, 1900, pp. 132–140+plate VI (with pictures of all of the solids). See [107].

84. J. SCOTT, "On the burning mirrors of Archimedes, with some propositions relating to the concentration of light produced by reflectors of different forms," R. Soc. Edinburgh, *Trans.*, v. 25, 1868, pp. 123–149+plate III.

85. R. A. FISHER, "Reconstruction of the sieve of Eratosthenes," *Math. Gazette*, v. 14, pp. 565–566, 1929. In E. HOPPE'S *Mathematik und Astronomie im klassischen Alterthum* (Heidelberg, 1911), it is stated on p. 284 that the method of the sieve of ERATOSTHENES for finding prime numbers was set forth completely in section 52 of Plato's *Phaedo.* This statement has been quoted more than once. On consulting the section in question, I find not the slightest basis for the statement's substantiation.

86. The best first introduction to APOLLONIUS would be through *Apollonius of Perga, Treatise on Conic Sections, edited in Modern Notation with Introductions including an Essay on the Earlier History of the Subject* by T. L. HEATH. Cambridge, 1896, cxxii, 254 pp. The earlier history of conics includes the names MENAECHMUS, EUCLID, and ARCHIMEDES. APOLLONIUS, and his own account of the conics is discussed, pp. lxvii–lxxxvi. Then follow further HEATH surveys: (a) "The construction of a conic by means of tangents," pp. cxxx–cxxxvii; (b) "The three-line and four-line locus," pp. cxxxviii–cl; (e) "The construction of a conic through five points," pp. cli–clvi. In the *Conics* of APOLLONIUS are the essentials for the construction of all three conics by the method (a). It seems certain that APOLLONIUS was the first to solve the problem (b), although there are no propositions leading directly to the results. Assuming that APOLLONIUS was in possession of a complete solution of the problem of constructing the four-line locus referred to the sides of a quadrilateral of any form, he had in fact solved the problem of constructing a conic through five points. But the problem of the construction of a conic through five points is, however, not found in the work of APOLLONIUS any more than the actual determination of the four-line locus. The PAPPUS generalization of the four-line locus may thus be stated: If $p_1, p_2, p_3, \cdots, p_n$ be the lengths of straight lines to meet n given straight lines at given angles (where n is odd) and a be a given line, then if

$$\frac{p_1}{p_2}\cdot\frac{p_3}{p_4}\cdots\frac{p_n}{a}=\lambda,$$

where λ is a constant, the point will be on a curve given in position. This will also be true if n is even, and

$$\frac{p_1}{p_2}\cdot\frac{p_3}{p_4}\cdots\frac{p_{n-1}}{p_n} = \lambda.$$

See THOMAS **[46a]**, v. 2, p. 603.

See also HEATH [22], *History*, v. 2, pp. 126–196. There is also the German edition of another excellent work, H. G. ZEUTHEN, *Die Lehre von den Kegelschnitten im Altertum.* Copenhagen, 1886, xiv, 511 pp. A monumental edition of the complete Conics of Apollonius, the eighth book restored, was published in 1710 by EDMOND HALLEY, to whom we shall later **[211]** return. Reference may also be given to O. NEUGEBAUER, "Apollonius—Studien (Studien zur Geschichte der antiken Algebra II.)," *Quellen u. Studien . . .* , v. 2B, pp. 215–253, 1932. See also O. NEUGEBAUER, "The astronomical origin of the theory of conic sections," Amer. Phil. Soc., *Proc.*, v. 92, 1948, pp. 136–138.

86a. See THOMAS **[46a]**, v. 1, pp. 362–363, 495–496.

87. For example, C. HELLWIG, *Das Problem des Apollonius* . . . Halle, 1856, iv, 43 pp.+4 plates. B. ALVORD, *The Tangencies of Circles and Spheres* (*Smithsonian Contribs. to Knowledge*). Washington, 1856, 16 pp.+9 plates.

88. See R. C. ARCHIBALD, (a) "Centers of similitude and certain theorems attributed to Monge. Were they known to the Greeks?," *Amer. Math. Monthly*, v. 22, 1915, pp. 6–12; (b) "Historical note on centers of similitude of circles," *idem*, v. 23, 1916, pp. 159–161.

89. R. C. ARCHIBALD, "Discussion and history of certain geometrical problems of Heraclitus and Apollonius," Edinburgh Math. Soc., *Proc.*, v. 28, 1910, pp. 152–178+5 folding plates. A hundred solutions of the problems of Heraclitus are given in a notable book: A. MAROGER, *Le Problème de Pappus et ses cent Premières Solutions.* Paris, 1925, viii, 386 pp. For an account of the important work of APOLLONIUS in connection with epicycles and eccentrics see J. L. E. DREYER, *History of the Planetary Systems from Thales to Kepler.* Cambridge, England, 1906, pp. 151 f.

90. FOTHERINGHAM **[31]**; the exact dating here of these astronomers is without foundation, since the reasoning of SCHNABEL, upon which the statements are based, is entirely wrong.

91. HEATH [22], *History*, v. 2, pp. 253–260; *Manual*, pp. 393–399.

92. There is not the slightest evidence that the Babylonians observed the precession of the equinoxes. Various writers, such as F. CAJORI, "Babylonian discovery of the precession of the equinoxes," *Science*, n.s. v. 65, 1927, p. 184, were misled by the wholly erroneous monograph of P. SCHNABEL, "Kidenas, Hipparch und die Entdeckung der Präzession," *Z. v. Assyriologie*, n.s., v. 3, 1926, pp. 1–60.

93. H. VOGT, "Versuch einer Wiederherstellung von Hipparchs Fixsternverzeichnis," *Astron. Nach.*, v. 225, 1925, cols. 17–54.

94. O. NEUGEBAUER, *Quellen u. Studien . . .* , v. 2B, pp. 41–44, where both Babylonian and Greek sources are quoted. See also THOMAS **[46a]**, v. 2, pp. 395–397; and F. THUREAU-DANGIN, "La division du cercle," *Revue d'Assyriologie*, v. 25, 1928, pp. 177–188 [see ARCHIBALD **[44]** (Supplement), 1928].

95. HEATH **[22]**, *History*, v. 2, pp. 298–354; *Manual*, pp. 415–433.

96. A. ROME, "Le problème de la distance entre deux villes dans la Dioptra de Héron," Soc. Sci. de Bruxelles, *Annales*, v. 42, 1923, mémoires pp. 234–258; O. NEUGEBAUER, "Über eine Methode zur Distanzbestimmung Alexandria-Rom bei Heron," Danske Vidensk. Selskab., *hist.-filol. Meddelelser*, v. 26, no. 2, 1938, 26 pp.+5 plates. Part 2, v. 26, no. 7, 1939, 11 pp. NEUGEBAUER discusses the dating of HERON and makes it seem plausible that he flourished in the first century of the Christian era.

97. J. E. HOFMANN, "Über die Annäherung von Quadratwurzeln bei Archimedes und Heron," Deutsche Math.-Ver., *Jahresb.*, v. 43, pp. 187–210, 1934.

98. Attention may be drawn to an interesting summary article by J. TROPFKE, "Zur Geschichte der quadratischen Gleichungen über dreieinhalb Jahrtausend," Deutsche Mathem.-Ver., *Jahresb.*, v. 43, pp. 98–107, Dec. 1933; v. 44, pp. 26–47, June, and pp. 95–119, Aug. 1934. The sections

dealt with in these parts are as follows: I. Introduction. Survey of Known Results; II. Babylonian Mathematics; III. EUCLID; IV. HERON; V. DIOPHANTUS; VI. Muhammed ibn Mūsā AL-KHOWĀRAZMÎ; VII. ABÛ KÂMIL SHOJÂ; VIII. AL-KARKHÎ; IX. 'OMAR KHAYYAM; X. ABRAHAM BAR CHIJA, named SAVASORDA; XI. LEONARDO VON PISA; XII. Anonymous arrangement of the Algebra of AL-KHOWĀRAZMĪ; XIII. MICHAEL STIFEL; XIV. GIROLAMO CARDANO; XV. RAFAEL BOMBELLI; XVI. CHRISTOPH CLAVIUS; XVII. SIMON STEVIN.

99. Perhaps the best first introduction to DIOPHANTUS is T. L. HEATH, *Diophantus of Alexandria. A Study in the History of Greek Algebra. Second ed., with a Supplement containing an Account of Fermat's Theorems and Problems connected with Diophantine Analysis and some Solutions of Diophantine Problems by Euler*. Cambridge, 1910, viii, 387 p. See also HEATH **[22]**, *History*, v. 2, pp. 448–517, and *Manual*, pp. 472–509.

100. J. KLEIN, *Quellen u. Studien* . . . , v. 3B, pp. 133–135, 1936. This is a long footnote about the date of DIOPHANTUS. See also Heath **[22]**, *History*, v. 2, pp. 306; dedication to Dionysius.

101. The history of the development of this subject is most elaborately set forth in L. E. DICKSON, *History of the Theory of Numbers*, v. 2, *Diophantine Analysis*. Washington, D. C., 1920, xxxv, 803 pp. ORE **[7b]**, "Diophantine problems," pp. 165–208.

102. HEATH **[22]**, *History*, v. 2, pp. 260–273; *Manual*, pp. 399–402. The question as to whether MENELAUS prepared a Catalogue of fixed stars is discussed by A. A. BJÖRNBO, *Bibl. Math.*, v. 2, 1901, p. 196–212. See also DREYER **[106]**, 1917, pp. 533–535.

103. A Latin edition of the *Sphaerica* based on Hebrew and Arabic mss. was prepared by EDMOND HALLEY and published posthumously in 1758; note HALLEY'S earlier Greek work in **[86]**. There are also the following two German translations based on a considerable body of Arabic material: (1) A. A. BJÖRNBO, *Studien über Menelaos' Sphärik, Beiträge zur Geschichte der Sphärik und Trigonometrie der Griechen.* (*Abh. z. Gesch. d. Math. Wissen.*, v. 14). Leipzig, 1902, viii, 154 pp. (2) M. KRAUSE, *Die Sphärik von Menelaus aus Alexandria. . . .* (Gesell. d. Wissen. z. Göttingen, *Abh.*, *phil.-hist. Kl.*, s. 3, no. 17). Berlin, 1936, viii, 254, 112 pp.+7 plates. This latter edition is by far the best.

104. E. H. BUNBURY & C. R. BEVAN, *Encycl. Britannica*, eleventh ed., v. 22, 1911, pp. 618–626; HEATH **[22]**, *History*, v. 2, pp. 273–297; *Manual*, pp. 402–414.

105. A. G. BURGESS, "Ptolemy's theorem and certain trigonometrical formulae," Edinburgh Math. So., *Math. Notes*, no. 4, 1910, pp. 42–43; *School Science and Math.*, v. 17, 1917, pp. 784–786; R. C. ARCHIBALD, "Ptolemy's theorem and formulae of trigonometry," *Amer. Math. Monthly*, v. 25, 1918, pp. 94–95. That PTOLEMY had given a solution of the so-called SNELL-POTHENOT Problem in connection with astronomy rather than geodesy, navigation, or elementary geometry, was recorded in J. A. C. OUDEMANS, (a) "Lösung des sog. Pothenot'schen, besser Snellius'schen Problems von Ptolemaeus," Astron. Gesell., *Vierteljahrschrift*, v. 22, 1887, pp. 345–349; (b) "Het problema van Snellius obgelost door Ptolemaeus," Akad. v. Wetens., *Verslagen, afd. Natuur.*, s. 2, v. 19, 1884, pp. 436–441. This problem has a very extended literature including discussions by C. F. GAUSS (1816–1852). The solution of WILLEBRORD SNELL (1580 or 1581–1626) was given in his *Eratosthenes Batavus de terrae ambitus vera quantitate*. Leyden, 1717, pp. 199–200. To him is also due (1624) the name "loxodrome" for the rhumb line in navigation, a rediscovery of the Hindu formula for the area of a circumscribed quadrilateral in terms of its sides, and the law of refraction of light (see J. A. VOLLGRAFF, "Snellius' notes on the reflection and refraction of rays," *Osiris*, v. 1, 1936, pp. 718–725.)

106. The best edition of this Catalogue is *Ptolemy's Catalogue of Stars, a Revision of the Almagest*, by C. H. F. PETERS and E. B. KNOBEL. Carnegie Institute of Washington, 1915, iv, 207 pp.+portrait. On account of various misleading statements made in histories, the following papers are of importance: J. L. E. DREYER, "On the origin of Ptolemy's catalogue of the stars," R. A. S., *Mo. Not.*, v. 77, 1917, pp. 528–539; v. 78, 1918, pp. 343–349. See also **[92]**.

106a. Pages 1–34 of the following scholarly work is mainly devoted to discussion of PTOLEMY'S Geography: A. E. NORDENSKIÖLD, *Facsimile—Atlas to the Early History of Cartography with Reproductions of the Most Important Maps Printed in the XV and XVI Centuries. Translated from the*

Swedish Original by J. A. EKELÖF and C. R. MARKHAM. Stockholm, 1889, viii, 141 pp., 51 double map-plates. The first English translation of the Geography was published by the New York Public Library in 1932, in an edition of 250 copies and sold at $1000 each. See *Isis*, v. 20, 1933, pp. 270–274. On account of the text chosen for translation this edition was a most unfortunate one and highly unsatisfactory as a work of scholarly achievement.

106b. See ALBERT LEJEUNE, (a) "Les tables de réfraction de Ptolémée," Soc. Sci. de Bruxelles, *Annales*, s. 1, v. 60, 1946, pp. 93–101; (b) "Les lois de la réflexion dans l'optique de Ptolémée," *L'Antiquité Classique*, v. 15, 1947, pp. 241–256.

106c. A somewhat defective text of this work, together with a Latin translation by JOHN WALLIS was published at Oxford in 1680, and reprinted in his Collected Works, 1699. Compare F. J. FÉTIS, *Biographie Universelle des Musiciens . . .* , Paris, v. 7, 1870; also *The Oxford History of Music*, v. 1, by H. E. WOOLDRIDGE, Oxford, 1901, pp. 15–22; H. WYLDE, *The Evolution of the Beautiful in Sound*, Manchester and London, 1888, chap. XI, *etc.*; and D. B. MUNRO, *The Modes of Ancient Greek Music*, Oxford, 1894, pp. 108–112. Such works may also be consulted with regard to the musical work of EUCLID and PYTHAGORAS.

107. HEATH [22, 28], *History*, v. 2, pp. 355–459; *Manual*, pp. 434–465. Reference may also be given to J. H. WEAVER (a) "Pappus, Introductory Paper," Amer. Math. Soc., *Bull.*, v. 23, pp. 127–135, 1916; (b) "On foci of conics," *idem*, pp. 357–365, 1917; (c) "Some extensions of the work of PAPPUS and STEINER on tangent circles," *Amer. Math. Monthly*, v. 27, 1920, doctoral dissertation, Univ. of Pennsylvania, with "Life," 10 pp. This last deals with properties of the figure called "Shoemaker's Knife" studied by ARCHIMEDES and discussed by PAPPUS from whom we also learn of the interest of Archimedes in semi-regular polyhedra (see [83]). PAPPUS was also the author of a notable work: *The Commentary on Book X of Euclid's Elements. Arabic Text and Translation by William Thomson, with Introductory Remarks, Notes, and a Glossary of Technical Terms by* GUSTAV JUNGE & WILLIAM THOMSON. Cambridge, Harvard Univ. Press, 1930, 294 pp.

108. Of this work, originally in 8 books, Book 1 and the first 13 of 26 propositions in Book 2 are missing. There is a French translation of the Collection by PAUL VER EECKE. Paris and Bruges, 1933, cxxviii, 885 pp. There are also translations of two portions: (1) the latter part of Book 3, by J. S. MACKAY, "Pappus on the progressions," Edinb. Math. Soc., *Proc.*, v. 6, 1888, pp. 48–58; (2) a German translation of books 7–8 by C. I. GERHARDT, Halle, 1871, ii, 381 pp.

109. For example P. VER EECKE, "Le theorème dit de Guldin considéré au point de vue historique," *Mathesis*, v. 46, 1932, pp. 395–397. In *Scripta Mathematica*, v. 1, 1933, p. 268, it is shown that there is no foundation in fact for the argument of VER EECKE that GULDIN was an independent discoverer; rather does it confirm the belief that GULDIN had taken his result from the work of PAPPUS.

110. HEATH [22], *History*, v. 2, pp. 389–390, gives a translation of PAPPUS' "Preface on the sagacity of bees." See also the valuable survey, with references to literature, in D. W. THOMPSON, *On Growth and Form*, new edition. Cambridge, 1942, pp. 526–539. Also THOMAS [46a], v. 2, pp. 589–593.

111. We have been making numerous references to astronomy since it involved mathematics and, until long after the period we are considering, it was regarded as a part of mathematics. An old work of importance, almost wholly devoted to Greek astronomy, is J. B. J. DELAMBRE, *Histoire de l'Astronomie Ancienne*, Paris, 1817, v. 1, lxxii, 556 pp., 1 plate; v. 2, vii, 640 pp., 16 plates. The amount of presented detail, elaborated from source material, is especially valuable.

111a. THEON OF ALEXANDRIA, fourth century mathematician and astronomer, prepared editions of EUCLID's Elements and Optics [76], and with his daughter HYPATIA, mathematician and philosopher, prepared an elaborate commentary on PTOLEMY's Almagest. There is a biography of HYPATIA in *Encycl. Britannica*, eleventh ed., v. 14, 1910. The story of her life was the basis of CHARLES KINGSLEY's romance *Hypatia* (1853). Scholarly biographies of both Hypatia and her father are in PAULY-WISSOWA, *Real-Encyclopädie*, HYPATIA by K. PRAECHTER, v. 9, 1914, cols. 242–249; THEON by K. ZIEGLER, s. 2, v. 10, 1934, cols. 2075–2080. The following is the best text of *Commentaires de Pappus et de Théon d'Alexandrie sur l'Almageste. Text établi et annoté par A.* ROME. Tome 1: *Pappus d'Alexandrie Commentaire sur les livres 5 et 6 de l'Almageste.* Tome II–III: *Théon*

d'Alexandrie, Commentaire sur les livres 1 et 2 et 3 et 4 de l'Almageste. 3 v. *Studie e Testi* 54, 72, 106. Rome, Vatican, 1931, 1936, 1943, paged continuously cxl, 1085 pp. See also [**172**].

111b. The Astrolabe is an instrument used not only for stellar but also for solar and lunar altitude observation. Its earliest forms may go back to the time of HIPPARCHUS; many varieties were developed by the Arabs and others up to the nineteenth century. The first Greek treatise on the astrolabe, which has been preserved, was by JOHN PHILOPONUS of Alexandria (*c.* 525). There is a French translation by P. TANNERY, "Jean le grammairien d'Alexandrie (Philopon), sur l'usage de l'astrolabe et sur les tracés qu'il présente," in his *Mémoires Scientifiques*, Toulouse and Paris, v. 9, 1929, pp. 341–367. A rather worthless English translation, by H. W. GREENE who evidently knew nothing of Greek scientific terminology, appears in v. 1, pp. 61–81 of R. T. GUNTHER, *The Astrolabes of the World*, 2 v., Oxford, 1932; this is a sumptuous and interesting work illustrated with over 150 plates. A recent excellent work discussing the history, theory and use of such instruments is HENRI MICHEL, *Traité de l'Astrolabe*, Paris, 1947, viii, 202 pp., with 24 plate reproductions. On pp. 164–165 are paragraphs about BÜRGI as a notable instrument maker. There are also references to the use of the linear astrolabe of NAŞÎR ED-DÎN AL-TÛSÎ (pp. 21, 115–122) and to the astrolabe and "Horizontall Dyall" of OUGHTRED (pp. 24, 129–130). The astrolabe was superseded by JOHN HADLEY'S much more accurate reflecting quadrant of 1731–34.

Of an earlier Greek work on the astrolabe instrument, by THEON of Alexandria, only the contents, in Syriac, have been preserved, but these are sufficient to indicate that the original was a main source for later writing. From PTOLEMY'S *Planesphaerium* (German translation by J. DRECKER in *Isis*, v. 9, 1927, pp. 255–278) it is suggested that HIPPARCHUS was familiar with stereographic projection, and it is clear that PTOLEMY knew of the construction and use of the plane astrolabe or Theon's "Little astrolabe." The use of the term astrolabe by PTOLEMY in the *Almagest* does not apply to this, but, as in the case of the *Hypotyposis* of PROCLUS, to the "armillary sphere."

The great fourteenth century poet GEOFFREY CHAUCER wrote in 1391 for his son, "Little Lowrys," a famous "Treatise on the Astrolabe. Bread and Milk for Children,"" of which an edition was published in R. T. GUNTHER, *Early Science in Oxford*, v. 5, Oxford, 1929. A better edition, perhaps, is that of W. W. SKEAT, London, 1872.

For many of the facts in this note I am indebted to Professor NEUGEBAUER, who has set forth the early history of the astrolabe in a memoir to be published in *Isis* in the near future. A reference may be given to MARCIA L. LATHAM, "The Astrolabe," *Amer. Math. Monthly*, v. 24, 1917, pp. 162–168, even though the distinction between astrolabe and "armillary sphere" has not been recognized.

112. P. K. HITTI, *History of the Arabs*, London, 1937, p. 166, gives information concerning Alexandrian libraries. "The great Ptolemaic Library was burnt as early as 48 B.C. by JULIUS CAESAR. A later one, referred to as the daughter Library, was destroyed about A.D. 389 at the command of Roman Emperor THEODOSIUS. At the time of the Arab conquest, therefore, no library of importance existed in Alexandria." A whole chapter is devoted to "The Library of Alexandria" in A. J. BUTLER, *The Arab Conquest of Egypt and the last Thirty Years of the Roman Dominion*, Oxford, 1902.

113. References may be given to the following recent publications dealing with Hindu mathematics before 600 A.D.:

B. DATTA, *The Science of the Śulba. A Study in Early Hindu Geometry*, Calcutta, 1932, xvi, 240 pp. The "śulbas" or "śulba-sūtras" are manuals for the construction of altars necessary in connection with sacrifices of vedic Hindus. While seven of these manuscripts, now available, are comparatively modern there is a wide range to the claims for antiquity of the originals—one authority placing 200 A.D. as an upper limit, another 500 B.C. The śulbas contain considerable discussion of mathematical interest, including a rule for squaring the circle. This rule, involving a segment of a circle and its apothem, suggests a possible connection with the problems showing that the Babylonians were familiar with the Pythagorean theorem about 2000 B.C. See also C. MÜLLER, "Die Mathematik der Śulvasûtra, eine Studie zur Geschichte indischer Mathematik," Hamburg Univ., Math. Sem., *Abhandlungen*, v. 7, 1929, pp. 173–204.

ĀRYABHATA (c. 476–c. 550) the elder was a noted Hindu astronomer whose celebrity rests on a work entitled *Āryabhaṭīya*, of which the second section, *Ganita* (66 lines) is devoted to mathematics

and is simply a set of rules which for the volume of a sphere is very inaccurate. It has been surmised that he attempted the general solution of a linear indeterminate equation. The following are references to literature: P. C. Sengupta, (a) "Āryabhaṭa, the father of Indian epicyclic astronomy," Univ. of Calcutta, *Jn. Dept. Letters*, v. 18, no. 3, 1929, 56 pp. (b) "The *Aryabhatiyam*, Translation," *idem*, v. 16, no. 6, 1927, 56 pp.; the *Ganita* with commentary occupies pp. 13–30. *The Āryabhatya of Āryabhaṭa. An Ancient Indian Work on Mathematics and Astronomy. Translated with notes* (and the use of SENGUPTA's works), by W. E. CLARK, Chicago, 1930, xxx, 90 pp. C. MÜLLER, "Volumen und Oberfläche der Kugel bei Āryabhaṭa I," *Deutsche Mathematik*, v. 5, 1940, pp. 244–255. S. GANGULI, "India's contribution to the theory of indeterminate equations of the first degree," Indian Math Soc., *Jn.*, v. 19, 1931, pp. 110–120, 129–142, 153–168; the first two parts deal with work of ĀRYABHAṬA the elder and the younger. Other references may be found in [**114–115**]. Had it not been for the lengthy review in *Math. Reviews*, v. 9, Feb. 1948, of the following miserable book, no reference to it would have been made here: L. V. GURJAR, *Ancient Indian Mathematics and Vedha*. Poona, 1947, vi, 202 pp. Another phase of its worthlessness is brought out in *Nature*, v. 161, Apr. 17, 1948, p. 580.

114. More general treatments of Hindu mathematics are to be found in the following: D. M. MEHTA, *Theory of Simple Continued Fractions (with special Reference to the History of Indian Mathematics)*. Diss. Heidelberg, Bhavnagar, India, 1931, iv, 164, 4 pp. G. CHAKRAVARTI, (a) "Growth and development of progressive series in India," (b) "The Hindu term for area," (c) "Surd in Hindu mathematics," (d) "On Hindu treatment of fractions," Univ. of Calcutta, *Jn. Dept. Letters*, v. 24, nos. 6–9, 76 pp. A. N. SINGH, "Use of series in Hindu mathematics," *Osiris*, v. 1, 1936, pp. 606–628.

115. Other general works of value are: G. R. KAYE, (a) *Indian Mathematics*. Calcutta and Simla, 1915, iv, 74 pp., 2 plates. (b) "Indian Mathematics," *Isis*, v. 2, 1919, pp. 326–356. B. DATTA & A. N. SINGH, *History of Hindu Mathematics. A Source Book. Part 1, Numerical Notation and Arithmetic*. 1935, xx, 261 pp. *Part 2, Algebra*. 1938, xvi, 314 pp. Lahore. Unreliable features of the first part are set forth in NEUGEBAUER's review in *Quellen u. Studien* . . . , v. 3B, pp. 263–271. See also DELAMBRE [**111**], v. 1, pp. 400–556.

116. H. G. ZEUTHEN, "Sur l'arithmétique géométrique des Grecs et des Indiens," *Bibl. Mathematica*, s. 3, v. 5, 1904, pp. 97–112.

117. Cajori [**2**], pp. 99–112; Smith [**4**] 1, pp. 164–177.

118. The great reference work for Arabian mathematicians, more than 500 of whom are listed, is H. SUTER, *Die Mathematiker und Astronomen der Araber und ihre Werke* (*Abhand. z. Gesch. d. mathem. Wissen.* Heft 10), Leipzig, 1900, x, 278 pp. (Also as Supplement to *Z. f. Math. u. Physik*, v. 45); "Nachträge und Berichtigungen," Heft 14, 1902, pp. 155–185. See also H. P. J. RENAUD, "Additions et corrections à Suter, 'Die Mathematiker . . . ,' " *Isis*, v. 18, pp. 166–183, 1932. Another Arabic work of first importance is H. SUTER, *Das Mathematiker-Verzeichnis in Firhrist des Ibn abî Ja'ḳûb an-Nadîm, zum erstenmal vollständig ins deutsche übersetzt und mit Anmerkungen versehen*. (*Abhand. z. Gesch. d. Mathem. Wissen.*, Heft 6), 1892, 87 pp. Also as Suppl. to *Z. f. Math. u. Physik*, v. 37. Ibn Ja'ḳûb an-Nadîm = Abu'l FaragMah.b: Ishâq.

119. We have already listed five works of this kind, [**72, 81, 86, 103**]. Among others were Arabic translations of works of APOLLONIUS, ARCHIMEDES, DIOPHANTUS, EUCLID, HERON, MENELAUS, PTOLEMY.

120. One of the main sources in English for the text of the chief mathematical work of BRAHMAGUPTA is *Algebra, with Arithmetic and Mensuration from the Sanscrit of Brahmegupta and Bháscara, translated by* H. T. COLEBROOKE. London, 1817, lxxxiv, 378 pp. In the long introduction, pp. xxv–xxxvii deal mainly with BRAHMAGUPTA's astronomical work; pp. lxxx–lxxxiv contain a note on "communications of the Hindus with western nations on astrology and astronomy." BHĀSKARA's work on Arithmetic (Lilávatí) and Algebra (Vija-ganíta) fills pp. 1–276. BRAHMAGUPTA's Arithmetic (Ganíta) and Algebra (Pulverizer) occupies pp. 277–378.

121. In DATTA & SINGH [**115**], v. 2, pp. 204–245, are sections on rational triangles and rational quadrilaterals. See also DICKSON [**101**], pp. 191, 216; and L. E. DICKSON, "Rational triangles and quadrilaterals," *Amer. Math. Monthly*, v. 28, 1921, pp. 244–245.

122. It is said that Lilavati was the name of BHĀSKARA's daughter and that the book of that

name was written in her honor. The first English translation was *Bija Ganita: or the Algebra of the Hindus*. By E. STRACHEY, London, 1813, viii, 119 pp. There are also two revised editions of *Colebrooke's Translation of the Lîlâvatî with Notes*. By H. C. BANERJI, Calcutta, (a) 1893, viii, 174, 120 (Sanskrit text appendix) pp.; second ed., 1927, x, 202, 114 pp.

123. BHĀSKARA's methods lead to $x=226153980$, $y=1766319049$ as the solution for the equation when $A=61$. Finding this result was proposed as a problem by FERMAT to FRÉNICLE DE BESSY in a letter of February 1647. EULER solved it in 1732. See DATTA and SINGH [115], v. 20, pp. 166–170. The cattle problem of ARCHIMEDES called for the solution of an equation of this type, with $A=4729494$; compare ARCHIBALD [80], or HEATH [22], *History*, v. 2, pp. 97–98. For details regarding tables of solutions of the equation $x^2-Ay^2=1$, $A\leqq 2000$ by DEGEN, CAYLEY, WHITFORD, D. H. LEHMER, see D. H. LEHMER, *Guide to Tables in the Theory of Numbers*, Washington, 1941.

124. D. E. SMITH & L. C. KARPINSKI, *The Hindu-Arabic Numerals*. Boston, 1911, vi, 160 pp.

125. F. CAJORI, "The controversy on the origin of our numerals," *Scientific Mo.*, v. 9, 1919, pp. 458–464.

126. For many years the chief information in English regarding the first Arabic arithmetic, geometry, and algebra, was *The Algebra of Mohammed ben Musa edited and translated* by F. ROSEN. London, 1831, xvi, 208, 127 (Arabic text) pp. See S. GANDZ, "The sources of al-Khowârizmî's algebra," *Osiris*, v. 1, 1936, pp. 263–277. Dissatisfied with the ROSEN edition of the geometry section of his algebra, GANDZ brought out a new English edition, and showed that its rather trivial contents are mainly a reproduction of a Hebrew geometry of 150 A.D., displaying no evidence of Greek influence: *The Mishnat ha Middot the first Hebrew Geometry of about 150 C.E. and The Geometry of Muhammad ibn Musa al-Khowarizmi the first Arabic geometry (c. 820), representing the Arabic version of the Mishnat ha Middot. A new edition of the Hebrew and Arabic Texts with Introduction, Translation and Notes, Quellen u. Studien . . .*, v. 2A, 1932, pp. 61–85+certain parts of pages 86–96 (Literature and Index, etc.). The v. consists of 96 pp.+4 plates. GANDZ has also made elaborate correction of the "book on the legacies" in another section of ROSEN's edition of al-Khowârizmî's Algebra, in "The algebra of inheritance, a rehabilitation of al-Khowarizmi," *Osiris*, v. 5, 1938, pp. 319–391. See also L. C. KARPINSKI, (a) "Robert of Chester's translation of the algebra of al-Khowarizmi," *Bibl. Math.*, s. 3, v. 11, pp. 125–131, 1911; (b) *Robert of Chester's Latin Translation of the Algebra of Al-Khowarizmi, with an Introduction, Critical Notes and English Version by L. C. Karpinski.* (*Univ. of Michigan Studies, Humanistic Series*, v. 11). New York, 1915, viii, 164 pp.+4 plates. The Introduction occupies pp. 1–63. See also SARTON [8], v. 1, pp. 563–564; and a popular article, DUMONT, "Un professeur de mathématiques au IX^e siècle: M. hammed ibn Mousa Al-Khowarizmi," *Rev. gén. d. Sci. Pures et Appl.*, v. 54, no. 2, 1947, pp. 7–13.

127. SMITH & KARPINSKI [124], p. 52. See also B. DATTA, "Early literary evidence of the use of zero in India," *Amer. Math. Monthly*, v. 33, 1926, pp. 449–454.

128. F. CAJORI, "The zero and principle of local value used by the Maya of Central America," *Science*, n.s. v. 44, 1916, pp. 714–717.

129. GANDZ [21], pp. 509–541, has the following subsections: 11. "AL-KHOWÂRIZMÎ's algebra. Its character and its contributions"; 12. "The algebraic analysis of the Arabic types"; 13. "The geometric demonstrations, Al-K. and EUCLID"; 14. "The Babylonian problems in Al-K. algebra"; 15. "The analysis of type A II in Al-K. algebra"; 16. "Al-K. and DIOPHANTUS"; 17. "Al-K. and HERO." See also TROPFKE [98], pp. 95–99.

F. THUREAU-DANGIN, in "Un problème algébrique babylonien," *Halil Edhem Hatira Kitabi* (*Recueil offert à la Mémoire de Halil Edhem*), Ankara, Türk Tarih Kurumu Basimeri, 1947, pp. 44–47, presents a Babylonian text dated about 1700 B.C. to show that of the three types of trinomial equations of the second degree distinguished by AL-KHOWÂRIZMÎ (1) $x^2+ax=b$, (2) $x^2+b=ax$, (3) $ax+b=x^2$, type 2 is also found in Babylonian mathematics. Al-K. employed no symbols but an algebraic geometry discussion. The Babylonians like Diophantus had numerical algebra. Types (1) and (3) were found there earlier (GANDZ [21], p. 142, and THUREAU-DANGIN [15], pp. xxi–xxiv, 1–10). See also especially THUREAU-DANGIN [34].

130. H. SUTER, *Die astronomischen Tafeln des Muhammed ibn Mûsâ al-Khowârizmî in der Bearbeitung des Maslama ibn Aḥmed al-Madjrîṭî unter der Latein. Uebersetzung des Athelhard von*

Bath auf Grund der Vorarbeitung von A. Bjørnbo *und* R. Besthorn *herausgegeben und commentiert.* Copenhagen, 1914 (Danske Vidensk, Selsk., *Skrifter*, s. 7, *Hist. of Filos.*, v. 3, no. 1), Tables 58, 58a, 60. See also A. A. Bjørnbo, "Al-Chwârizmî's trigonometriske Tavler," pp. 1–17. of *Festskrift til* H. G. Zeuthen, Copenhagen, 1909.

131. Sengupta **[113]** (b) *Aryabhatiyam*, p. 11, a table of the values of 3438 sin A, with first differences for 24 angles $A = 3°45'(225')90°$. The values vary from 225 to 3438. See also A. von Braunmühl, *Vorlesungen ü. Gesch. d. Trigon.*, 2 v. Leipzig, 1900, 1903; v. 1, p. 34.

132. H. Suter, *Encyclopaedia of Islam*, v. 1, Leyden and London, 1908, pp. 112–113. Sarton [8], v. 1, pp. 666–667. J. B. J. Delambre, *Histoire de l'Astronomie au Moyen Age*, 1819, pp. 156–170.

133. There are French and German translations of his geometry based on different manuscripts, as follows: F. Woepcke, "Analyse et extraits d'un receuil de constructions gé métriques par Aboûl Wefâ," *Journal Asiatique*, s. 5, v. 5, 1855, pp. 218–256, 309–359, also as a reprint 89 pp. H. Suter, "Das Buch der geometrischen Konstruktionen des Abu'l Wefâ" (*Abh. z. Gesch. d. Naturwiss. u. d. Medizin*, Heft 4). Erlangen, 1922, pp. 94–109. The history of various attempts at geometrical constructions with compasses with a single opening is given in W. M. Kutta, "Zur Geschichte der Geometrie mit constanter Zirkelöffnung," K. Leop.-Carol. deutsch. Akad. Naturf., *Nova Acta*, v. 81, 1897, pp. 71–101+3 plates. See also L. Rodet, *Bull. d. Bibl. d. Storia d. Sci. Matem. e Fis.*, v. 16, 1883, pp. 534–542.

134. Sin $30' = 0^p31'24''55'''54^{iv}55^v = .0087265373$, which should have been $0^p31'24''55'''$-$54^{iv}0^v17^{vi} = .0087265355$. Braunmühl **[131]**, v. 1, p. 57, in this connection is decidedly erroneous. The value of sin 30′ was first given by Rodet, in *Journal Asiatique*, s. 5, v. 15, 1860, p. 303.

135. W. E. Story, *Omar Khayyàm as a Mathematician, read at a meeting of Omar Khayyàm Club of America, 6 April 1918.* Privately printed, Needham, Mass., Rosemary Press, 1919, 13 leaves +plates; also in *Twenty Years of the Omar Khayyàm Club of America*, 1921. Needham, Mass., leaves 70–72, 74, 75, 77, 78, 80, 81. G. Sarton, "The tomb of Omar Khayyâm," *Isis*, v. 29, 1938, pp. 15–19 with plate illustration of the magnificent tomb now at Nishâpûr; also J. Fleming, "A pilgrim to Omar's forgotten tomb," *Travel*, v. 58, 1932, pp. 9–14. Among several imaginary sketches of the life of Omar the following two may be mentioned: the attractively written H. Macfall, *The Three Students*, London, 1926, viii, 351 pp.; and H. Lamb, *Omar Khayyam, a Life*. Garden City, N.Y., 1934, viii, 316 pp. The chief general English articles for scholars are in *Encyclopaedia of Islam*, v. 3, 1936, p. 985–989 by V. Minorsky, and v. 1, 1912, p. 1006, art. "Djalālī," by H. Suter; and in Sarton [8], v. 1, pp. 759–761.

136. The famous English translation of this by Edward Fitzgerald first appeared in 1859 and was successively revised in 1868, 1872, 1879. A beautifully illustrated edition, "set forth in meter" by D. E. Smith, and said to be more complete and more literal than Fitzgerald's, was published at New York in 1933.

137. *L'Algèbre d'Omar Alkhayyâmî, publiée traduite et accompagnée d'extraits de manuscrits inédits* by F. Woepcke. Paris, 1851, xx, 128, 56 (Arabic text) pp.+5 folding plates. The following English translation by D. S. Kasir, professes to be a translation of an Arabic ms. in the D. E. Smith Library of Columbia University: *The Algebra of Omar Khayyam.* Doctoral diss., Teachers' College, Columbia Univ., New York, 1931, vi, 126 pp. Kasir states that the Arabic ms. he used is practically identical with the one basic in Woepcke's work. See also Tropfke [98], pp. 105–106. Special attention should be directed to a recent extensive paper concerning the binomial theorem and the extraction of cube and higher roots in Islamic mathematics: P. Luckey, "Die Ausziehung der n-ten Wurzel und der binomische Lehrsatz in der islamischen Mathematik," *Mathem. Annalen*, v. 120, 1948, pp. 217–274. The names of Omar Khayyàm and al-Bêrûnî, among many others, arise in the discussion.

138. Sarton [8], v. 2, pp. 1001–1013. R. Strothmann & T. Ruska, *Encyclopaedia of Islam*, v. 4, 1934, pp. 980–982. Suter [118], pp. 146–153.

139. *Traité du Quadrilatère attribué à Nassiruddin-el-Toussy . . . traduit par* A. Carathéodory. Constantinople, 1891, ii, 214, 157 (Arabic text) pp. Review by (a) H. Suter, *Bibl. Math.*, s. 2, v. 7, pp. 1–8, the best; (b) P. Tannery, *Bull. Sci. Math.*, s. 2, v. 16, 1892, pp. 147–152; (c) Carra de Vaux, *Jn. Asiatique*, s. 8, v. 20, 1892, pp. 176–181. See also Sister Mary Claudia Zeller, *The Development of Trigonometry from Regiomontanus to Pitiscus*, Diss. Univ. Mich., 1946, pp. 9–13.

A. VON BRAUNMÜHL, "Nassîr Eddîn Tûsi und Regiomontan," K. Leopold-Carol. deutsch. Akad. d. Naturf., *Nova Acta*, v. 81, 1897, pp. 39–68+2 plates. In his discussion of celestial spheres NASÎR ED-DÎN notes incidentally that the locus of a point on the circumference of a circle C_1 rolling inside the circumference C_2, whose radius is twice that of C_1 is a diameter of C_2; see P. TANNERY, *Recherches sur l'Histoire de l'Astronomie Ancienne.* Paris, 1893, viii, 370 pp. p. 348.

140. D. E. SMITH, "Euclid, Omar Khayyâm, and Saccheri," *Scripta Mathematica*, v. 3, 1935, pp. 5–10. In order to have here a true picture of what is known, the following facts should also be borne in mind. At least three editions of NASÎR ED-DÎN'S *Euclidis elementorum libri XII studii Nassiredini*, an Arabic work with Latin title, were published (1594, 1657, 1801). The part of it dealing with EUCLID'S fifth postulate was translated into Latin, in JOHN WALLIS, *Opera*, v. 2, 1693, pp. 669–673. See HEATH [73], v. 1, pp. 208–210; but especially BONOLA [78], where it is made clear that SACCHERI'S work on noneuclidean geometry was started by his knowledge of NASÎR ED-DÎN'S writings. If scholars later agree that in this discussion NASÎR ED-DÎN is reporting on a work of OMAR KHAYYÂM, it would certainly be a matter of exceptional interest.

141. L. BOUVAT, *Encycl. of Islam*, v. 4, 1934, pp. 994–996; DELAMBRE [132], pp. 204–211. *Ulugh Beg's Catalogue of Stars revised from all Persian Manuscripts existing in Great Britain . . .* by E. B. KNOBEL, as published by the Carnegie Institution of Washington in 1917, 109 pp., is the best. There is information about ULUGH BEG in this introduction. This Catalogue was made mainly from original observations. See also DELAMBRE [132], pp. 204–211.

142. AL-BÎRÛNÎ-SCHOY [81], pp. 92–108. Tables of sin A, $A=0(1')44°59'$, and of tan A, $A=0(1')45°$ are given here; the latter table was continued at $5'$ interval to 90°, and of sin A, at interval $1'$. See also C. SCHOY, "Beiträge zur arabischen Trigonometrie," *Isis*, v. 4, 1923, pp. 398–399, where 7 values of sin A, from $A=57°4'$ to $A=89°59'$ are given. This article of SCHOY is otherwise of great interest and value.

143. See BRAUNMÜHL [131], v. 1, pp. 72–75. If the radius $=60^p$ the cubic equation is $2700^p x = x^3+900^p \sin 3°$, whence $x=\sin 1°=1^p2'49''43'''11^{IV}$ is found. The more accurate value is $\sin 1°=1^p2'49''43'''11^{IV}14^{V}44^{VI}=0.017\,452\,406\,437$. It seems probable that the Arabs, long before the time of ULUGH BEG, knew how to find sin 1°. See WOEPCKE, "Discussion de deux méthodes Arabes pour déterminer une valeur approchée de sin 1°," *Jn. d. Math. Pures et Appl.*, v. 19, 1854, pp. 153–176; and also WOEPCKE [137], p. 125. If in Ulugh Beg's cubic we set $r=1$, we get the well-known relation $\sin 3°=3\sin 1°-4\sin^3 1°$.

144. CANTOR [12], v. 2, pp. 1–53; SARTON [8], v. 2, pp. 611–613; ARCHIBALD [72]; TROPFKE [98]; *Gli Scienziati Italiani*, v. 1, part 1, Rome, 1921, "Leonardo Fibonacci," by G. LORIA, pp. 4–12, excellent sketch of life and works, with bibliography. Most of LEONARDO'S writings have been brought together in two very large volumes, edited by Prince BALDASSARE BONCOMPAGNI: *Scritti di Leonardo Pisano*, Rome; v. 1, 1857, 459 pp., contains the *Liber Abaci*; v. 2, 1862, with *Practica Geometriae*, pp. 1–224; *Flos*, pp. 227–247; *Liber Quadratorum*, pp. 253–283.

145. See [144] and CANTOR [12], v. 2, pp. 5–35; K. VOGEL, "Zur Geschichte der linearen Gleichungen mit mehreren Unbekannten," *Deutsche Mathematik*, v. 5, 1940, pp. 217–240. This paper deals with 29 problems on pp. 228–258 of section 12 (extending from pp. 166 to 318) in the 15 sections of *Liber Abaci*. The contents of the other sections are suggested by the following headings: 1. Reading and writing of numbers in the Hindu-Arabic system; 2. Multiplication of integers; 3. Addition of integers; 4. Subtraction of integers; 5. Division of integers; 6. Multiplication of integers by fractions; 7. Further work with fractions; 8. Prices of goods; 9. Barter; 10. Partnership; 11. Alligation; 13. Rule of false position; 14. Square and cube roots; 15. Geometry and algebra, the former devoted to mensuration.

146. D. E. SMITH, "On the origin of certain typical problems," *Amer. Math. Monthly*, v. 24, 1917, pp. 64–71.

147. If u_n denotes the nth term of this series, DANIEL BERNOULLI in 1724, and EULER in 1726, found that

$$u_n=\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right]:\sqrt{5}.$$

The literature connected with this series is very extended, and in different directions, such as

golden section, decomposition of large numbers, mathematical puzzles, and older discussions of leaf arrangement or phyllotaxis. Some suggestions in this connection may be found in R. C. ARCHIBALD, (a) "Golden section" and "Fibonacci series," *Amer. Math. Monthly*, v. 25, 1918, pp. 232–238; (b) "Notes on the logarithmic spiral, golden section and the Fibonacci series," pp. 146–157 of J. HAMBIDGE, *Dynamic Symmetry*, New Haven, 1920. See also D. W. THOMPSON, *On Growth and Form*, second ed., Cambridge, 1942, pp. 912–933; L. E. DICKSON, *History of the Theory of Numbers*, Washington, D.C., v. 1, 1919, pp. 393–411; and D. YARDEN, "A bibliography of the Fibonacci sequence," *Riveon Lematematika*, v. 2, Jan. 1948, pp. 36–45, covering the period 1202–1947.

148. R. B. MCCLENON, "Leonardo of Pisa and his Liber Quadratorum," *Amer. Math. Monthly*, v. 26, 1919, pp. 1–8.

149. *Flos*, *Scritti*, p. 234 **[144]**; $x = 1^{p}22'7''42'''33^{IV}4^{V}40^{VI} = 1.36880\ 81078\ 532$ instead of 1.36880 81078 213. But more remarkable than the approximation of this root is LEONARDO's discussion of the equation in attempting to prove that a geometrical construction of a root with ruler and compasses was impossible. As BELL [5] remarks, there was nothing in algebra like the inspiration for this attempted proof until the nineteenth century. See J. P. GRAM, "Essai sur la restitution du calcul de Leonard de Pise sur l'équation $x^3+2x^2+10x=20$," Danske Vidensk. Selskabs, *Meddelelser*, no. 1, 1893, pp. 18–28; F. WOEPCKE, "Sur un essai de déterminer la nature de la racine d'une équation du troisième degré . . . ," *Jn. d. Math. Pures et Appl.*, v. 19, 1854, pp. 401–406; A. GENOCCHI, *Annali di Scienze Matem.*, *Fisiche*, v. 6, 1855, pp. 161–168; Q. VETTER, (a) "Nota alla risoluzione dell' equazione cubica di Leonardo Pisano," R. Accad. d. Sci. d. Torino, *Atti*, *Cl. d. Sci. Fis.*, *Matem. e Nat.*, v. 63, 1928, pp. 296–299; (b) also in Czechish, *Časopis pro pěstování Matem. a Fysiky*, v. 58, 1928, pp. 149–151. A solution is obtained by the method of false position. To illustrate Arabic solutions of cubics, similar to Leonardo's, but two centuries earlier, see AL-BÊRÛNÎ-SCHOY **[81]**, pp. 19, 21, where for the equation $x^3 = 1+3x$ the root is given as $1^{p}52'45''47'''13^{IV} = 1.879\ 385\ 246\ 8$ whereas the more accurate approximation is 1.879 385 241 8; for $x^3+1=3x$, $x = 0^{p}20'50''16'''1^{IV} = 0.347\ 296\ 373\ 5$, where the better value is 0.347 296 355 3. These equations come up in AL-BÊRÛNÎ's discussion of the construction of a regular polygon of nine sides. For Greek solutions of cubic equations see THOMAS **[64a]**, v. 2, pp. 133, 137–159, 163, 539–541.

150. J. G. HAGEN, *The Catholic Encyclopedia*, v. 10, New York, 1911, pp. 628–629; A. M. CLERKE, *Encycl. Britannica*, eleventh ed., v. 23; Delambre **[132]**, pp. 288–365; CANTOR **[12]**, v. 2, pp. 252–289; A. ZIEGLER, *Regiomontanus (Joh. Müller aus Königsberg in Franken) ein geistiger Vorläufer des Columbus*. Dresden, 1874, iv, 104 pp. Astronomical Ephemerides (1473–74) of REGIOMONTANUS with positions of the sun, moon and planets and the eclipses 1475 to 1506, guided COLUMBUS to America. See also pp. 304–305, 320–323 of J. D. BOND, "The development of trigonometrical methods down to the close of the XVth century (with a general account of the methods of constructing tables of natural sines down to our days)," *Isis*, v. 4, pp. 295–323, 1922. Also Sister M. C. ZELLER **[139]** pp. 17–36; *A Source Book in Mathematics*, ed. by D. E. SMITH, New York, 1929, pp. 427–433, "On the law of sines for spherical triangles" and "On the relations of the parts of a triangle."

151. In an essentially astrological work, *Tabulae Directionum Professionumque* (1490) REGIOMONTANUS gives a table of tangents for each degree of the quadrant of a circle of radius 100 000. Up to 45° the error in the last figure does not exceed 2, but later, the error is much larger.

152. D. E. SMITH, "The first printed arithmetic (Treviso, 1478)," *Isis*, v. 6, 1924, pp. 311–331. See also D. E. SMITH, *Rara Arithmetica*, Boston, 1908; *Addenda*, 1939.

153. For various printed editions of Euclid's *Elements*, see P. RICCARDI, *Saggio di una Bibliografia Euclidea*, 5 parts, Bologna, 1887–1893; C. THOMAS-STANFORD, *Early Editions of Euclid's Elements*. London, 1926, vi, 67 pp.+13 pls.

154. CANTOR, v. 2, pp. 306–344; H. STAIGMÜLLER, "Lucas Paciuolo. Eine biographische Skizze," *Z. f. Math. u. Physik, Hist. Abt.*, v. 34, 1889, pp. 81–102, 121–128. Of PACIOLI's *Sũma* there was a second edition in 1523 and of the divine proportion there was a German edition with commentary by C. WINTERBERG (Vienna, 1896, iv, 365 pp.). See also S. MORISON, *Fra Luca de Pacioli of Borgo S. Sepolcro*. New York, The Grolier Club, 1933, x, 106 pp.

155. J. B. GEIJSBEEK, *Ancient Double-Entry Book-keeping Lucas Pacioli's Treatise (A.D.*

1494—The earliest known writer on bookkeeping) reproduced and translated with reproductions, notes and abstracts from Manzoni, Pietra, Mainardi, Ympyn, Stevin, and Dafforne. Denver, Colo., 1914, iv, 182 pp.

156. In the main I have here followed CANTOR [12], v. 2, pp. 482–496, and H. WIELEITNER, "Über Cardan's Beweis für die Lösung der kubischen Gleichung," Physik.-medizinische Sozietät, *Sitzb.*, Erlangen, v. 58–59, 1928, pp. 173–176. Another point of view is set forth by TROPFKE [11], v. 3, p. 136–139; perhaps his is the more correct.

157. J. CARDAN, *The Book of My Life* (*De Vita Propria Liber*) translated from the Latin by Jean STONER, New York, 1930, xx, 331 pp.; SMITH [54]; R. B. LINDSAY, "Jerome Cardan, 1501–1576," *Amer. Jn. Physics*, v. 16, May, 1948, pp. 311–317; W. G. WALTERS, *Jerome Cardan, a Biographical Study.* London, 1898, vi, 301 pp.; H. MORLEY, *The Life of Girolamo Cardano, of Milan, Physician.* 2 v. London, 1854, xii, 304, iv, 308 pp.; S. GÜNTHER, *Allgemeine Deutsche Biographie*, v. 28, 1889, pp. 388–390; TROPFKE [98]; D. E. SMITH, "Medicine and mathematics in the sixteenth century," *Annals of Medical History*, 1917, p. 130. Of the 528 mathematicians and astronomers listed in SUTER [118], at least 87 are known to have been physicians; see also [162]. ARCHIBALD [57] tells about Cardan's work on music. JAMES ECKMAN, *Jerome Cardan.* (*Supplements* to the *Bulletin of the History of Medicine*, no. 7). Baltimore, 1946, xiv, 120 pp. Under "References" are 335 titles.

158. An English translation, with commentary, of the solutions of cubic and quartic equations occurring in CARDAN's work, is given by R. B. MCCLENON in SMITH [150], pp. 203–212. CARDAN gave the modern transformation for reducing a four-term equation of the third degree to one lacking the second-degree term; he recognized that a cubic equation had three roots, which might be negative or irrational, and he knew that the sum of the roots of a cubic equation was equal to the coefficient of the term of the second degree (see TROPFKE [11], v. 3, pp. 138–139).

159. For some details in this regard see J. W. L. GLAISHER, "Report on mathematical tables," BAAS *Report 1873*, pp. 43–45, 158; A. DEMORGAN, (a) art. "Table" in *The English Cyclopaedia, Arts and Science Section*, v. 7, 1861, cols. 984, 988; (b) "On the almost total disappearance of the earliest trigonometrical canon," RAS *Mo. Not.*, v. 6, 1845, pp. 221–228; reprinted with a small addition in *Phil. Mag.*, s. 3, v. 26, 1845, pp. 517–526. This last article is of exceptional interest and value. See also DELAMBRE [160], v. 2, pp. 1–35. In a 22-page *Canon Doctrinae Triangulorum*, Leipzig, 1551, RHETICUS has a 7-place table of the trigonometric functions at interval 10′.

160. K. LUNDMARK, "Nicolaus Kopernikus (Kopernik) and his astronomical reformation, introductory observations," Lund, Universitet, Observatoriet, *Meddelanden*, s. 2, no. 112, 1944, 18 pp. (*Historical Notes and Papers*, no. 19). A. ARMITAGE, *Copernicus, the Founder of Modern Astronomy.* London, 1938, ii, 183 pp. *Nicholas Copernicus, a Tribute of Nations.* Ed. by S. P. MIZWA. New York, 1945, xix, 268 pp. O. J. LODGE, *Pioneers of Science*, London, 1919, "Copernicus, and the motion of the earth," pp. 3–31. DREYER [89], pp. 305–344. SMITH [54]. L. C. KARPINSKI, "The progress of the Copernican theory," *Scripta Math.*, v. 9, 1943, pp. 139–154, illustr. DELAMBRE, *Histoire de l'Astronomie Moderne*, 2 v., Paris, 1821, v. 1, pp. 85–142. My reasons for calling Copernicus a Polish astronomer are that he was born in Thorn, Poland; that his father was born in Cracow, Poland; that his mother's brother was a bishop of Ermeland, Poland. Ermeland was Polish 1466–1772, and Thorn 1466–1793.

161. There was a German translation of the *De Revolutionibus* in 1879, a French translation in 1934, and an English translation by C. G. WALLIS, *On the Revolution of the Celestial Spheres*, 1939. This last is not, however, available for general distribution, but is for consultation in the Library of St. John's College, Annapolis, Md. There is available, however, *Nicolaus Copernicus, De Revolutionibus Preface and Book I translated by* J. F. DOBSON & S. BRODETSKY, *with a Biographical Note and Notes on the Translation.* RAS, *Occasional Notes*, no. 10, May 1947, ii, 32 pp., 2 plates. Even though it is not easy for a library to procure this publication, it is to be hoped that this translation, although somewhat abridged, may be continued. In order to prevent useless search the following notes may be added: In *Nature*, v. 106, p. 515, 16 Dec. 1920, a note states that DOBSON & BRODETSKY had nearly completed their translation of *De Revolutionibus.* This note was elaborated by F. E. BRASCH in *Science*, n.s., v. 64, 13 Aug. 1926, stating that printers' proof of the trans-

lation was being read but that the v. was to be published by the Oxford Univ. Press. The v. has not yet appeared. Many years ago I saw the original manuscript of *De Revolutionibus* in the City Hall at Prague. Although this Hall was destroyed during the recent war, I learn that the manuscript is still intact in Prague. See also G. McCOLLEY, "The universe of *De Revolutionibus*," *Isis*, v. 30, pp. 452–472. An admirable introduction to ideas of COPERNICUS is given by E. ROSEN in *Three Copernican Treatises: The Commentariolus of Copernicus, the Letter against Werner, the Narratio Prima of Rheticus; translated with Introduction and Notes.* New York, 1939, x, 211 pp.

162. ROBERT RECORDE (1542) was the author of four books on mathematics and one on medicine, for he was also physician to King EDWARD VI and Queen MARY. There were many editions of his *Ground of Arts* (1541?), arithmetic; his *Castle of Knowledge* (1551), astronomy, dealt with the Copernican System; the *Whetstone of Witte* (1557), algebra, was the source of our sign for equality; L. C. KARPINSKI, "The Whetstone of Witte (1557)", *Bibliotheca Math.*, s. 3, v. 13, 1913, pp. 223–238. See SMITH [4], v. 1, pp. 317–321. Also D. E. SMITH, "New information respecting Robert Recorde," *Amer. Math. Monthly*, v. 28, 1921, pp. 296–300; F. V. MORLEY, "Finis coronat opus," *Scientific Monthly*, v. 10, 1920, pp. 306–308; F. M. CLARKE, "New light on Robert Recorde," *Isis*, v. 8, 1926, pp. 50–70; F. R. JOHNSON and S. V. LARKEY, "Robert Recorde's mathematical teaching and the anti-Aristotelian movement," *The Huntington Library Bull.*, no. 7, 1935, pp. 59–87. A work including quaintly expressed biographical notes concerning seventeenth century mathematicians is: *"Brief Lives," Chiefly of Contemporaries, set down by* JOHN AUBREY *between the Years 1669 & 1696. Edited from the Author's MSS.* by ANDREW CLARK. 2 v. Oxford, 1898, xvi, 428. iv, 372 pp., 8 plates.

163. Admirable surveys of the life and work of STEVIN and of the history of decimal fractions have been given by G. SARTON, (a) "Simon Stevin of Bruges (1548–1620)," *Isis*, v. 21, 1934, pp. 214–303+2 plates; (b) "The first explanation of decimal fractions and measures (1585). Together with a history of the decimal idea and a facsimile (no. XVII) of STEVIN's Disme," *Isis*, v. 23, 1935, pp. 153–244. Then there's the recent fine volume, in Dutch, *Simon Stevin* by E. J. DIJKSTERHUIS. The Hague, 1943, x, 379 pp., 7 plates. He had earlier published "Simon Stevin und seine Bedeutung für die Geschichte der Mathematik und Naturwissenschaften," *Unterrichtsblätter f. Mathem. u. Naturw.*, v. 38, 1932, pp. 148–150. R. C. ARCHIBALD, *Mathematical Table Makers. Portraits, Paintings, Busts, Monuments, Bio-Bibliographical Notes.* New York, 1948, pp. 74–76, 82. See also F. CAJORI, *A History of Mathematical Notations*, 2 v., Chicago, 1928–1929, v. 1, pp. 154–158, 314–315, etc.

164. R. C. ARCHIBALD, *Math. Tables and Other Aids to Computation*, v. 1, pp. 400–402, 1945; v. 2, pp. 91–92, 1946.

165. A good account of VIETA and his works, with other material, is given by A. DEMORGAN in the *Penny Cyclopaedia*, v. 26, 1843, pp. 311–317; and also in *The English Cycl.-Biog.*, v. 6, 1858, cols. 361–371. See also C. HUTTON, *A Philosophical and Mathematical Dictionary*, new ed., v. 2, London, 1815; J. L. F. BERTRAND, *Éloges Académiques*, n.s., Paris, 1902, "La vie d'un savant au xvi siècle," pp. 143–146; ARCHIBALD [163], pp. 81–82; SMITH [54]; and CAJORI [163], v. 1–2.

166. Q. VETTER, "Sur l'équation du quarante-cinquième degré d'Adriaan van Roomen," *Bull. d. Sciences Math.*, s. 2, v. 54, 1930, pp. 277–283; and BRAUNMÜHL [131], v. 1, pp. 169–170. DELAMBRE [132], pp. 455–483; C. HUTTON, *Mathematical Tables*, sixth ed., London, 1822, "History of trigonometrial tables," pp. 4–9. These last two items tell something of VIETA's remarkable tables of trigonometric functions, *Canon Mathematicus seu ad Triangula*, 1679. See also DEMORGAN (a) [159], cols. 985–986 and K. HUNRATH, "Des Rheticus Canon doctrinae triangulorum und VIETA's Canon Mathematicus," *Abh. z. Gesch. d. Math.*, v. 9, 1899, pp. 211–240.

167. In a publication of 1593 VIETA derived an expression for the area of a unit circle, which was equivalent to finding the following relation:

$$2/\pi = \sqrt{\tfrac{1}{2}}\cdot\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}}}\cdot\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}}}}\cdots.$$

(The unconditional convergence of this product was proved by F. RUDIO, *Z. Math. Phys.*, v. 26, 1891, *hist.-lit. Abt.*, pp. 139–140.) This is a particular case of the relation obtained by EULER in 1737

$$\sin\theta/\theta = \cos\tfrac{1}{2}\theta\cdot\cos\tfrac{1}{4}\theta\cdot\cos\tfrac{1}{8}\theta\cdots,\ \theta<\pi.$$

168. *The Sumario Compendioso of Brother Juan Diez, the Earliest Mathematical Work of the New World.* Boston, 1921, vi, 65 pp., frontispiece, is a little work edited by D. E. SMITH. It gives a facsimile reproduction and translation of pages of chief mathematical interest, with introductory material. See also D. E. SMITH, "The first work on mathematics printed in the New World," *Amer. Math. Monthly*, v. 26, 1921, pp. 10–15. L. C. KARPINSKI, *Bibliography of Mathematical Works Printed in America through 1850*, Ann Arbor, Mich., 1940, p. 25, lists this work by "Juan Diez freyle" (Brother Juan Diez) as by Freyle, Juan Diez. The earliest arithmetic published in America was that of PEDRO DE PAZ at Mexico City in 1623; see F. CAJORI, *Isis*, v. 9, 1927, pp. 391–401+1 plate.

It is a striking fact that in the Old World not until the nineteenth century (1825) was the first book published in Athens, Greece.

169. W. R. MACDONALD, *Dict. Nat. Biog.*, v. 40, 1894, pp. 59–65. J. W. L. GLAISHER, "Napier, John," *Encycl. Britannica*, eleventh ed., v. 19, 1911, pp. 171–175; also "Logarithm," v. 16, 1911, pp. 868–877. ARCHIBALD [163], pp. 58–63. *Napier Tercentenary Memorial Volume*, ed. by C. G. KNOTT. Publ. by the R. Soc. Edinburgh, London, 1915. xii, 442 pp. [Partial contents: LORD MOULTON, "The invention of logarithms, its genesis, and growth," pp. 1–32+6 pp. of plates; P. H. BROWN, "John Napier of Merchiston," pp. 33–51; F. CAJORI, "Algebra in Napier's day and alleged prior inventions of logarithms," pp. 93–109; D. M. Y. SOMMERVILLE, "Napier's rules and trigonometrically equivalent polygons," pp. 169–176]. *Modern Instruments and Methods of Calculation. A Handbook of the Napier Tercentenary Exhibition*, edited by E. M. HORSBURGH, London, 1914, viii, 343 pp.+plates; G. A. GIBSON, "Napier and the invention of logarithms," pp. 1–16. SMITH [54]. The word logarithm, employed by NAPIER in 1614, is found much earlier in a 1553 work on divination by CASPAR PEUCER; "among the kinds of arithmanteia there is one we call by the new name logarithmanteia." For further details see F. CAJORI, *Archeion*, v. 12, 1930, pp. 229–233.

170. J. GINSBURG, "The Napier bones," SMITH [150], pp. 182–185; W. D. CAIRNS, "On the table of logarithms," SMITH [150], pp. 149–155.

171. E. W. HOBSON, *John Napier and the Invention of Logarithms, 1914*, Cambridge, 1914, 48 pp. H. S. CARSLAW, "The discovery of logarithms by Napier," *Math. Gazette*, v. 8, 1915, pp. 76–84, 115–119. R. C. ARCHIBALD, "Napier's *Descriptio* and *Constructio*," Amer. Math. Soc., *Bull.*, v. 22, pp. 182–187, 1916. *The Construction of the Wonderful Canon of Logarithms by* JOHN NAPIER. *Translated from Latin into English with Notes and a Catalogue of the Various Editions of Napier's Work by* W. R. MACDONALD. Edinburgh and London, 1889, xx, 169 pp.

172. TROPFKE [11], v. 2, pp. 217–218. The only rival of NAPIER in the invention of logarithms was the Swiss JOOST BÜRGI, who published a table of logarithms *Arithmetische und Geometrische Progress Tabulen*, at Prague in 1620. This table was conceived and constructed independently of NAPIER. The base of these logarithms was $(1.001)^{\cdot 1}$. For sources of information about BÜRGI see ARCHIBALD [163], pp. 15, 82. See also CAJORI [169] and KNOTT [169], R. A. SAMPSON, "The discovery of logarithms by Jobst Buergi," pp. 208–218+plate. Also L. DEFOSSEZ, *Les Savants du XVIII^e S. et la Mesure du Temps.* Lausanne, 1946.

173. The gap from 20000 to 90000, to ten places of decimals, was filled in by DEDECKER & ADRIAEN VLACQ in a work published at Gouda, Holland in 1628 (English edition, London, 1631). It was a comparatively easy matter to fill this gap to 10 places from what BRIGGS had already done in his *Arithmetica Logarithmica* to 14. Rushing into print with this as a second edition of the work of BRIGGS, at a time when it was known that BRIGGS had nearly completed his table to 14 places is not easily condoned. For a biography of BRIGGS by THOMAS WHITTAKER, see *Dict. Nat. Biog.* v. 6, 1886, pp. 326–327. On pp. 10–11 of his 1624 volume BRIGGS found the value of $\log_{10} 1.(000)^{5}1 = 0.(0)^{16}43429\ 44819\ 03251\ 804$. Since $\log_{10} e = .43429\ 44819\ 03251\ 82765\ldots$, and $e \approx (1+10^{-16})^{10^{16}}$, Briggs really gave $\log_{10} e$ correct to 16D.

174. *Logarithmica Britannica, being a Standard Table of Logarithms to Twenty Decimal Places.* By A. J. THOMPSON, Cambridge, Engl., Parts 1–10, 1924–1949, each part containing the logarithm of 10000 numbers. In several of the parts are some interesting facsimile reproductions, such as the will of HENRY BRIGGS, the title-pages of his work of 1624, the title and specimen pages of his tract of 1617, *Logarithmorum Chilias Prima*, and pages illustrating the relation of BRIGGS to NAPIER'S *Constructio Canonis*, 1619. See also ARCHIBALD [163], "A. J. Thompson," pp. 78–79 and portrait.

175. See E. BORTOLOTTI, *L'École Mathématique de Bologne, Aperçu historique.* Bologna, 1928,

pp. 35–38; among mathematicians at Bologna before 1800 were the following: PACIOLI, TARTAGLIA, CARDAN, FERRARI, CAVALIERI, RICCATI, AGNESI. L. BRUNSCHVICG, *Les Étapes de la Philosophie Mathématique.* Paris, 1912, pp. 162–167.

176. *Johann Kepler 1571–1630, A Tercentenary Commemoration of his Work . . . prepared under the Auspices of the History of Science Society.* Baltimore, 1931. [Contents: W. C. RUFUS, "Kepler as an astronomer," pp. 1–38; D. J. STRUIK, "Kepler as a mathematician," pp. 39–57 (admirable sketch); F. E. BRASCH, "Bibliography of Kepler's Works," pp. 86–133]. The best Kepler bibliography, with many facsimile reproductions of title pages, is *Bibliographia Kepleriana. Ein Führer durch das gedruckte Schriftum von Johannes Kepler mit 80 Faksimile.* By M. CASPAR, Munich, 1936, 160, 86 pp. Many other references to material about KEPLER including a discussion of numerous false portraits are to be found in ARCHIBALD **[163]**, pp. 34–42. DREYER **[89]**, pp. 372–412. There are exactly 9 regular solids: the 5 convex solids of the ancient Greeks, and the four star polyhedra, two discovered by KEPLER; see H. S. M. COXETER, "The nine regular solids," *Proc. of the First Canadian Math. Congress Montreal 1945*, Toronto, 1946, pp. 252–264. The best biography of BRAHE is J. L. E. DREYER, *Tycho Brahe, A Picture of Scientific Life and Work in the Sixteenth Century.* Edinburgh, 1890, xvi, 405 pp. T. BRAHE, *Opera Omnia*, Copenhagen, 15 v. 1913–1929.

177. HART **[60]**, pp. 102–124. J. J. FAHIE, (a) *Galileo his Life and Work.* New York, 1903, xvi, 401 pp. *Memorials of Galileo, 1564–1642. Portraits and Paintings, Medals, Medallions, Busts, and Statues, Monuments, Mural Inscriptions.* London, 1929. (b) "The scientific work of Galileo (1564–1642) with some account of his life and trial," *Studies in the History of Science*, ed. by C. SINGER, v. 2, Oxford, 1921, pp. 206–284+3 plates; this is a most admirable revealing of Galileo's greatness as a scientist. An excellent brief summary is to be found in chapter 6, pp. 75–95 of H. F. GIRVIN, *A Historical Appraisal of Mechanics*, Scranton, Pa., 1948, xii, 275 pp. A. KOYRÉ, "Galileo and the scientific revolution of the seventeenth century," and L. OLSCHKI, "Galileo's philosophy of science," *Philosophical Review*, v. 52, 1943, pp. 333–348 and 349–365. *Portraits of Famous Physicists, with Biographical Accounts* by HENRY CREW, New York, 1942. [An excellently edited portfolio: GALILEO, HUYGENS, NEWTON, AMPÈRE, FRESNEL, FARADAY, JOULE, CLAUSIUS, MAXWELL, GIBBS, HERTZ, ROWLAND].

LANE COOPER, *Aristotle, Galileo, and the Tower of Pisa.* Ithaca, N. Y., 1935. R. T. GUNTHER, "Galileo and the leaning tower of Pisa," *Nature*, v. 136, 1935, pp. 6–7 (review). A. S. EVE, "Galileo and scientific history. The leaning tower and other stories," *Nature*, v. 137, 1937, pp. 8–10. That GALILEO really experimented with falling bodies at the leaning tower is here made to appear somewhat uncertain, so far as evidence available is concerned.

178. The bridge, by ESSEX, across the Cam in the grounds of Trinity College, Cambridge, has cycloidal arches (BALL **[1]**, p. 287).

179. A. M. CLERKE, *Dict. Nat. Biog.*, v. 24, 1890, pp. 437–439. H. STEVENS, *Thomas Hariot the Mathematician, the Philosopher and the Scholar . . .* , London, Privately printed, 1900. Also F. V. MORLEY, "Thomas Hariot—1560–1621", *Scientific Monthly*, v. 14, 1922, pp. 59–65; F. CAJORI, "A revaluation of Harriot's *Artis Analyticae Praxis*," *Isis*, v. 11, 1928, pp. 316–324; and *Supplement to Dr. [James] Bradley's Miscellaneous Works with an Account of Harriot's Astronomical Papers*, Oxford, 1833, pp. 17–70+5 plates with facsimiles of Harriot manuscripts. Anyone desiring carefully to study HARRIOT'S *Artis Analyticae Praxis* would, in view of conflicting history statements, naturally read the survey in J. WALLIS, *A Treatise of Algebra.* Oxford, 1685, pp. 126–200, before turning to CAJORI'S "Revaluation," referred to above. The first paragraph (p. 126) is as follows: "Mr. *Harriot* in his Postumous Treatise of *Algebra* or *Analytica*, (published by Mr. *Walter Warner*, in the year 1631; soon after the first edition of Mr. *Oughtred's Clavis*, in the same year;) doth in divers things vary from the Method of *Vieta* and *Oughtred*. And hath made very many advantagious improvements in this art; and hath laid the foundation on which *Des Cartes* (though without naming him,) hath built the greatest part (if not the whole) of his *Algebra* or *Geometry.* Without which, that whole superstructure of *Des Cartes* (I doubt) had never been." Then on p. 198: "The Improvements of *Algebra* to be found in *Harriot* (as appears from what is already said,) and which (all or most of them) we owe to him; (of which it will not be amiss, before I leave him, to give a brief Recapitulation;) are chiefly these." Then follow 25 numbered statements of which no.

5 is: "Determining the Number of Roots (Affirmative, Negative, or Imaginary), in every equation. *viz.* So many as are the Dimensions of its Highest Term." This is contrary to HARRIOT's own statement in his *Artis*, pp. 89–90, Lemma, that equations have only positive roots. Directly and indirectly CAJORI shows that hardly any of WALLIS's statements quoted above are correct. But he also sums up the various contributions of importance which HARRIOT made. In H. W. TURNBULL, *Theory of Equations*, second ed., Edinburgh and London, 1944, reference is made (pp. 99–100) to "The Harriot-Descartes Rule of Signs" because the rule is "implicit in the work of Harriot". There is not the slightest foundation in fact for this statement.

180. For "Pascal on the arithmetic triangle" ed. by A. SAVITSKY, see SMITH [150], pp. 67–79; and for triangular arrays before the publication by PASCAL in 1654 see SMITH [4], v. 2, pp. 508–511.

181. F. CAJORI, (a) *William Oughtred, a Great Seventeenth-Century Teacher of Mathematics*, Chicago, 1916, 100 pp.; (b) "A list of Oughtred's mathematical symbols with historical notes," Univ. of California, *Publs. in Math.*, v. 1, pp. 171–186, 1920; (c) "On the history of Gunter's scale and the slide rule during the seventeenth century," *idem.*, pp. 187–209; (d) CAJORI [163], v. 1–2, see indices. AUBREY [162], Oughtred, v. 2, pp. 105–114. H. BOSMANS, "La première édition de la Clavis Mathematica d'Oughtred; son influence sur la géométrie de Descartes," Soc. Sci. de Bruxelles, *Annales*, v. 35, 1911, pp. 24–78; Quotation: "Celle-ci reste . . . un ouvrage original et de haute valeur, en progrès notable sur les travaux de Viète et ayant exercé une grande influence. Il n'est pas permis de la passer sous silence dans l'histoire des mathématiques."

181a. This publication is the interesting and remarkable anonymous Appendix (16 pp.) of the so-called second English edition by EDWARD WRIGHT, 1618, of Napier's *Descriptio*. The Appendix is reprinted in full in J. W. L. GLAISHER, "The earliest use of the radix method for calculating logarithms, with historical notices relating to the contributions of Oughtred and others to mathematical notation," *Quart. Jn. Math.* v. 46, pp. 125–197, 1915. The Oughtred abbreviations for the trigonometric functions are: s = sine, t = tangent, s_* = cosine, t_* = cotangent. In the Table are given the 54 values of $10^6 \log_e x$, for $x = [1(1)10(10)100(100)1000(1000)10\,000(10\,000)100\,000$ $(100\,000)900\,000;\ 6D]$; in the "Supplement of the Table for tenth and hundredth parts" are numbers from which the values for 18 more x's may be found. The errors of this table are listed in *Math. Tables and Other Aids to Computation*, v. 3, Jan. 1949.

182. F. CAJORI, *A History of the Logarithmic Slide Rule and Allied Instruments*. New York, 1909, viii, 128, x pp.

183. ARCHIBALD [57], pp. 15–16.

184. L. G. SIMONS, "Desargues on perspective triangles" and "Desargues on the 4-rayed pencil," SMITH [150], pp. 307–314. W. M. IVINS, JR., "A note on Girard Desargues," *Scripta Math.*, v. 9, 1943, pp. 33–48. G. VACCA, *Enciclopedia Italiana di Scienze Lettere ed Arti*, v. 12, 1931, p. 660. H. T. PLEDGE, *Science since 1500. A Short History of Mathematics, Physics, Chemistry, Biology.* London, 1939; "Kepler and Desargues," pp. 74–75.

185. BELL [53]; SMITH [54]; T. P. ARMSTRONG, "Pascal in England," *Notes and Queries*, London, v. 170, 8 Feb. 1936, pp. 102–103; A. WOLF, *A History of Science, Technology, and Philosophy in the 16 & 17th Centuries*. New York, 1935, pp. 223–225, 560–561 (picture of PASCAL's calculating machine). The definitive edition of the *Oeuvres* of Pascal is the one edited by L. BRUNSCHVICG, P. BOUTROUX, and F. GAZIER, 14 v., Paris, 1914–1925 (some v. second ed.). Biographies of PASCAL, his father and his sister are given pp. 1–164 of v. 1. H. BOSMANS, "Sur les Oeuvres mathématiques de Blaise Pascal," *Revue d. Questions Scientifiques*, 1929. É. PICARD, *Éloges et Discours Académiques*. Paris, 1931, "Pascal mathématicien et physicien," pp. 1–21. *Encyclopaedia of the Social Sciences*. New York, v. 12, 1934, "Pascal" by L. BRUNSCHVICG. A DESBOVES, *Étude sur Pascal et les Géomètres Contemporains, suivie de plusieurs Notes Scientifiques et Littéraires*. Paris, 1878, iv, 175 pp. +1 plate. W. H. BUSSEY, "The origin of mathematical induction," *Amer. Math. Monthly*, v. 24, 1917, pp. 199–297; "Pascal's use of complete induction," pp. 203–205. M. STUYVAERT, "Sur l'auteur de l'histoire de la roulette publiée par Blaise Pascal" *Bibl. Math.*, s. 3, v. 8, pp. 170–172, 1908.

186. ARCHIBALD [69], "limaçon."

187 PASCAL, *Oeuvres*, v. 1, pp. 243–260: "Essay pour les coniques," 1639, published on a single sheet 1640. Facsimile of this in *Isis*, v. 10, 1928 oppo. p. 16 and a translation by F. M.

CLARKE, pp. 17–20. In revised form this translation was given in SMITH [150], pp. 326–330. An earlier English edition was given by W. J. MACDONALD in Edinb. Math. Soc., *Proc.*, v. 2, 1884, pp. 19–24. And again by J. J. MILNE in *Math. Gazette*, v. 12, 1924, pp. 53–56.

188. D. BAXANDALL, *Catalogue of the Collection in the Science Museum. 1. Calculating Machines and Instruments*. London, 1926, pp. 8, 12, 13. The Museum has a replica (made in 1925–26) of Pascal's original calculating machine. DIDEROT has given a detailed description of this machine and its use in *Encyclopédie Méthodique, Mathématiques*, v. 1, Paris, 1784, pp. 136–142. See also PASCAL [185], *Oeuvres*, v. 1, pp. 291–321+2 plates of the machine; SMITH [150], "Pascal on his calculating machine," ed. by L. L. LOCKE, pp. 165–172.

189. SMITH [150], "Fermat and Pascal on probability," ed. by V. SANFORD, pp. 546–565. I. TODHUNTER, *A History of the Mathematical Theory of Probability from the Time of Pascal to that of Laplace*. Cambridge and London, 1865, and New York reprint, 1931; "Pascal and Fermat," pp. 7–21, *etc.*

190. G. MOHR, *Euclides Danicus, Amsterdam, 1672*, with an introduction by J. HJELMSLEV, and a German translation from the Danish by J. PÁL. Copenhagen, 1928, 8, 36, 41 pp. +3 folding plates. This work contains a facsimile of the Danish original. There was also a Dutch edition in 1672. See also J. HJELMSLEV, ed., "Beiträge zur Lebensbeschreibung von Georg Mohr (1640–1697)," K. Danske Vidensk. Selskab, *mathem.-fysiske Meddelelser*, v. 11, no. 4, 1931, 22 pp.+1 plate.

191. R. E. LANGER, "René Descartes," *Amer. Math. Monthly*, v. 44, 1937, pp. 495–512. BELL [53], "Gentleman, soldier, mathematician—Descartes," pp. 35–55. I. B. HART, *Makers of Science. Mathematics, Physics, Astronomy*. London, 1923, "René Descartes and coördinate geometry," pp. 125–137. TURNBULL [51], pp. 70–76. SMITH [54], KEYSER [54]. G. S. MILHAUD, *Descartes, Savant*. Paris, 1921, 249 pp. E. S. HALDANE, *Descartes: His Life and Times*. New York, 1925, 398 pp. The standard edition of Descartes' *Oeuvres* is that edited by C. ADAM & P. TANNERY, 12 v., Paris, 1897–1910. In v. 12 there is a life of Descartes by ADAM, xix, 646 pp. Of the important *Descartes correspondance publiée avec une Introduction et des Notes* by C. ADAM & G. S. MILHAUD, 4 v. have so far appeared, Paris, 1937–1947, almost 400 pp. per v.

192. In the *Geometrie* steps in the demonstrations are often omitted and we meet with such statements as "I shall not stop to explain this in more detail because I should deprive you of the pleasure of mastering it yourself"; and "I find nothing here so difficult that it cannot be worked out by any one at all familiar with ordinary geometry and with algebra." This was done purposely by Descartes so that some professed mathematicians were unable to open their mouths in criticism of him, because they were unable to follow his arguments. These deficiencies were made good in the second edition of the *Geometrie*, in Latin, with notes and additions by F. DE BEAUNE and F. v. SCHOOTEN, published as a volume of 400 pp. at Amsterdam in 1659. There was an English edition, *The Geometry of René Descartes, Translated from the French and Latin* by D. E. SMITH and M. L. LATHAM, *with a Facsimile of the First Edition 1637*. Chicago, 1925, xiv, 246 pp. In the review of this v. by T. L. HEATH in *Nature*, v. 118, 1926, pp. 400–401 occurs the following: "The translation . . . is in many places inaccurate and sometimes wholly misleading."

This publication of DESCARTES gave him priority of publication over everyone else in connection with analytic geometry from the modern point of view. It is to be borne in mind, however, that several years before 1637 FERMAT had also independently come to the conception of an analytic geometry, and gone further than DESCARTES in consideration of tangents and maxima and minima of curves. Reference may be given to TROPFKE [11], v. 6, pp. 92–169; J. L. COOLIDGE (a) "The origin of analytic geometry," *Osiris*, v. 1, 1936, pp. 231–250; (b) *A History of Geometrical Methods*, Oxford, 1940, pp. 122–128. We know that DESCARTES read HARRIOT's work [179] but that his own work was thereby in any way benefited seems decidedly doubtful; but it is barely possible that he may have been influenced by OUGHTRED's *Clavis* [181]. J. TROPFKE, "Das x Symbol der unbekannten bei Descartes und seinen Nachfolgern," *Archeion*, v. 13, 1931, pp. 300–319.

193. ARCHIBALD [59], DESCARTES' Ovals and Folium.

194. BELL [53], "The prince of amateurs—Fermat," pp. 56–72. SMITH [54]. SMITH [150], "Fermat on analytic geometry," ed. by J. SEIDLIN, pp. 389–396; "Fermat and Pascal on probability,"

ed. by V. SANFORD, pp. 546–565; "Fermat on maxima and minima," ed. by V. SANFORD, pp. 610–612. TODHUNTER [189]. H. WIELEITNER, *Mathematische Quellenbücher*, v. 1: *Rechnen und Algebra*, Berlin, 1927, vi, 75 pp.; v. 2: *Geometrie u. Trigonometrie*, 1927, viii, 68 pp.; v. 3: *Analytische u. Synthetische Geometrie*, 1928, vii, 89 pp.; v. 4: *Infinitesimalrechnung*, 1929, vii, 160 pp.; "Fermat and the equations of a line and ellipse," v. 3, pp. 15–23; "Fermat's quadrature of higher hyperbolas and the method of maxima and minima," v. 4, pp. 82–96. G. MILHAUD, "Les querelles de Descartes et de Fermat au sujet des tangentes," *Rev. générale d. Sci.*, v. 28, 1917, pp. 332–337. J. F. SCOTT, *The Mathematical Work of John Wallis*. London, 1938: "Controversy with Fermat and other members of the French School," pp. 71–82. G. WERTHEIM, "Pierre Fermat's Streit mit John Wallis, ein Beitrag zur Geschichte der Zahlentheorie," *Abh. z. Gesch. d. Mathem.*, v. 9, 1899, pp. 555–576. CANTOR [12], v. 2, "Descartes, Fermat," pp. 851–876. The best edition of the *Oeuvres* of FERMAT is that edited by P. TANNERY and C. HENRY, 4 v., Paris, 1894–1912. ORE [7b]," The converse of Fermat's theorem," pp. 326–339.

195. T. L. HEATH, *Diophantus of Alexandria, a Study in the History of Greek Algebra*. Second ed., Cambridge, 1910; "Additional notes, theorems and problems of Fermat," pp. 267–328. J. M. CHILD, "Did Fermat have a solution of the so-called Pellian equation?", *Isis*, v. 3, 1920, pp. 255–262; CHILD concludes that FERMAT did, and that it was the one which he gives. DICKSON [101], many references in the index.

196. KLEIN [68], pp. 81–85.

197. A. DEMORGAN, "Cavalieri, Buonaventura," *Penny Cyclopaedia*, v. 6, 1836, and *English Cycl.—Biography*, v. 1, 1856. F. CAJORI, "Indivisibles and 'ghosts of departed quantities,'" *Scientia*, v. 37, 1925, pp. 301–306. *The Works of Aristotle translated into English under the editorship of* W. D. ROSS, v. 6, Oxford, 1913; "De lineis insectabilibus" (concerning indivisible lines), ed. by H. H. JOACHIM, Oxford, 1908, iv, 38 pp. H. BOSMANS, "Un chapitre de l'oeuvre de Cavalieri. Les propositions XV–XXVII de *l'Exercitatio quarta*," *Mathesis*, v. 36, 1922, pp. 365–373, 446–456. M. G. SITTIGNANI, "Sulla geometria degli indivisibile di B. Cavalieri," *Periodico d. Matematiche*, s. 4, v. 13, 1933, pp. 266–288. E. WALKER, *A Study of the Traité des Indivisibles of Gilles Persone de Roberval with a View to answering, insofar as possible, the two Questions: Which Propositions contained therein are his own, and which are due to his Predecessors or Contemporaries? and what Effect if any, had this work on his Successors?* New York, 1932, vi, 272 pp. ROBERVAL (1602–1675), not of noble birth but a native of the village of Roberval, for 49 years professor at the Collège de France, Paris, is best known by the *Traité* (1693), referred to above, where the subject is treated more scientifically than by CAVALIERI in a similar work (1635). See also H. M. WALKER, "An unpublished hydraulic experiment of Roberval, 1668," *Osiris*, v. 1, 1936, pp. 726–732. Information about EVANGELISTA TORRICELLI (1608–1647) may be found in *Encycl. Britannica*, eleventh ed., v. 27, 1911; and TORRICELLI, *Opere Matematiche*, 3 v., Fienza, 1919. C. R. WALLNER, "Die Wandlungen des Indivisibilienbegriffs von Cavalieri bis Wallis," *Bibl. Mathem.*, s. 3, v. 2, 1901, pp. 230–234. C. B. BOYER, *The Concept of the Calculus. A Critical and Historical Discussion of the Derivative and of the Integral.* New York, 1939, viii, 346 pp.

198. G. W. EVANS, "Cavalieri's theorem in his own words," *Amer. Math. Monthly*, v. 24, 1917, pp. 447–451. SMITH [150], "Cavalieri's approach to the calculus," ed. by E. WALKER, pp. 605–609.

199. *James Gregory. Tercentenary Memorial Volume. Containing his Correspondence with John Collins and His hitherto Unpublished Mathematical Manuscripts together with Addresses and Essays Communicated to the Royal Society* [of Edinburgh] *July 4, 1938*. Edited by H. W. TURNBULL, London, 1938, xii, 524 pp.+5 plates. The papers here include (pp. 468–509): M. DEHN & E. HELLINGER, "On James Gregory's *Vera Quadratura*"; E. J. DIJKSTERHUIS, "James Gregory and Christiaan Huygens" (controversy); A. PRAG, "On James Gregory's *Geometriae Pars Universalis*." A. M. CLERKE, *Dict. Nat. Biog.*, v. 23, 1890, pp. 98–99. F. CAJORI, "On an integration ante-dating the integral calculus," *Bibl. Math.*, s. 2, v. 14, 1915, pp. 316–318. G. HEINRICH, "James Gregory's *Vera Circuli et Hyperbolae Quadratura*," *Bibl. Math.*, s. 2, v. 2, 1901, pp. 77–85. H. W. TURNBULL, "James Gregory; a study in the early history of interpolation," Edinb. Math. Soc., *Proc.*, s. 2, v. 3, 1933, pp. 151–172. G. HEINRICH, "Notiz zur Geschichte der Simpsonschen Regel," *Bibl. Math.*, s. 2, v. 1,

1900, pp. 90–92; discussion of a passage in GREGORY'S *Exercitationes Geometricae* (1668); parts of this work are published in F. MASERES, *Scriptores Logarithmici*, v. 2, 1796, pp. 1–19. G. GIBSON (a) "James Gregory's mathematical work: a study based chiefly on his letters," Edinb. Math. Soc. *Proc.*, v. 41, 1923, pp. 2–25; (b) "Sketch of the history of mathematics in Scotland to the end of the 18th century," *idem*, s. 2, v. 1, 1927, pp. 12–17.

200. Of the function u there have been many tables of which the first was by EDWARD WRIGHT (1550) in his *Certaine Errors in Navigation* with special reference to MERCATOR charts; see R. C. ARCHIBALD, Lambertian or lambda function," *Math. Tables and Other Aids to Computation*, v. 3, pp. 222–225, 1948.

201. Machin's result (in expanded form) was first published in W. JONES, *Synopsis palmiorum matheseos*, London, 1706, p. 263; it was also in this book, pp. 243, 263, that the symbol π for the ratio of the circumference of a circle to its diameter was first used. Making use of GREGORY'S series for $\tan^{-1} x$, MACHIN found that if $\tan \alpha = \frac{1}{5}$, $\tan 4\alpha = \frac{120}{119}$, and that if $y = 4\alpha - \frac{1}{4}\pi$, $\tan y = \frac{1}{239}$; whence MACHIN'S formula. The details of this derivation were first published by MASERES, *A Dissertation on the Use of the Negative Sign in Algebra*, London, 1758, pp. 289–290; see also MASERES, *Scriptores Logarithmici*, v. 3, 1796, pp. 157–161. It was WILLIAM SHANKS who computed π to $707D$, and D. F. FERGUSON who discovered that this value was erroneous beyond $526D$, and carried the computation along to $808D$. Using MACHIN'S formula JOHN W. WRENCH, JR., and LEVI S. SMITH checked the accuracy of this computation, details concerning which may be found in *Math. Tables and Other Aids to Computation* (*MTAC*), v. 2, 1947, pp. 245–248; v. 3, 1948, pp. 18–19. This calculation to $808D$ was intended as a companion to P. PEDERSEN'S calculation of e to $808D$; see *MTAC*, v. 2, pp. 68–69. There is a sketch of MACHIN, a professor at Gresham College, by A. M. CLERKE, in *Dict. Nat. Biog.*, v. 34, 1893.

202. A. DEMORGAN, "Huyghens, Christian," *Penny Cycl.*, v. 12, 1839, and *Engl. Cycl.—Biography*, v. 3, 1856. A. M. CLERKE, *Encycl. Brit.*, eleventh ed., v. 14, 1910. CREW **[177]**. P. LENARD, *Great Men of Science*, transl. from the second German ed., by H. S. HATFIELD. New York, 1933, "Huygens," pp. 67–83. A. E. BELL, *Christian* [*sic*] *Huygens and the Development of Science in the Seventeenth Century*. London, 1948, 220 pp.+portr., is a work of value. *The Contribution of Holland to the Sciences. A Symposium* edited by A. J. BARNOUW & B. LANDHEER, New York, 1943; "Astronomy" by JAN SCHILT, pp. 267–280; "Mathematics" by D. J. STRUIK, pp. 281–295. D. J. KORTEWEG, "La solution de Christiaan Huygens du problème de la chainette," *Bibl. Math.*, s. 3, v. 1, 1900, pp. 97–108. R. C. ARCHIBALD (a) "Discussion and history of certain geometrical problems of Heraclitus and Apollonius," Edinb. Math. Soc., *Proc.*, v. 28, 1910, pp. 152–178+5 plates; HUYGENS gave 13 solutions of these problems. (b) "Problems discussed by Huygens," *Amer. Math. Monthly*, v. 28, 1921, pp. 468–480. TODHUNTER **[189]**, Huygens, pp. 22–25, etc. ARCHIBALD **[69]**, catenary, cycloid, evolute. H. BOSMANS "Galilée ou Huygens. À Propos d'un épisode de la première application du pendule aux horlogues," *Rev. d. Questions Scient.*, s. 3, v. 22, 1912, pp. 573–586. The edition of Huygens' *Oeuvres*, of which 21 v. had been published by the Dutch Society of Sciences at Leiden, 1888–1944, is the most magnificent in appearance and detailed editing of the work of any scientist.

203. *Archimedes, Huygens, Lambert, Legendre. Vier Abhandlungen über die Kreismessung. Deutsch herausgegeben und mit einer Übersicht über die Geschichte des Problems von den ältesten Zeiten bis auf unsere Tage, versehen von* R. RUDIO. Leipzig, 1892, viii, 166 p.; text statement on p. 40.

204. J. F. SCOTT, (a) *The Mathematical Work of John Wallis D.D., F.R.S.* (*1616–1703*). London, 1938, xii, 240 p., portrait frontispiece. (b) "John Wallis as a historian of mathematics, *Annals of Science*, v. 1, 1936, pp. 325–337. A. DEMORGAN, "Wallis, John," *Penny Cycl.*, vol. 27, 1843, and *Engl. Cycl.—Biography*, v. 6, 1858. AUBREY **[162]**, Wallis, v. 2, pp. 280–283. T. P. NUNN, "The Arithmetic of Infinities," *Math Gazette*, v. 5, pp. 345–346, 1910 and 377–386, 1911; an analysis of WALLIS's greatest work, *Arithmetica Infinitorum*, first published in 1655. We here find for the first time the familiar symbol, ∞, to denote infinity. SMITH **[150]**, "Wallis on imaginary numbers" ed. by D. E. SMITH, pp. 46–54; "Wallis on general exponents" ed. by E. M. SANFORD, pp. 217–218; "Wallis and Newton on the binomial theorem for fractional and negative exponents," ed. by D. E. SMITH, pp. 219–223. G. ENESTRÖM, "Die geometrische Darstellung imaginärer Grössen bei Wallis,"

Bibl. Math., s. 3, v. 7, 1907, pp. 263–269. A. CAYLEY, "The investigation by Wallis of his expression for π," *Quart. Jn. Math.*, v. 23, 1889, pp. 165–169, and *Math. Papers*, v. 13, 1897, pp. 22–25. G. A. DICKINSON, "Wallis's product for $\frac{1}{2}\pi$," *Math. Gazette*, v. 21, 1937, pp. 135–139. H. WIELEITNER, "Die Verdienste von John Wallis um die analytische Geometrie " *Das Weltall*, v. 29, 1930, pp. 56–60. W. LOREY, "Bruchaufgaben und Reihensumme nach Wallis," *Unterrichtsblätter f. Mathem. u. Naturw.*, v. 41, 1935, pp. 57–58. W. KUTTA, "Elliptische und andere Integrale bei Wallis," *Bibl. Math.*, s. 3, v. 2, 1901, pp. 230–234. A. PRAG, "John Wallis 1616–1703. Zur Ideengeschichte der Mathematik im 17. Jahrhundert," *Quellen und Studien* . . . , v. 1B, 1930, pp. 381–412. F. CAJORI, "Controversies on mathematics between Wallis, Hobbes, and Barrow," *Math. Teacher*, v. 22, 1929, pp. 146–151. D. E. SMITH, "John Wallis as a cryptographer," Amer. Math. Soc. *Bull.*, v. 24, 1917, pp. 82–96; WALLIS seems to have served his country as expert cryptographer for 60 years; VIETA—gave similar service in France. WALLIS's *Opera Mathematica*, 3 large v., Oxford, 1695, 1693, 1699.

Apart from the circle, the first three curves (see ARCHIBALD [69]) to be rectified were as follows: logarithmic spiral by TORRICELLI in 1640; semi-cubical parabola by WILLIAM NEILE (a pupil of Wallis and a friend of CHRISTOPHER WREN) in 1657; and cycloid by WREN in 1658. Neile's rectification was based on principles set forth in Wallis's *Arithmetica Infinitorum*. In 1657 HUYGENS first showed that the rectification of a parabola depended upon the quadrature of a rectangular hyperbola. Learning of this result, HENDRIK VAN HEURAET, a Dutch disciple of DESCARTES, made advances, published in the 1659 edition of DESCARTES' geometry. The names of NEWTON, FERMAT, GREGORY also come up in the admirable survey of J. E. HOFMANN, "Über die ersten logarithmischen Rektifikationen. Eine historisch-kritische Studie in vergleichender Darstellung," *Deutsche Math.*, v. 6, 1941, pp. 283–303. This article is based on his earlier important discussions: "Nicolaus Mercators Logarithmotechnia (1669)" and "Weiterbildung der logarithmischen Reihe Mercators in England," *Deutsche Math.*, v. 3, 1938, pp. 446–466, 598–605; v. 4, 1939, pp. 556–562; v. 5, 1940, pp. 358–375. JAMES GREGORY in 1667 seems to have been the first to show (or at least to publish the result) that if $xy=1$ is the equation of a rectangular hyperbola, the area between the two ordinates $y=1$ and $y=a$, is $\log_e a$; see G. ENESTRÖM, *Bibl. Math.*, s. 3, v. 11, 1911, p. 239.

205. HISTORY OF SCIENCE SOCIETY, ed., *Sir Isaac Newton, 1727–1927. A Bicentenary Evaluation of His Work*, Baltimore, 1928, 9+351 pp. [Contents: "Newton in the light of modern criticism" by D. E. SMITH, pp. 3–11; "Newton and optics" by D. C. MILLER, pp. 15–48; "Newton's philosophy of gravitation with special reference to modern relativity ideas" by G. D. BIRKHOFF, pp. 51–64; "Newton's influence upon the development of astrophysics" by W. W. CAMPBELL, pp. 67–86; "Newton's dynamics" by M. I. PUPIN, pp. 89–97; "Newton as an experimental philosopher" by P. R. HEYL, pp. 101–108; "Developments following from Newton's works" by E. W. BROWN, pp. 111–124; "Newton's twenty years' delay in announcing the law of gravitation" by F. CAJORI, pp. 127–188; "Newton's fluxions" by F. CAJORI, pp. 191–200; "Newton's work in alchemy and chemistry" by L. C. NEWELL, pp. 203–255; "Newton's place in the history of religious thought" by G. S. BRETT, pp. 259–273; "Newton in the Mint" by G. E. ROBERTS, pp. 277–298; "Newton's first critical disciple in the American colonies—John Winthrop" by F. E. BRASCH, pp. 301–338; . . . Material displayed in the Newton exhibit at the Amer. Museum of Nat. Hist., pp. 342–351.]

206. MATHEMATICAL ASSOCIATION, England, ed., *Isaac Newton, 1642–1727. A Memorial Volume*, London, 1927, 8+181 pp.+plates. [Contents: "Two unpublished documents of Newton" by D. E. SMITH, pp. 16–34; "Letters from Newton in Corpus Christi College, Oxford," ed. by J. L. E. DREYER, pp. 35–44; "Newton and interpolation" by D. C. FRASER, pp. 45–69; "Newton's work in optics" by E. T. WHITTAKER, pp. 70–74; "Newton's problem of the solid of least resistance" by A. R. FORSYTH, pp. 75–86; "Newton's work on the theory of the tides" by J. PROUDMAN, pp. 87–95; "Newton's contribution to the geometry of conics" by J. J. MILNE, pp. 96–114; "Newton on plane cubic curves" by H. HILTON, pp. 115–116; "Newton and the art of discovery" by J. M. CHILD, pp. 117–129; "Newton's influence on method in the physical sciences" by A. E. HEATH, pp. 130–133; "Plagiarism in the seventeenth century, and Leibniz" by L. J. RUSSELL, pp. 134–136; "The contemporary significance of Newton's metaphysics" by E. A. BURTT, pp. 137–140; "Newton and his homeland; the haunts of his youth" by J. A. HOLDEN, pp. 141–143; "Trinity College in the time of Newton" by G. N. WATSON, pp. 144–147; "A Newton Bibliography" by H.

ZEITLINGER, pp. 148–170; "Portraits of Sir Isaac Newton" by D. E. SMITH, pp. 171–178; "The portrait medals of Newton" by D. E. SMITH, pp. 179–180.]

207. THE ROYAL SOCIETY *Newton Tercentenary Celebrations 15–19 July 1946.* Cambridge, University Press, 1947, xvi, 92 p.+6 plates (3 portraits of Newton, Newton's rooms at Trinity, Woolsthorpe Manor, Newton letter). [Contents: "Address of welcome to the delegates" by R. ROBINSON, President of the R. S., pp. 1–2; "Newton" by E. N. DA C. ANDRADE, pp. 3–23; "Address of welcome to the delegates" by G. M. TREVELYAN, Master of Trinity, pp. 24–26; "Newton, the man" by the late *Lord* KEYNES (in 1942), pp. 27–34; "Newton and the infinitesimal calculus" by J. HADAMARD, pp. 35–42; "Newton and the atomic theory" by S. I. VAVILOV, pp. 43–55; "Newton's principles and modern atomic mechanics" by N. BOHR, pp. 56–61; "Newton: the algebraist and geometer" by H. W. TURNBULL, pp. 62–72; "Newton's contributions to observational astronomy" by W. ADAMS, pp. 73–81; "Newton and fluid mechanics" by J. C. HUNSAKER, pp. 82–90.]

208. R. E. LANGER, "Isaac Newton," *Scripta Mathem.*, v. 4, pp. 241–255, 1936. H. W. TURNBULL, *The Mathematical Discoveries of Newton.* London and Glasgow, 1945, viii, 68 pp.+frontispiece portr.; admirable little book. W. W. BALL, "Newton," *Math. Gazette*, v. 7, pp. 349–360, 1914. CREW [177]. HART [60], pp. 138–172. BELL [53], "On the seashore—Newton," pp. 90–116. SMITH [54]. A. DEMORGAN, *Essays on the Life and Work of Newton. Edited with Notes and Appendices* by P. E. B. JOURDAIN. Second ed., Chicago, 1914, xiv, 198 pp. S. BRODETSKY, *Sir Isaac Newton a Brief Account of His Life and Work.* London, 1927, xii, 161 pp.+plate. J. CRAIG, *Newton at the Mint.* Cambridge, Engl., 1946, vi, 128 pp., 4 plates, J. W. N. SULLIVAN, *Isaac Newton, 1642–1727. . . . With a memoir of the author* by C. SINGER. London, 1938, xx, 275 pp.; very attractively written. L. T. MORE, *Isaac Newton, a Biography.* New York and London, 1934, xiv, 675 pp.; of the long biographies of Newton this is a decided advance on Brewster's, but since the author is hampered by the fact that he is out of sympathy with modern developments in physics, he fails to see Newton's work in perspective.

In R. DE VILLAMIL, *Newton: The Man.* London, 1931, vi, 112 p.+portrait frontispiece, is a list of 1896 books originally in Newton's library. (Brown University has the fine copy of DEMOIVRE's *Annuities upon Lives*, 1725, here listed.) In July 1943, 860 of these volumes, including the septuagint Old Testament in Greek, and BARROWS' edition of EUCLID's Elements, were purchased by The Pilgrim Trust and presented to the Library of Trinity College, Cambridge, which had been founded by ISAAC BARROW, who, before resigning his chair (1669) in NEWTON's favor, and later becoming Master of Trinity, declared that he could not make a Bible out of his EUCLID or a pulpit out of his mathematical chair. See *The Times*, London, 1943, Apr. 12, p. 6d; 14, p. 5d; June 30, p. 2c. In "Sir Isaac Newton's early study of the Apocalypse," *Pop. Astron.*, v. 34, 1926, pp. 75–78, F. CAJORI describes NEWTON's much annotated copy of HENRY MORE's work in the Library of the University of California at Berkeley.

Newton's last London residence (1710–1725) on St. Martin's St., Leicester Square, was demolished in 1913 by order of the London City Council, but the woodwork of the "fore parlour" was in 1939 re-erected in America, in the Library of the Babson Institute, BABSON Park, Mass., where there is also a large collection of Newtoniana. A picture of this "parlour" appears in F. E. BRASCH, "Newton's portraits and statues," *Scripta Math.*, v. 8, 1941, pp. 199–227, 24 of the pages being 12 plates, 10 of which are portraits. See also "Sale of Newtoniana," *Nature*, v. 138, 1936, p. 195.

An interesting item is MARJORIE H. NICOLSON, *Newton Demands the Muse; Newton's Opticks and the Eighteenth Century Poets.* Princeton, N. J., 1946, xi, 177 pp.

209. F. CAJORI, *A History of the Conceptions of Limits and Fluxions in Great Britain from Newton to Woodhouse.* Chicago and London, 1919, viii, 299 pp.+frontispiece. A. DEMORGAN, "Fluxions, fluents, method, notation and early history," *Penny. Cycl.*, v. 10, 1838. F. CAJORI, "The spread of Newtonian and Leibnizian notations of the calculus," Amer. Math. Soc., *Bull.*, v. 27, 1921, pp. 453–458. G. A. GIBSON, (a) "Berkeley's Analyst and its critics: an episode in the development of the doctrine of limits," *Bibl. Math.*, s. 2, v. 13, 1899, pp. 65–70; (b) Edinb. Math. Soc., *Proc.*, v. 17, 1899, pp. 9–32. "Berkeley and Newton," *Math. Gazette*, v. 7, pp. 418–421, 1914. GEORGE BERKELEY (1685–1753), bishop of Cloyne, mathematician, and one of the most subtle and original meta-

physicians, has considerable interest for both American and mathematical history. He lived in Newport, Rhode Island, Jan. 1729–Aug. 1731 and there wrote his longest work *Alcephron or the Minute Philosopher* (publ. 1732). He was raised to the bishopric of Cloyne in 1734. His first printed work (1707) consisted of two mathematical tracts. In 1734 he published *The Analyst: or a Discourse addressed to an infidel Mathematician* [Halley]. *Wherein it is examined whether the Object, Principles, and Inferences of the Modern Analysis are more distinctly conceived, or more evidently deduced, than religious Mysteries and Points of Faith. By the Author of The Minute Philosopher.* This precipitated the *Analyst* controvery in which many participated. See "Bibliography of George Berkeley," *Scripta Math.*, v. 3, 1935, pp. 81–83.

SMITH [150], "Wallis and Newton on the binomial theorem for fractional and negative exponents," ed. by D. E. SMITH, pp. 219–223; "Newton on the binomial theorem for fractional and negative exponents," ed. by E. M. SANFORD, pp. 224–228; "Newton on fluxions," ed. by E. WALKER, pp. 613–618.

210. A. DEMORGAN, "Barrow, Isaac," *Penny Cycl.*, v. 3, 1835, and *English Cycl.—Biography*, v. 1, 1856. The Soc. for the Promotion of Christian Knowledge published a popular biography, with slighting of scientific matters: P. H. OSMOND, *Isaac Barrow, His Life and Times*. London, 1944, vi, 230 pp., 4 plates. AUBREY [162], BARROW, v. 1, pp. 87–94. BARROW was professor of geometry at Gresham College (1662–64) and first Lucasian professor of mathematics at Cambridge (1664–69). His *Mathematical Works* (Cambridge, 1860, xx, 414, ii, 320 pp., 27 folding plates) were edited by W. WHEWELL, and his *Geometrical Lectures . . . Translated, with Notes and Proofs, and a Discussion on the Advance made therein on the Work of his Predecessors in the Infinitesimal Calculus*, edited by J. M. CHILD, was published at Chicago, 1916, xiv, 218 pp.

211. The composition and publication of the *Principia* was due to a young man about thirty years of age, **EDMOND HALLEY**, who became the successor of WALLIS as Savilian professor of geometry at Oxford. He has been called "flapper HALLEY" that is, according to SWIFT, one who flaps another to remind him of a task to be performed. The huge task of seeing the work through the press involved him in considerable personal financial outlay. The first edition was very small and probably sold for ten or twelve shillings a copy. It went out of print so quickly it was difficult to obtain even in 1691. Before World War II a copy could be occasionally picked up for $100, but recently five times that amount has been obtained. Many corrections and developments of the first edition occurred in the second edition published in 1713 (750 copies), which sold unbound at fifteen shillings a copy. (Of this edition there were pirated Amsterdam reprints in 1714 and 1723.) There was a reprint of the second edition at Glasgow in 1871. For interesting information about the second ed., see W. G. HISCOCK (a) "The war of the scientists; new light on Newton and Gregory [David]," *Times Lit. Suppl.*, v. 35, 1936, p. 34; (b) *David Gregory, Isaac Newton and their Circle*. Oxford, 1937, ix, 48 pp. The printing of the third edition begun in 1723, when NEWTON was 80 years old, was not finished till 1726, the year before he died. The student of the *Principia* may find the following useful: W. W. R. BALL, *An Essay on Newton's 'Principia.'* London, 1893, x, 175 pp. The *Principia* was first translated into English (from the third ed.) by ANDREW MOTTE (1729); R. THORPE published a translation of book I (1777). These are the bases of the English ed. by F. CAJORI, posthumously published by the Univ. of California in 1934, xxvi, 680 pp.; second printing 1946. A rather scathing review by R. A. SAMPSON appeared in *Nature*. v. 135, 1935, pp. 128–129. See also *Scripta Math.*, v. 3, 1935, pp. 69–74, 186–187; and *Eureka, The Archimedean's Jn.*, no. 10, Mar. 1948, pp. 3–5. To the second edition of the *Principia*, HALLEY contributed a Latin poem entitled (in translation): "To this mathematico-physical work of the illustrious Mr. Isaac Newton, an achievement which is the greatest glory of our age and nation." Of this poem four English editions have been published: (1). 1759, by "Eugenio," in *General Mag. of Arts and Sciences*, London, v. 1, p. 4, and E. F. MACPIKE, *Correspondence and Papers of Edmond Halley*. Oxford, 1932, pp. 207–208. (2). 1934, by L. J. RICHARDSON, in Cajori's edition of the *Principia*, noted above, pp. xiii–xv. (3). Jan. 1943, by A. Weinstein, *Science*, n.s., v. 97, p. 70. (4). Aug. 1943, by Mrs. S. CHAPMAN, *Nature*, v. 152, p. 231. See also R. C. ARCHIBALD, "Mathematicians, and poetry and drama," *Science*, n.s., v. 89, 1939, pp. 48–49, footnote 74.

Good short sketches of Halley are those of A. DEMORGAN, *Penny Cycl.*, v. 12, 1839; or *English*

Cycl.—Biog., v. 3, 1856; and A. M. CLERKE, *Dict. Nat. Biog.*, v. 24, 1890, pp. 104–109. But the most comprehensive and up-to-date source is *Correspondence and Papers of Edmond Halley, preceded by an Unpublished Memoir of his Life by one of his Contemporaries and the Éloge by D'Ortous de Mairan, arranged and edited* by E. F. MACPIKE. Published for the [Amer.] History of Science Society, Oxford, Clarendon Press, 1932, xiv, 300 pp., 11 plates. We have already referred [**86**, **103**] to publications of HALLEY editions of works by APOLLONIUS and MENELAUS. He was also a notable contributor to the foundation of statistical and actuarial theory (see H. BÖCKH, "Halley als Statistiker," Inst. Intern. de Statistique, *Bull.*, Rome, v. 7, 1893, pp. 1–24; and C. WALFORD, *Insurance Cyclopaedia*, London, v. 5, 1878, pp. 616–618, and v. 1, 1871, pp. 360–362). One of his outstanding contributions in the field of astronomy was when, in 1682, he identified, and computed the orbit of a comet with a period of about 76 years, last seen in 1910. He observed that the orbit was nearly identical with those of the comets in 1607 and 1531, and he also found that this was the same comet which was recorded as appearing in 1456, 1301, 1145, 1066. Compare *Pop. Astron.*, v. 42, 1934, pp. 191–201, where 29 "observations," 240 B.C.–1910 A.D. are discussed.

212. "Gauss's line" is a term also used in this connection since GAUSS showed (Zach's *Monatliche Korrespondenz*, 1810, p. 120; GAUSS, *Werke*, v. 4, pp. 385–392; see also *Mess. Math.*, v. 4, 1868, p. 137) that the locus of the centers of ellipses tangent to the sides of a complete quadrilateral is the line through the centers of the three diagonals. TERQUEM noted in 1845 (*Nouv. Annales de Math.*, v. 4, p. 545) that the five NEWTON lines, of the quadrilaterals formed by omitting one of five tangents to a conic, are concurrent. That the middle points of the three diagonals of a complete quadrilateral are collinear was first pointed out by F. J. CONNOR in *Ladies Diary*, 1795, p. 47.

213. *Sir Isaac Newton's Enumeration of Lines of the Third Order, Generation of Curves by shadows, . . . , translated from the Latin with notes and examples*, by C. R. M. TALBOT, London, 1861, xii, 5–140 pp., 12 folding plates; W. W. R. BALL, "Newton's classification of cubic curves," London Math. Soc., *Proc.*, v. 22, pp. 104–143, 1890; *Encykl. d. math. Wissen.*, v. III_2, pp. 461–462. See also R. H. GRAHAM, "Newton's influence on modern geometry," *Nature*, v. 42, 1890, pp. 139–142; C. TAYLOR, "On a section of Newton's Principia in relation to modern geometry," Cambridge Phil. Soc., *Proc.*, v. 3, 1880, pp. 359–360.

214. The memoranda of DAVID GREGORY for the period 1696–1708 give interesting information about the publication of the first edition of *Arithmetica Universalis*, edited and published by WM. WHISTON in 1707; see HISCOCK [**211**]. Other Latin editions were published in 1732 and 1761, at London, Leiden, and Milan. The first English edition appeared in 1720, the second in 1728, and another in 1769. There were also a French edition (1802) and various volumes of illustrations. For further details see G. J. GRAY, *A Bibliography of the Works of Sir Isaac Newton*, second ed. rev. and enl., Cambridge, Engl., 1907.

215. The most remarkable passage in Newton's *Arithmetica Universalis* is the Rule for the discovery of Imaginary Roots, stated without any proof (see H. W. TURNBULL, *The Mathematical Discoveries of Newton*, 1945, pp. 49–51). How NEWTON discovered such a complicated Rule is a mystery. It was not until nearly 160 years later that a proof of the rule was finally discovered by J. J. SYLVESTER, in 1864–65; see R. Soc. London, *Trans.*, v. 154, 1864, pp. 579–666+2 pls. and *Proc.*, v. 13 (1863–64), pp. 179–183, and v. 14 (1865), pp. 268–270; also in SYLVESTER, *Coll. Math. Papers*, v. 2, Cambridge, 1908, pp. 376–479, 493–494. See also the summary by H. J. PURKISS, *Oxford, Cambr. and Dublin Mess. Math.*, v. 3, 1865, pp. 129–142. The Rule is a generalization of DESCARTES' Rule of Signs, but it is a much deeper result. In a leaflet of 1896 SYLVESTER noted

> "Descartes' and Newton's law lay hid in night:
> Heaven touched my heart with fire, and all was light."

The American poet SIDNEY LANIER, in his "Ode to The Johns Hopkins University" has the following (lines 10–14):

> "From far the sages saw, from far they came
> And ministered to her,
> Led by the soaring-genius'd Sylvester
> That, earlier, loosed the knot great Newton tied,
> And flung the door of Fame's locked temple wide."

F. CAJORI has given "A history of the arithmetical methods of approximation to the roots of numerical equations of one unknown quantity," Colorado College, *Publication, General Series*, nos. 51–52, *Science Series*, v. 12, 1910, pp. 171–287. The first part, before VIETA, pp. 171–182; the second part, pp. 182–215, deals with VIETA, NEWTON, LAGRANGE; the third part, pp. 217–288, modern times, discusses the NEWTON-RAPHSON method and allied processes, RUFFINI-HORNER method, WEDDLE's method, DANDELIN-GRÄFFE method, method by infinite series, etc. See also the rather trivial M. A. NORDGAARD, *A Historical Survey of Algebraic Methods of Approximating the Roots of Numerical Higher Equations up to the Year 1819*. Diss. Columbia, Teachers College, New York, 1922, vi, 65 pp. "Vieta's extension of the Hindu method of approximating to complete equations," pp. 24–32; "Newton's method of approximation," pp. 33–46; "Certain effective but non-practicable methods," pp. 47–52; "Horner's method, similar methods by Ruffini and by the Chinese of the thirteenth century," pp. 53–57. F. CAJORI, "Newton's solution of numerical equations by means of slide rules," Colorado College, *Publication, General Series* no. 95, *Engineering Series*, v. 1, no. 18, 1917, pp. 245–253. C. RUNGE, "Separation und Approximation der Wurzeln," *Encykl. d. mathem. Wissen.*, v. 1, pp. 404–448, 1899. The so-called "Gräffe method" is really due to earlier writers: EDWARD WARING (1734–1798), *Meditationes Analyticae*, 1776, pp. 311 f., or GERMINAL DANDELIN (1794–1847), "Recherches sur la résolution des équations numériques," Acad. roy. d. Sci. d. Belgique, *Mémoires*, v. 3, 1826—pp. 7–71, 153–159, but especially p. 48 f. The method was also suggested independently by N. I. LOBACHEVSKY (1793–1856), *Algebra ili Ischislenie Konechnykh Velichin* [Algebra or Calculus of Finite Quantities]. Kazan, 1834, 528 p.; p. 157, §257. A few other accounts mainly expository, but some expansive, are: E. T. WHITTAKER & G. ROBINSON, *The Calculus of Observations*, third ed., seventh impression, London, 1940, pp. 106–120. S. BRODETSKY and G. SMEAL, "Method for complex roots of algebraic equations," Camb. Phil. Soc., *Proc.*, v. 22, pp. 83–87, 1924. C. A. HUTCHINSON, *Amer. Math. Monthly*, v. 42, 1935, pp. 149–163. D. H. LEHMER, "The Graeffe process as applied to power series," *Math. Tables and Other Aids to Comput.*, v. 1, pp. 377–383, 1945, and v. 3, pp. 227, 1948.

It was the Belgian professor of Physics, G. P. DANDELIN, referred to above, who discovered the beautiful theorem that the foci of the conic cut by a plane, p, from a right circular cone, are the points of contact of the spheres tangent to the cone and to p. Furthermore, the planes of the contact circles of the spheres intersect p in the directrices of the conic. See DANDELIN, Acad. d. Sci. d. Bruxelles, *Nouv. Mém.*, v. 2, 1822, pp. 171–173.

216. As yet I have not been able to trace this quotation to its source: the earliest reference I can give is to F. R. MOULTON, *Introduction to Astronomy*, New York, 1906, p. 199. If this story is true it would seem as if the Queen in question must have been SOPHIA DOROTHEA, wife of FREDERICK WILLIAM I, and mother of FREDERICK the Great. She was the first Queen of Prussia during the last 10 years that LEIBNIZ lived.

217. BELL [**53**], "Master of all trades—Leibniz," pp. 117–130. SMITH [**54**]. KEYSER [**54**]. CAJORI [**4**], pp. 205–219, etc. LENARD [**202**], "Leibniz," pp. 111–118. SMITH [**150**], "Leibniz and the Bernoullis on the polynomial theorem," ed. by J. GINSBURG, pp. 229–231; "Leibniz on the calculus," ed. by E. WALKER, pp. 619–626. J. M. CHILD, *The Early Mathematical Manuscripts of Leibniz. Translated from the Latin Texts published by Carl Immanuel Gerhardt with critical and historical Notes*. Chicago and London, 1920, iv, 238 pp. F. CAJORI. "Grafting the theory of limits on the calculus of Leibniz," *Amer. Math. Monthly*, v. 30, 1923, pp. 223–234. BOYER [**197**], "Newton and Leibniz," pp. 187–223.

218. E. STEPHENS, "Bibliography on general (or fractional) differentiation," Washington Univ., *Studies, Scientific Series*, v. 12, pp. 149–152, 1925.

219. T. MUIR, (a) *The Theory of Determinants in the Historical Order of its Development*, v. 1, up to 1841, second ed. London, 1906, xii, 491 pp.; v. 2, 1841–1860, 1911, xvi, 476 pp.; v. 3, 1861–1880, 1920, xxvi, 503 pp.; v. 4, 1880–1900, 1923, xxxi, 508 pp. (b) *Contributions to the History of Determinants*, [v. 5], 1906–1920, 1930, xxiv, 408 pp. Leibniz, v. 1, pp. 6–10. SMITH [**150**], "Leibniz on determinants," ed. by T. F. COPE, pp. 267–270.

220. L. L. LOCKE, "The contribution of Leibniz to the art of mechanical calculation," *Scripta Math.*, v. 1, 1933, pp. 315–321+1 plate. SMITH [**150**], "Leibniz on his calculating machine," ed. by M. KORMES, pp. 173–181. LEIBNIZ began work on the calculating machine about 1671 and had

completed one machine in 1694 and a second about 1706. See R. MEHMKE & M. D'OCAGNE, *Encycl. d. Sci. Math.*, tome 1, v. 4, fasc. 2, 1908, pp. 248–251.

221. F. CAJORI, "Leibniz the master-builder of mathematical notations," *Isis*, v. 7, 1925, pp. 419–429. D. MALMKE, "Leibniz als Begründer der symbolischen Mathematik," *Isis*, v. 9, 1927, pp. 279–293. Also CAJORI [**163**], see indices, v. 1–2.

222. A. DEMORGAN, *Penny Cycl.*, v. 4, 1835, "Bernoullis." BELL [**53**], "Nature or Nurture?—The Bernoullis," pp. 131–138. *Encycl. Britannica*, eleventh ed., v. 3, 1910, pp. 803–805. James=Jacques=Jakob=Giacomo; John=Jean=Johann=Giovanni. *Gedenkbuch der Familie Bernoulli zum 300. Jahrestage ihrer Aufnahme in das Bürgerrecht 1622–1922.* Basel, 1922, viii, 287 pp.+4 plates. Interesting portraits, facsimiles of writing, etc. C. v. BEHR-PINNOW, "Begabungsvererbung der Familie Bernoulli," *Naturwissenschaften*, v. 22, 1934, pp. 717–721. R. WOLF, *Biographien zur Kulturgeschichte der Schweiz*, 4 v. Zürich, v. 1, 1858, pp. 133–166. CANTOR [**12**], v. 3, "Jakob und Johann Bernoulli," pp. 89–98, 215–248, etc. JAMES BERNOULLI, *Unendliche Reihen (1689–1704) . . . aus dem Lateinischen übersetzt und herausgegeben*, by G. KOWALEWSKI. (*Ostwald's Klassiker* series). Leipzig, 1909, 142 pp. A. PROCISSI, "Il Problema bernoulliano 'De quadrisectione trianguli scaleni per duas normales rectas,'" *Per. d. Matem.*, s. 4, v. 14, 1934, pp. 1–27. This problem was propounded and solved by BERNOULLI in *Acta Eruditorum*, 1687, pp. 617–623. It was also solved by L'Hospital before 1704. See LORIA's history of the problem, *Bibl. Math.*, s. 3, v. 4, 1903, pp. 48–51.

223. See, for example, D. N. LEHMER, "Cornu's Spiral as transition curve," *California Jn. Technology*, v. 3, 1904, pp. 71–82. A. L. HIGGINS, *The Transition Spiral and its Introduction to Railway Curves*. London, 1921, viii, 112 pp. A. N. TALBOT, *The Railway Transition Spiral*. Sixth ed. New York, 1927, viii, 96 pp.

224. R. C. ARCHIBALD, "Euler's integrals and Euler's spiral—sometimes called Fresnel integrals and the clothoïde or Cornu's spiral," *Amer. Math. Monthly*, v. 25, 1918, pp. 276–282.

225. R. C. ARCHIBALD, "The logarithmic spiral," pp. 146–151 of J. HAMBIDGE, *Dynamic Symmetry*. New Haven, 1920.

226. TODHUNTER [**189**], "James Bernoulli," pp. 56–77; "Daniel Bernoulli," pp. 213–238, etc. Of BERNOULLI's *Ars Conjectandi* there was a French edition of the first part, "*L'art de Conjecturer*." Caen, 1801, iv, 180 p.; "Observations, eclaircissements et additions" occupy pp. 101–180. The German edition of all four parts, *Wahrscheinlichkeitsrechnung* (*Ars Conjectandi*), transl. and ed. by R. HAUSSNER, appeared in the *Ostwald's Klassiker* series, 2 v., Leipzig, 1899, 162, 172 pp. An English translation of the first three chapters of the second book of the *Ars*, by F. MASERES appeared in his *The Doctrine of Permutations and Combinations*. London, 1795, pp. 35–213; also in MASERES, *Scriptores Logarithmici*. London, v. 3, 1796, pp. 25–133 (pp. 100–133 being "An appendix to the foregoing translation").

227. J. L. COOLIDGE, "The beginnings of analytic geometry of three dimensions," *Amer. Math. Monthly*, v. 55, 1948, pp. 76–86.

228. *Euclides ab omni naevo vindicatus*, Milan, 1733, 142 pp. SACCHERI's work was translated into English (book 1; book 2, pp. 102–142 omitted) and edited by G. B. HALSTED, Chicago, 1920, xx, 246 pp. E. MCCLINTOCK, "On the early history of the non-euclidean geometry," N. Y. Math. Soc., *Bull.*, v. 2, pp. 144–147, 1893. R. BONOLA, *Non-Euclidean Geometry. A Critical and Historical Study of its Development*. Engl. transl. by H. S. CARSLAW. Chicago, 1912, pp. 22–44, etc. E. CARRUCCIO, "Saccheri, Giovanni Girolamo," *Enciclopedia Italiana*, Rome, v. 30, 1936.

229. HELEN M. WALKER, "Abraham De Moivre," *Scripta Math.*, v. 2, pp. 311–333, 1934; exceedingly interesting and comprehensive sketch. A. M. CLERKE, "Abraham De Moivre," *Dict. Nat. Biog.*, v. 38, 1894, pp. 116–117. C. HUTTON, *Phil. and Math. Dict.*, second ed., London, 1815, v. 2, p. 402. C. WALFORD, *The Insurance Cyclopaedia*, London, v. 1, 1871, "Annuities on Lives, Hist. of," pp. 120–122; v. 2, 1873, "De Moivre" and "De Moivre's hypothesis," pp. 180–183. A. DEMORGAN, "Demoivre, Abraham" and "Demoivre's hypothesis," *Penny Cycl.*, v. 8, 1839, pp. 380–381. D. E. SMITH, "Among my autographs: De Moivre expresses himself," *Amer. Math. Monthly*, v. 29, 1922, pp. 340–343. H. M. WALKER, *Studies in the History of Statistical Method, with Special Reference to Certain Educational Problems*, Baltimore, 1929, viii, 229 pp.; "De Moivre," pp. 12–19, *etc.*

230. TODHUNTER [189], pp. 134–193, etc. SMITH [150], "De Moivre on the laws of normal probability," ed. by H. M. WALKER, pp. 566–575. See also [229].

231. The formula in question is the following: If m is very large

$$m! \approx (2\pi m)^{1/2} e^{-m} m^m. \tag{1}$$

This is due to DE MOIVRE (1730–1733) and was not anywhere given by STIRLING or implied by anything he stated. We shall briefly indicate STIRLING'S small contribution in the direction of this result. JAMES STIRLING (1692–1770) was a prominent Scottish mathematician who wrote two books: A commentary on NEWTON'S lines of the third order, *Lineae tertii ordinis Newtonianae sive illustratio tractatus Newtoni de enumeratione linearum tertii ordinis*. Oxford, 1717, and added two new kinds (J. P. DE GUA subsequently detected four others) to the 72 which NEWTON had remarked among curves of the third order. At p. 32 of this work occurs the theorem which usually goes by the name of MACLAURIN'S Theorem. His second, and most important book was one on the calculus of finite differences. *Methodus differentialis: sive Tractatus de Summatione et Interpolatione Serierum Infinitarum*. London, 1730, viii, 154 pp. There was a second edition in 1764 and an English edition in 1749. Information about his life and work may be found in A. DEMORGAN, *Engl. Cycl.-Biog.*, v. 5, 1857, "Stirling"; and C. TWEEDIE, (a) "Life of James Stirling, the Venetian," *Math. Gazette*, v. 10, pp. 119–128, 1920. (b) *James Stirling. A Sketch of his Life and Works along with his Scientific Correspondence*. Oxford, 1922, xii, 214 pp.+2 folding plates.

In the *Methodus* (prop. XXVIII, pp. 135–137) STIRLING gave the equivalent of the formula

$$\ln m! = \tfrac{1}{2}\ln(2\pi) + z\ln z - \frac{1}{2\cdot 12z} + \frac{7}{8\cdot 360z^3} - \frac{31}{32\cdot 1260z^5} + \frac{127}{128\cdot 1680z^7} - \cdots \tag{2}$$

where $z = m+\frac{1}{2}$. This is all that STIRLING gives. Nowhere in this connection does he consider m very large, or any formula such as (1). DE MOIVRE'S *Miscellanea Analytica* was published in Jan. 1730. This did not then contain the "Supplementum" (21 p.) which was apparently written after the appearance of STIRLING'S *Methodus*, and published with bound copies of his *Miscellanea* before the end of September 1730. DE MOIVRE here showed (pp. [8]–[9]) that

$$\ln m! = \tfrac{1}{2}\ln B + (m+\tfrac{1}{2})\ln m - m + \frac{1}{12m} - \frac{1}{360m^3} + \frac{1}{1260m^5} - \frac{1}{1680m^7} + \cdots \tag{3}$$

where

$$\tfrac{1}{2}\ln B = 1 - \frac{1}{12} + \frac{1}{360} - \frac{1}{1260} + \frac{1}{1680} - \cdots = \tfrac{1}{2}\ln(2\pi),$$

(the fact that $B = 2\pi$ being due to STIRLING), or

$$\log m! = \tfrac{1}{2}\log(2\pi m) + m\log m - m\log e + \frac{\log e}{12m} - \frac{\log e}{360m^3} + \frac{\log e}{1260m^5} - \frac{\log e}{1680m^7} + \cdots. \tag{4}$$

This may be written

$$\log m! = \log\left[(2\pi m)^{1/2} m^m e^{-m}\right] + \left[\frac{B_1}{1\cdot 2}\cdot\frac{1}{m} - \frac{B_2}{3\cdot 4}\cdot\frac{1}{m^3} + \frac{B_3}{5\cdot 6}\cdot\frac{1}{m^5} - \cdots\right]\log e, \tag{5}$$

$B_1, B_2, \ldots$ denoting the BERNOULLI numbers. When m is very large, the relation (1) readily follows from (5). The values of the first 71 of the coefficients $(-1)^{n-1}B_n/[(2n-1)2n]$, for $n=1(1)71$, were given in H. S. UHLER, "The coefficients of Stirling's series of log $\Gamma(x)$," Nat. Acad. Sci., *Proc.*, v. 28, 1942, pp. 59–62. On p. (22) of the *Miscellanea*, DE MOIVRE gives a table(15–19S) of log $m!$ $m=10(10)900$, not without errors. DE MOIVRE returned to (1) in a private publication of Nov. 12, 1733: *Approximatio ad Summam Terminorum Binomii* $(a+b)^n$ *in Seriem expansi*. (7 p.), printed in facsimile in R. C. ARCHIBALD, "A rare pamphlet of Moivre and some of his discoveries," *Isis*, v. 8, pp. 671–676+7 plates, 1926. A somewhat elaborated English edition of this pamphlet is given in the second and third editions of DE MOIVRE'S *The Doctrine of Chances* (1738, 1756). See also K. PEARSON (a) "Historical note on the origin of the normal curve of errors," (b) "James Bernoulli's

theorem," *Biometrika*, (a) v. 16, pp. 402–404, 1924; (b) v. 17, pp. 201–210, 1925. It is of special interest that DE MOIVRE tells us that the part of his 1733 publication into which we are here inquiring dates back to 1721 at least ("It is now a dozen years or more since I had found what follows"). Writing 23 years later as on p. 334 of the Appendix to the third edition of his *Doctrine of Chances* the genial De Moivre was evidently somewhat forgetful of the exact setting of his friend's relationship to one of his own discoveries.

PEARSON's articles make clear that to DE MOIVRE in 1733 is also due: (a) the first treatment of the probability integral, and essentially of the normal curve (later given by LAPLACE in 1774 and 1778; and by GAUSS in the 19th century); (b) the theorem that the measure of accuracy depends on the inverse square root of the size of the sample, so often called BERNOULLI's theorem, although wholly DE MOIVRE's.

232. SMITH [**150**], "De Moivre on his formula", pp. 440–450, "Euler," pp. 450–454, ed. by R. C. ARCHIBALD. BRAUNMÜHL [**131**], v. 2, pp. 75–78. A. v. BRAUNMÜHL, "Zur Geschichte der Entstehung des sogenannten Moivrischen Satzes," *Bibl. Math.*, v. 2, 1901, pp. 97–102.

233. E. I. CARLYLE, "Taylor, Brook," *Dict. Nat. Biog.*, v. 55, 1898. B. TAYLOR, *Contemplatio Philosophica: a posthumous work. . . . To which is prefixed a life of the author by his grandson Sir William Young*. London, 1793, iv, 150 pp. H. BATEMAN, "The correspondence of Brook Taylor," *Bibl. Math.*, v. 7, pp. 367–371, 1907. One of the mss. "dated July 26, 1712 contains an enunciation of the theorem which bears his name." H. AUCHTER, *Brook Taylor der Mathematiker und Philosoph. Beiträge zur Wissenschaftsgeschichte der Zeit des Newton-Leibniz-Streites*. Diss. Marburg. Würzburg, 1937, iv, 112 pp.+portrait plate reproduced from previous item; extracts from Taylor's correspondence, pp. 72–97; "Contemplatio philosophica, an essay by Brook Taylor," pp. 98–110. A. PRINGSHEIM, "Zur Geschichte des Taylorschen Lehrsatzes," *Bibl. Math.*, s. 3, v. 1, 1900, pp. 433–479. Of TAYLOR's *Methodus Incrementorum*, there was a second ed., or print, in 1717 and a facsimile print of this at Berlin, in 1862. F. J. FÉTIS, *Biographie Universelle de Musiciens*, Paris, v. 8, 1884, p. 194.

234. ARCHIBALD [**57**], B. Taylor, p. 19.

235. C. PLATTS, "Maclaurin, Colin," *Dict. Nat. Biog.*, v. 35, 1893. C. TWEEDIE, "A study of the life and writings of Colin Maclaurin," and "Notes on the life and works of Colin Maclaurin," *Math. Gazette*, v. 8, pp. 133–151+4 plates, 1915, and v. 9, pp. 303–305, 1919. H. W. TURNBULL, "Colin Maclaurin," *Amer. Math. Monthly*, 1947, v. 54, pp. 318–322. CAJORI [**209**], chap. VI, "Maclaurin's Treatise of Fluxions, 1742," pp. 181–189. C. TWEEDIE, "The 'Geometria Organica' of Colin Maclaurin: a historical and critical survey," R. Soc. Edinb., *Proc.*, v. 36, pp. 87–150, 1916.

236. "Un chef-d'oeuvre de Géométrie, qu'on peut comparer à tout ce qu'Archimède nous a laissé de plus beau et de plus ingénieux" (*Mém. de l'Acad. de Berlin*, 1773, p. 121). A reference may also be given to G. A. GIBSON, "Sketch of the history of mathematics in Scotland to the end of the 18th century," Edinb. Math. Soc., *Proc.*, s. 2, v. 1, pp. 79–84, 1928.

237. ARCHIBALD [**69**], "pedal curves," "cissoid," *etc.*

238. R. E. LANGER, "The life of Léonard Euler," *Scripta Math.*, v. 3, 1935, pp. 61–66, 131–138. BELL [**53**], "Analysis incarnate-Euler," pp. 139–152. SMITH [**54**]. R. FUETER, *Leonhard Euler*. (*Kurze Mathematiker-Biographien*, no. 3). *Z. d. Elemente d. Mathematik, Beiheft*, Basel, Jan. 1948, 24 pp. There are 3 illustrs. (2 portraits and a facsimile of a page of Euler's writing). L. G. DU PASQUIER, *Léonard Euler et ses Amis*. Paris, 1927, ix, 125 pp. R. WOLF, *Biographien zur Kulturgeschichte der Schweiz*, 4 v., Zürich, v. 4, 1862, pp. 87–134. *Festschrift zur Feier des 200. Geburtstages Eulers*. (*Abh. z. Gesch. d. math. Wissen.*, Heft 25). Leipzig, 1907, iv, 137 pp.+2 portraits. [Contents: G. VALENTIN, "Leonhard Euler in Berlin," pp. 1–20; A. KNESER, "Euler und die Variationsrechnung," pp. 21–60; F. MÜLLER, "Über bahnbrechende Arbeiten Leonhard Eulers aus der reinen Mathematik," pp. 61–116; E. LAMPE, "Zur Entstehung der Begriffe der Exponentialfunktion und der logarithmischen Funktion eines komplexen Arguments bei Leonhard Euler," pp. 117–137.] G. ENESTRÖM, *Verzeichnis der Schriften Leonhard Eulers*. Leipzig, 1910–1913, 388 pp. EULER's works are now being published by the Society of Swiss Naturalists in *Opera Omnia* (33 v., 1911–1947, there are to be about 40 more v.).

SMITH [**150**], "Euler, proof that every integer is a sum of four squares," ed. by E. T. BELL, pp. 91–94 (see also [**99, 101**]; "Euler's use of the letter e to represent 2.718" ed. by F. CAJORI, pp. 95–98; "Euler on differential equations of the second order," ed. by F. CAJORI, pp. 638–643. U. G. MITCHELL & M. STRAIN, "The number e," *Osiris*, v. 1, 1936, pp. 476–496. D. H. LEHMER, "On the value of the Naperian base," *Amer. Jn. Math.*, v. 48, pp. 139–143, 1936; the value of e is here found to 707D to match the then supposedly accurate SHANKS value of π to 707D—see [**201**]. R. C. ARCHIBALD, "Euler's integrals and Euler's spiral—sometimes called Fresnel integrals and the clothoïde or Cornu's spiral," *Amer. Math. Monthly*, v. 25, 1918, pp. 276–282. This spiral is the elastic curve in the following: L. EULER, "Elastic curves," transl. and annotated by W. A. OLDFATHER, C. A. ELLIS & D. M. BROWN, *Isis*, v. 20, pp. 72–160, 1933; this is a complete translation of "Additamentum I. De Curvis Elasticis," pp. 245–310 of EULER's Methodus inveniendi Lineas Curvas Maximi Minimive proprietate gaudentes. (Lausanne and Geneva, 1744). Everything in H. LISENBARTH's edition of a German transl. of the text, in OSTWALD's *Klassiker d. exakten Wissen.*, v. 175 (Leipzig, 1910), has been incorporated into this translation; and there is little else. ORE [**7b**], "Euler's theorem and its consequences," pp. 272–310.

238a. F. CAJORI, "Frederick the Great on mathematics and mathematicians," *Amer. Math. Monthly*, v. 34, 1927, pp. 122–130.

239. TODHUNTER [**189**], chap. XII, Euler, pp. 239–257; *etc.*

240. A. ENNEPER, *Elliptische Functionen. Theorie und Geschichte.* Second ed. by F. MÜLLER, Halle a. S., 1890, xx, 598 pp. More than a dozen papers of EULER are here analyzed.

241. ARCHIBALD [**57**], pp. 20–21.

242. CAJORI [**163**], v. 1–2, see indices.

243. F. LINDEMANN, "Ueber die Zahl π," *Mathem. Annalen*, v. 20, 1882, pp. 213–225.

244. H. S. UHLER, "On the numerical value of i^i" and R. C. ARCHIBALD, "Historical notes on the relation $e^{-\frac{1}{2}\pi}=i^i$," *Amer. Math. Monthly*, v. 28, 1921, pp. 114–121.

245. J. S. MACKAY, "An abstract of one of Euler's papers," Edinb. Math. Soc., *Proc.*, v. 4, 1886, pp. 51–55. See also SMITH [**150**], "Feuerbach on the theorem which bears his name," ed. by R. A. JOHNSON, pp. 339–345. And J. L. COOLIDGE, *A History of Geometrical Methods*, Oxford, 1940, xviii, 452 pp.; pp. 66, 147.

246. A. DEMORGAN, "Fourier, Joseph," *Penny Cycl.*, v. 10, 1838; and *Engl. Cycl.-Biog.*, v. 2, 1856. BELL [**53**], "Friends of an emperor, Monge and Fourier," pp. 183–203. BELL [**5**], pp. 292–294, etc. F. ARAGO, "Joseph Fourier" (read 1833) transl. into English, Smithsonian Institution, *Annual Report*, **1871**, pp. 137–176; also in F. ARAGO, *Biographies of Distinguished Men*, Boston, 1859, v. 1, pp. 374–444. M. BÔCHER, "Historical summary," pp. 267–275, of W. E. BYERLY, *An Elementary Treatise on Fourier Series and Spherical, Cylindrical, and Ellipsoidal Harmonics.* Boston, 1893. See also the admirable survey, R. E. LANGER, *Fourier's Series, the Genesis and Evolution of a Theory.* (*Herbert Ellsworth Slaught Memorial Papers*, no. 1). Suppl. *Amer. Math. Monthly*, v. 54, 1947, vi, 86 pp. Of Fourier's great work *Théorie Analytique de la Chaleur*, Paris, 1822, there was an English translation by A. FREEMAN, *The Analytic Theory of Heat, translated with Notes.* Cambridge, Univ. Press, 1878, xxviii, 466 pp. The French edition filled the first of two v. of *Oeuvres de Fourier*, ed. by G. DARBOUX. Paris 1888–1890. These v. do not include the following work not completely published at the time of Fourier's death: *Analyse des Équations Déterminées. Première Partie* [no more publ.]. Paris 1831, xxiv, 258 pp.+1 folding plate. This v. deals with topics which had interested him from early manhood, and the noted theorem (perfected by STURM in 1829, publ. in 1835) concerning the number of roots of an algebraic equation in a given interval, is certainly due to him both because it was taught to his pupils in the École Polytechnique in 1796, 1797, and 1803, and because he first printed the theorem and its proof in 1820. (*Oeuvres*, v. 2, pp. 291–309). Many writers (including ARAGO, above) were wholly incorrect in imagining that F. D. BUDAN gave the theorem in his book of 1806. BUDAN did finally give a complete demonstration in 1822. P. E. B. JOURDAIN, "Note on Fourier's influence on the conceptions of mathematics," Intern. Congress Mathems., Cambridge 1912, *Proc.*, 1912, v. 2, pp. 526–527. N. NIELSEN, *Géomètres Français sous la Révolution.* Copenhagen, 1929, viii, 251 pp. "Fourier," pp. 88–96; "Budan," pp. 38–39, with ac-

curate details regarding Fourier's theorem. The 80 sketches in this volume include also the following: LAGRANGE, pp. 136–152, LAPLACE, pp. 157–163, LEGENDRE, pp. 166–174, MONGE, pp. 182–190, J. R. ARGAND (1768–1822), pp. 6–9, J. C. DE BORDA (1733–1799), pp. 19–24, J. C. CALLET (1744–1799) pp. 40–44, J. B. J. DELAMBRE (1749–1822), pp. 70–72, J. J. LE F. DE LA LANDE (1732–1807), pp. 43–45, J. E. MONTUCLA (1725–1799), pp. 190–192. Concerning the last six we shall add some notes:

ARGAND—We referred to WALLIS's attempt to represent imaginary quantities graphically. ARGAND gave our modern method in 1806. He was, however, anticipated by CASPAR WESSEL (1745–1818) whose memoir, presented to the Danish Academy in 1797, was published in 1799. For a good history of geometric representation of imaginary quantities, see W. W. BEMAN, "A chapter in the history of mathematics," Amer. Assoc. Adv. Sci., *Proc.*, v. 46, 1897, pp. 33–50. An English translation of ARGAND's paper is: *Imaginary Quantities: their Geometrical Interpretation. Translated from the French of M. Argand by* A. S. HARDY. New York, 1881, 135 p.

BORDA—See the sketch of BORDA by A. DEMORGAN, in *Penny Cycl.*, v. 5, 1836, and *Engl. Cycl.-Biog.*, v. 1, 1856. BORDA, LAGRANGE, LAPLACE, MONGE & CONDORCET, in a report, made in 1791 to the Academy of Sciences, Paris, recommended the decimal division of the quadrant of the circle (100^g, $1^g=100'$, $1'=100''$). This subdivision is found in CALLET's tables, and, 1801, in BORDA's posthumous: *Tables Trigonométriques Décimales, ou Tables des Logarithmes*, with revision and an explanation by DELAMBRE (iv, 637 pp.). Through recommendation of the above committee the meter was in 1799 adopted as the standard of length.

CALLET—was the author of a famous volume: *Tables portatives de Logarithmes*, 1795, cxviii, 680 pp., later appearing in scores of editions and impressions.

DELAMBRE—see also the excellent sketch by A. DEMORGAN, in *Penny Cycl.*, v. 8, 1937, and *Engl. Cycl.-Biog.*, v. 2, 1856. D. was chosen an associate of almost every learned body in Europe. He was the author of 6 great v. of History of Astronomy (1817–1827), and many other v. And also the author of what are known as "GAUSS's analogies" **[272]** in trigonometry; see I. TODHUNTER, "Note on the history of certain formulae in spherical trigonometry," *Phil. Mag.*, s. 4, v. 45, 1873, pp. 98–100.

LALANDE—see also A. DEMORGAN, *Penny Cycl.*, v. 13, 1939, and *Engl. Cycl.-Biog.*, v. 3, 1856; and ARCHIBALD **[163]**, pp. 43–45. Director of the Paris Observatory (1768–1807). Author of a v. of mathematical tables which went through many editions down into the 20th century (ARCHIBALD **[163]**, pp. 43–45); of Navigation its history theory and practice (1793); and of the fourth v. of MONTUCLA's History (1802).

MONTUCLA—author of two noteworthy treatises on the history of mathematics. One was on the quadrature of the circle (1754, second ed. by LACROIX, 1831); the other and larger work was the first modern history of mathematics, and it may be called a classic (2 v. Paris, 1758; second edition 4 v. 1799–1802. The third v. was partly printed when M. died; the rest of it (after p. 336) was seen through the press by LALANDE who prepared the fourth v., mainly on the history of astronomy.) See G. SARTON, "Montucla (1725–1799): his life and works," *Osiris*, v. 1, 1936, pp. 519–567. Also D. E. SMITH, "Among my autographs": "The threatened loss of the second edition of Montucla's *History of Mathematics*"; "Montucla's closing years," *Amer. Math. Monthly*, v. 28, 1928, pp. 207–208; v. 29, 1929, pp. 253–255.

247. M. GODEFROY, *La Fonction Gamma: Théorie, Histoire, Bibliographie.* Diss. Paris. Paris, 1901, pp. 1–6, *etc.*, EULER's Beta Function, $\int_0^1 x^{p-1}(1-x)^{q-1}dx$, and the more general form, $\int_0^1 x^{p-1}dx/(1-x^n)^{(n-q)/n}$, were studied by EULER in papers of 1738, 1750, 1761, and 1789; see, ENESTRÖM **[238]**.

248. $$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{x}=\gamma+\ln x+\frac{1}{2x}-\frac{B_1}{2x^2}+\frac{B_2}{4x^4}-\frac{B_3}{6x^6}+\cdots.$$

J. W. L. GLAISHER, "On the history of Euler's constant," *Mess. Math.*, v. 1, pp. 25–30, 1871. The constant γ was calculated to 263D by J. C. ADAMS, B.A.A.S., *Report, 1877*, p. 15 and R. Soc. London, *Proc.*, v. 27, 1878, pp. 93–94.

249. G. Sarton, "Lagrange's personality (1736–1813)," Amer. Phil. Soc., *Proc.*, v. 88, 1944, pp. 457–496+portrait frontispiece. Bell [53], "A lofty pyramid, Lagrange," pp. 153–171. Smith [54], Nielsen [246], pp. 136–152. Turnbull [51], "Maclaurin and Lagrange," pp. 101–107. O. J. Lodge, *Pioneers of Science*, London, 1919; lecture XI: "Lagrange and Laplace—the stability of the solar system, and the nebular hypothesis," pp. 254–271. "Bicentenary of Lagrange (1736–1813)," *Nature*, v. 137, 1936, pp. 141–142. G. Loria, (a) "Nel secondo centenario della nascita di G. L. Lagrange 1736–1936," *Isis*, v. 28, 1938, pp. 366–375, with interesting illustrations. (b) "G. L. Lagrange nella vita e nelle opere," *Annali di Matem.*, s. 3, v. 20, 1913, pp. IX–LII. L. G. Simons, "The influence of French mathematicians at the end of the eighteenth century upon the teaching of mathematics in American Colleges," *Isis*, v. 15, 1931, pp. 104–106, 108, 109, 119.

250. *Engl. Cycl.-Biog.*, v. 3, 1856, "Lambert, John Henry." J. H. Lambert, *Opera Mathematica, Commentationes Arithmeticae Algebraicae Analyticae*, ed. Andreas Speiser, V. 1–2, Zürich, 1946–1948, xxxii, 348, xxx, 324 pp.; the introductory essays of v. 1–2 by the editor, are synopses of interest. Pp. 1–15 of v. 1 are occupied by "Éloge de M. Lambert" by J. H. S. Formey, reprinted from *Nouv. Mém. de l'Acad. r. de Berlin 1778, Histoire*, pp. 72–91. R. Wolf, *Biographien zur Kulturgeschichte der Schweiz.* 4 v., Zürich, 1858–1862; v. 3, 1860, pp. 317–356. A sumptuous work dealing primarily with Lambert's writings on perspective but listing all of his published and unpublished writings is *Johann Heinrich Lambert Schriften zur Perspektive. Herausgegeben und eingeleitet von* Max Steck. Berlin, 1943, xvii, 496 pp.+21 plates, folio format.

251. *Engl. Cycl.-Biog.*, "Wallace, William," v. 5, 1857. In Leybourn's *Math. Repository*, n.s., no. IV, Oct. 28, 1797, Wallace proposed the problem: "If three straight lines touch a parabola a circle described through their intersections shall pass through the focus of the parabola. Required the Demonstration." In his proof (no. VI, 1798, II, pp. 54–55) he proves that the feet of the perpendiculars from the focus of the parabola on the three tangents all lie on the tangent at the vertex of the parabola. Then in no. VII, 1799, II, p. 111 the Wallace Line theorem emerges. Poncelet, who doubtless never saw Wallace's articles, treats the theorems in the same way, see his *Propriétés Projectives*, 1822, §§466–468. In the latter paragraph is a generalization of the theorem, namely: that the perpendiculars on the sides may be replaced by obliques making (in cyclical order) equal angles with the sides. The same generalization was also made by Steiner in Gergonne's *Annales*, v. 19, 1828, pp. 37–64. Steiner remarks that the straight lines obtained in varying the angles of the obliques will envelop a parabola, whose focus is the point from which the obliques are drawn. That the envelope of all the Wallace Lines of a circle is a three-cusped hypocycloid was shown by W. K. Clifford in 1865, *Educ. Times Reprint*, v. 3, pp. 81–82. The number of theorems connected with the Wallace Line is very large. Some references may be found in J. S. Mackay, "The Wallace line and the Wallace point," Edinb. Math. Soc., *Proc.*, v. 9, 1891, pp. 83–91; and in the remarkable *Frère* Gabriel Marie, *Exercices de Géométrie*, fifth ed., Paris, 1912, §764–767. See also J. S. Mackay, "Bibliography of the envelope of Wallace line the three-cusped hypocycloid," Edinb. Math. Soc., *Proc.*, v. 23, 1905, pp. 80–88; and a bibliography of the extension of Wallace's Line to space by O. Degel, in *L'Intermédiaire des Mathém.*, v. 18, 1911, pp. 254–255.

252. F. Rudio, *Archimedes, Huygens, Lambert, Legendre. Vier Abhandlungen über die Kreismessung. Deutsch herausgegeben und mit einer Übersicht über die Geschichte des Problems von der Quadratur des Zirkels . . . versehen von* F. Rudio. Leipzig, 1892, pp. 54–58, 133–155. Compare A. Pringsheim, "Ueber die ersten Beweise der Irrationalität von e und π," Bayerische Akad. d. Wissen., *Sitzungsb., math.-phys. Classe*, 1898, pp. 325–337. It is here pointed out that Lambert did not know of Euler's work 30 years earlier; also that Lambert's proof of the irrationality of π is more rigorous than the later one of Legendre; that Lambert here gave the first, and for a long time the only, example of the rigorous development of a particular function into a converging continued fraction. See also J. Popken, "On Lambert's proof for the irrationality of π," Akad. v. Wetens., Amsterdam, *Proc.*, v. 43, 1940, pp. 712–714. The simplest proof yet found for the irrationality of π seems to be that of I. Niven, Amer. Math. Soc., *Bull.*, v. 53, 1947, p. 509. Hermite's proof that π and π^2 were irrational is of interest both in relation to the proof of Lambert and as containing the germ of the later proof of the transcendence of e and π (*Jn. f. d. reine u.*

angew. Math., v. 76, 1873, pp. 342–344); see also *Oeuvres de Charles Hermite*, v. 3, 1912, pp. 146–149. A simple proof of the irrationality of e was given in 1815 by FOURIER. D. E. SMITH, "The history of the transcendence of π," in J. W. A. YOUNG, *Monographs on Topics of Modern Mathematics*, New York, 1911, pp. 387–416.

253. G. LORIA, "J. Liouville and his work," *Scripta Math.*, v. 4, 1936, pp. 147–154, 257–262, 301–306, and portrait plate.

254. BELL [53], "The man, not the method—Hermite," pp. 448–465. G. PRASAD, *Some Great Mathematicians of the Nineteenth Century: Their Lives and Their Works*. Benares, India, v. 1, 1933 (Gauss, Cauchy, Abel, Jacobi, Weierstrass, Riemann), xv, 347 pp.+6 portraits; v. 2, 1934 (Cayley, Hermite, pp. 34–59, Brioschi, Kronecker, Cremona, Darboux, Cantor, Mittag-Leffler, Klein, Poincaré), xviii, 324 pp.+10 portraits. G. DARBOUX, *Éloges académiques et Discours*, Paris, 1912, pp. 116–172. M. NOETHER, *Math. Annalen*, v. 55, 1901, pp. 337–385. LOREY [276], JACOBI, pp. 43–44.

255. S. GÜNTHER, *Die Lehre von den gewöhnlichen und verallgemeinerten Hyperbelfunktionen*. Halle a S., 1881, x, 440 pp.

256. C. H. DEETZ & O. S. ADAMS, *Elements of Map Projection with Applications to Map and Chart Construction*. Fifth ed., rev., Washington, 1945, 256 pp.+14 plates.

257. BELL [53], "Friends of an Emperor—Monge and Fourier," pp. 183–205. A. DEMORGAN, *Engl. Cycl.-Biog.*, v. 4, 1857, "Monge, Gaspard." D. E. SMITH, "Gaspard Monge, politician," *Scripta Math.*, v. 1, pp. 111–122, 1932. D. E. SMITH, "Among my autographs: Monge and the American Colonies" and "Monge the lesser," *Amer. Math. Monthly*, v. 28, 1921, pp. 166, 208, 209. SIMONS [249], pp. 115, 119. NIELSEN [244], "Monge," pp. 182–190. L. DELAUNAY, *Monge, Fondateur de l'École Polytechnique*. Paris, 1933, 280 pp. D. F. J. ARAGO, "Biographie de Gaspard Monge," *Mémoires de l'Acad. d. Sci.*, Paris, v. 24, 1853, 157 pp.; also in ARAGO, *Notices Biographiques*, Paris, v. 2, 1853, pp. 427–592. C. DUPIN, *Essai Historique sur les Services et les Travaux Scientifiques de Gaspard Monge*. Paris, 1819, viii, 316 pp. For Descriptive Geometry used by PTOLEMY, see NEUGEBAUER [7a], and HEATH [22], *History*, v. 2, pp. 286–292.

258. D. J. STRUIK, "Outline of a history of differential geometry," *Isis*, v. 19, 1933, pp. 92–120 (1. "The time before Leibniz," pp. 92–96; 2. "The first systematic contributions," pp. 96–99; 3. "The eighteenth century," pp. 99–110; 4. "Monge and the École Polytechnique," pp. 110–113; 5. "Monge's pupils," pp. 113–120), and v. 20, pp. 161–191, 1933 (6. "Gauss," pp. 161–167; 7. "The French school of the forties," pp. 167–172; 8. "Riemann," pp. 173–177; 9. "The beginning of modern times," pp. 177–184; 10. "Differential geometry from 1870 to 1900," pp. 184–189; 11. "Sources," pp. 189–191). See also J. L. COOLIDGE, *A History of Geometrical Methods*. Oxford, 1940; "Differential geometry," pp. 318–421.

259. F. CAJORI, "Plans for a history of mathematics in the nineteenth century," *Science*, n.s., v. 48, 1948, pp. 279–284; address as retiring president of the Math. Assoc. Amer.

260. A. DEMORGAN, "Laplace," *Penny Cycl.*, v. 13, 1839, and *Engl. Cycl.-Biog.*, v. 3, 1856, cols. 800–805. BELL [53], "From peasant to snob—Laplace," pp. 172–182. K. PEARSON, "Laplace"; G. A. SIMON, "Les origines de Laplace: sa généalogie—ses études," *Biometrika*, v. 21, 1930, pp. 202–216; pp. 217–230+1 plate. D. F. J. ARAGO, "Eulogy on Laplace," transl. into English by B. POWELL, Smithsonian Inst., *Report, 1874*, pp. 129–168. H. MACPHERSON, *Makers of Astronomy*, Oxford, 1933; "Laplace," pp. 88–93. LENARD [202], pp. 218–223+portrait plate. SMITH [54]. NIELSEN [246], "Laplace," pp. 157–163. H. ANDOYER, "*L'Oeuvre Scientifique de Laplace*. Paris, 1922, 162 pp. STRUIK [6], pp. 192–199. E. S. SMITH, "The tomb of Laplace," *Nature*, v. 119, 1927, pp. 493–494; quotation (p. 494): "On 11 Dec. 1925, a fire completely destroyed the Chateau de Mailloc, and with this were lost all the papers and personal relics of Laplace."; see also *Nature*, v. 4, 1871, p. 108. E. ESCLANGON, "Hommage à Laplace, inauguration de sa statue, à Beaumont-en-Auge le 3 Juillet 1932," *L'Astronomie*, v. 46, 1932, pp. 402–404. *Oeuvres Complètes de Laplace*. 14 v., Paris, 1878–1912.

261. TODHUNTER [189], "Laplace," pp. 464–613, *etc.* E. C. MOLINA, "The theory of probability: some comments on Laplace's *Théorie Analytique [des Probabilités]*," Amer. Math. Soc., *Bull.*, v. 36, 1930, pp. 369–392. The fundamental parts of LAPLACE's *Théorie* are given by A.

DEMORGAN in his article "Theory of Probabilities" in *Encycl. Metropolitana*, 1837. LAPLACE's "Essai Philosophique sur les Probabilités" was first printed as an introduction to the second edition of the *Théorie*, 1814. An English translation from the sixth French edition of the *Essai* (*A Philosophical Essay on Probabilities*) was published at New York, in 1902, 196 pp. SMITH [150], "Laplace on the probability of the errors in the mean results of a great number of observations and on the most advantageous mean result," ed. by J. L. C. A. GŸS, pp. 588–604.

262. R. C. ARCHIBALD, "Bowditch, Nathaniel," *Dict. Amer. Biog.*, v. 2, 1929; also *A Catalogue of a Special Exhibition of Manuscripts, Books, Portraits and Personal Relics of Nathaniel Bowditch (1773–1838)*. Salem, Mass., 1937, iv, 40 pp.+7 plates. ("Nathaniel Bowditch," by H. BOWDITCH, pp. 1–6; "The scientific achievements of Nathaniel Bowditch," by R. C. A., pp. 7–16). A. STANFORD, *Navigator. The Story of Nathaniel Bowditch*, New York, 1937, xii, 308 pp. R. E. BERRY, *Yankee Stargazer: the Life of Nathaniel Bowditch*, London, and New York, 1941, 234 pp.

263. A. DEMORGAN, "Legendre," *Penny Cycl.*, v. 13, 1839, and *Engl. Cycl.-Biog.*, v. 3, 1856. J. W. L. GLAISHER, "Legendre," *Encycl. Brit.* eleventh ed., Cambridge, v. 16, 1911 [G. errs in attributing to Sir DAVID BREWSTER the transl. into English of LEGENDRE's geometry]. L. ÉLIE DE BEAUMONT, "Éloge historique de Adrien Marie Legendre," Acad. d. Sci., Paris, Mémoires, v. 32, 1864, pp. xxxvii-xciv, 56 pp.; Engl. transl. by C. A. ALEXANDER, Smithsonian Inst., *Report for 1867*, pp. 137–157. NIELSEN [246], "Legendre," pp. 166–174. ARCHIBALD [163], pp. 45–49. C. DORIS HELLMAN, "Legendre and the French reform of weights and measures," *Osiris*, v. 1, 1936, pp. 314–340. I. TODHUNTER, *History of the Mathematical Theories of Attraction and the Figure of the Earth*. London 1873; in v. 2, chapters 20, 22, 24, 25, pp. 20–139, contain complete accounts of LEGENDRE's memoirs on attractions of ellipsoids. I. TODHUNTER, *A History of the Progress of the Calculus of Variations during the Nineteenth Century*, Cambridge and London, 1861, pp. 229–233, 489. L. E. DICKSON, *History of the Theory of Numbers*. Washington, v. 1–3, 1919, 1920, 1927; also New York, Stechert ed., 1935; see various references to LEGENDRE in indices. SMITH [150], "Legendre's law of quadratic reciprocity," ed. by D. H. LEHMER, pp. 112 f. D. E. SMITH, "Among my autographs: Legendre and Cauchy sponsor Abel," *Amer. Math. Monthly*, v. 29, 1922, pp. 394–395.

264. Legendre's work in Elliptic Functions is summarized in ENNEPER [240], index, pp. 586–587. His final volumes appeared almost simultaneously with the complete theory of elliptic functions by the great Norwegian mathematical genius **NIELS HENRIK ABEL**, which Legendre recognized as a great advance over his own. See BELL [53], "Genius and poverty—Abel," pp. 307–326. PRASAD [254], v. 1, pp. 111–165+portrait. SMITH [150], "Abel on the quintic equation" ed. by W. H. LANGDON & O. ORE, pp. 261–266. [It is impossible to solve the general equation of the fifth, or higher, degree, in terms of radicals.]; "Abel on the continuity of functions defined by power series," ed. by A. A. BENNETT, pp. 286–291; "Abel on integral equations," ed. by J. D. TAMARKIN, pp. 656–662. G. MITTAG-LEFFLER, "Niels Henrik Abel," *La Revue du Mois*, 1907, pp. 34–38. J. BERTRAND, *Éloges Académiques*, Paris, 1902; "N. H. A. Tableau de sa vie et de son action scientifique," pp. 313–328.

LEGENDRE was similarly impressed with the work of **C. G. J. JACOBI** which appeared somewhat later (1829). BELL [53], "The great algorist—Jacobi," pp. 327–339. PRASAD [254], v. 1, pp. 166–219+portrait plate. SMITH [54]. STRUIK [6] pp. 227–232. "Correspondance mathématique entre Legendre et Jacobi," *Jn. f. d. reine u. angew. Math.*, v. 80, 1875, pp. 205–279; also in JACOBI, *Gesammelte Werke*, v. 1, Berlin, 1881, pp. 385–461. ENNEPER [240], pp. 581–583. MUIR [219], many references in indices and contents to "Jacobians," *etc.* I. TODHUNTER, *History of the Progress of the Calculus of Variations during the Nineteenth Century*. Cambridge, 1861; "Jacobi," pp. 243–253. "Commentators on Jacobi," pp. 254–332. L. KOENIGSBERGER, *Carl Gustav Jacob Jacobi. . . .* Leipzig, 1904, xviii, 554 pp.+1 plate.

265. M. MERRIMAN, "List of writings related to the method of least squares with historical and critical notes," Connecticut Acad., *Trans.*, v. 4, 1877, pp. 160–173. For a transl. of a portion of LEGENDRE's "On a method of least squares," ed. by H. A. RUGER & H. M. WALKER, see SMITH [150], pp. 576–579.

266. The first English translation of LEGENDRE's geometry was by JOHN FARRAR, of Harvard

University, Cambridge, Mass., 1819. The next English translation, together with an introductory chapter on proportion, was by THOMAS CARLYLE, Edinburgh, 1824. There were 39 American editions (6 of FARRAR and 33 of CARLYLE) 1819–1880, and the work led to the practical abandonment in the United States of EUCLID's *Elements*. See SIMONS [249], pp. 107–115; and L. C. KARPINSKI, *Bibliography of Mathematical Works Printed in America through 1850*. Ann Arbor, 1940, pp. 11, 228–229, 292–293, 449, 459. *The Thirteen Books of Euclid's Elements Translated from the Text of Heiberg* with Introd. and Commentary by T. L. HEATH, v. 1, Cambridge, 1908, pp. 112, 169, 213–219; second ed. rev. with additions, 1926. BONOLA [78], pp. 55–60, *etc.* In a copy of THOS. SIMPSON, *The Doctrine and Application of Fluxions*, 1776, in the Library of Brown University, is the signature: "Thomas Carlyle, Studt. Edinb. 1814." This is the year when Carlyle (19 years of age) was mathematical teacher at Annan.

267. BELL [53], "The Copernicus of geometry—Lobatchewsky," pp. 307–326, G. B. HALSTED, "Lobachevsky," *Amer. Math. Monthly*, v. 2, 1895, pp. 137–139. A. V. VASILIEV, *Nicolai Ivanovich Lobachevsky, Address pronounced at the commemorative Meeting of the Univ. of Kazan, Oct. 22, 1893*, transl. by G. B. HALSTED, Austin, Texas, 1894, viii, 40 pp. LOBACHEVSKY's noneuclidean geometry writings included: (1) *Ueber die Principien der Geometrie* (Kazan, 1829–1830); (2) *Geometrische Untersuchungen zur Theorie der Parallellinien*, Berlin, 1840, French transl. by HOÜEL, 1866 and English transl. by HALSTED, *Geometrical Researches on the Theory of Parallels*, Austin, Texas, 1891, 50 pp.; new ed. with portrait and bibl., Chicago, 1914, 50 pp. L. published also (3) *Pangéométrie ou Précis de Géométrie fondée sur une Théorie générale et rigoureuse des Parallèles*, Kazan, 1855. BONOLA [78], pp. 84–96, *etc.* In 1944 the Academy of Sciences, Moscow-Leningrad. publ. a 348-page biog. of LOBACHEVSKY by V. F. KAGAN.

268. E. T. BELL, "Father and son Wolfgang and Johann Bolyai," *Scripta Math.*, v. 5, 1938, pp. 37–44, 95–100. G. B. HALSTED, (a) "Bolyai Farkas," and (b) "Bolyai Janos," *Amer. Math. Monthly*, (a) v. 3, 1896, pp. 1–5; (b) v. 5, 1898, pp. 35–38. BONOLA [78], pp. 96–113, *etc.* JÁNOS BOLYAI's Latin Appendix of 1832 was translated into English: *The Science Absolute of Space, . . . independent of the Truth or Falsity of Euclid's Axiom XI (which can never be decided a priori). . . .* Austin, Texas, fourth ed., 1896, 71 pp. P. G. STÄCKEL, *Wolfgang und Johann Bolyai, geometrische Untersuchungen*. Leipzig, 1913, 2 v., viii, 274, xii, 282 pp. *Briefwechsel zwischen Carl Friedrich Gauss und Wolfgang Bolyai*, ed. by F. SCHMIDT u. P. G. STÄCKEL, Leipzig, 1899, 220 pp.+2 plates+14 facsimiles.

269. D. M. Y. SOMMERVILLE, *Bibliography of Non-Euclidean Geometry, including the Theory of Parallels, the Foundations of Geometry, and Space of n Dimensions*. London, 1911, xii, 404 pp.

270. BELL [53], "The prince of mathematicians—Gauss," pp. 218–269. PRASAD [254], v. 1, pp. vii–viii, 1–67+portrait frontispiece. LENARD [202], pp. 240–247. *Carl Friedrich Gauss, Inaugural Lecture on Astronomy and Papers on the Foundation of Mathematics*. Transl. and ed. by G. W. DUNNINGTON, Baton Rouge, La., 1937, xi, 91 pp.; "The early life of C. F. G." pp. 1–32. G. W. DUNNINGTON, "The historical significance of C. F. G. in mathematics and some aspects of his work," *Math. News Letter*, v. 8, 1934, pp. 175–179. F. CAJORI, (a) "C. F. G. and his children," *Science*, n.s., v. 9, 1899, pp. 697–704; (b) "Gauss and his American descendants," *Pop. Sci. Mo.*, v. 81, 1912, pp. 105–114. H. MACK, *Carl Friedrich Gauss und die Seinen. Festschrift zu seinem 150. Geburtstage*. Braunschweig, 1927, 12+150 pp.+12 plates; one finds here details concerning GAUSS's sons who settled in the United States, and their descendants. SMITH [150], "Gauss on the congruence of numbers," ed. R. G. ARCHIBALD, pp. 107–111; "Gauss, third proof of the law of quadratic reciprocity," ed. D. H. LEHMER, pp. 112–118; "Gauss, second proof of the fundamental theorem of algebra," ed. C. R. ADAMS, pp. 292–306; "Gauss, on conformal representation," ed. H. P. EVANS, pp. 463–475. BONOLA [78], pp. 64–75, *etc.* STRUIK [258], pp. 161–167. P. E. B. JOURDAIN, "The theory of functions with Cauchy and Gauss," *Bibl. Math.*, v. 6, pp. 190–207, 1905. A. PRINGSHEIM, "Kritisch-historische Bemerkungen zur Funktionentheorie. V–VI. Über einen Gaussischen Beweis der Irrationalität von tang x bei rationalem x," Akad. d. Wissen., Munich, *Sitzb., math.-naturw. Abt.*, 1932, pp. 193–200, 1933, pp. 61–70. J. MUNRO, *Heroes of the Telegraph*, London, 1891, pp. 269–272. R. W. POHL, "Zur Jahrhundertfeier des elektromagnetischen Telegraphen von Gauss

und Weber . . . ", Gesell. d. Wissen. zu Göttingen, *Nach., Jahresb.*, 1933–34, pp. 48–56. F. KLEIN, *Vorlesungen über die Entwicklung der Math. im* 19. *Jahrhundert*, part 1. Berlin, 1926, pp. 6–60, etc.; see also part 2, 1927, index. F. KLEIN, "Gauss's wissenschaftliches Tagebuch, 1796–1814" Göttingen Gesell. d. Wissen., *Festschrift*, Berlin, 1901, pp. 1–44 with portr. and facs. *Materialien für eine wissenschaftliche Biographie von G.* Leipzig, 8 parts, 1911–1920. [1. P. BACHMANN, *Über Gauss' zahlentheoretische Arbeiten*, 1911, iv, 54 pp. 2–3. GAUSS, *Fragmente zur Theorie des arithmetisch-geometrischen Mittels aus den Jahren 1797–1799*, and L. SCHLESINGER, *Über Gauss' Arbeiten zur Funktionentheorie.* 1912, iv, 34, 143 pp. 4–5. A. GALLE, *C. F. Gauss als Zahlenrechner*, and P. STÄCKEL, *C. F. Gauss als Geometer*, 1918, iv, 142 pp. 6. P. MAENNCHEN, *Die Wechselwirkung zwischen Zahlenrechnen und Zahlentheorie bei C. F. Gauss*, 1918, iv, 47 pp. 7. M. BRENDEL, *Über die astronomischen Arbeiten von Gauss, . . . theoretische Astronomie.* 1919, iv, 106 pp. 8. A. FRAENKEL, *Zahlbegriff und Algebra bei Gauss*, mit einem Anhang von A. OSTROWSKI *zum ersten und vierten Gausschen Beweise des Fundamentalsatzes der Algebra.* 1920, ii, 59 pp. L. BIEBERBACH, *C. F. Gauss, ein deutsches Gelehrtenleben.* Berlin, 1938, 178 pp. Of Gauss's *Werke*, 12 v., in 14, have been publ. 1863–1933.

271. SMITH [**150**], "Gauss on the division of the circle into *n* equal parts," ed. J. S. TURNER, pp. 348–350. F. KLEIN, *Famous Problems of Elementary Geometry*, transl. by W. W. BEMAN & D. E. SMITH. Second ed. rev. and enl. with notes by R. C. ARCHIBALD, New York, 1930; "Gaussian polygons," pp. 81–85. J. PIERPONT, "On an undemonstrated theorem of the *Disquisitiones Arithmeticae*," Amer. Math. Soc., *Bull.*, v. 2, 1895, pp. 77–83. R. C. ARCHIBALD, G. W. DUNNINGTON, & H. GEPPERT, "Gauss's *Disquisitiones Arithmeticae* and the French Academy of Sciences," *etc.*, *Scripta Math.*, v. 3, 1935, pp. 98, 193–196, 285–286, 356–358; v. 4, 1936, p. 106; and *Nat. Math. Mag.*, v. 9, 1935, pp. 187–192.

272. An American edition of one of GAUSS's famous astronomical works was: *Theory of the Motion of the Heavenly Bodies moving about the Sun in Conic Sections; a translation of Gauss's "Theoria Motus" with an Appendix.* By C. H. DAVIS, Cambridge, Mass., 1857, xvii, 326, 42 pp.+7 plates. In this work are found four formulae of spherical trigonometry, often called "GAUSS's analogies," although discovered considerably earlier by DELAMBRE [**246**].

273. JEAN LE ROND, dit D'ALEMBERT (1717–1783) mathematician, philosopher, author of many books and articles including *Opuscules Mathématiques* (8 v., Paris 1776–1780), *Encyclopédie Méthodique. Mathématiques.* (5 v., 1784–97, in collaboration with others), *Mélanges de Littérature, d'Histoire et de Philosophie* (5 v., new ed., Leyden, 1782–83), and *Éléments de Musique théorique et pratique suivant les Principes de M. Rameau éclaircis, développées et simplifiés* (Paris, 1752; fourth ed., Lyons, 1779). There is a very interesting sketch of D'Alembert by A. DEMORGAN, in *Penny Cycl.*, v. 1, 1833, and *Engl. Cycl.-Biog.*, "ALEMBERT," v. 1, 1856. L. LEMOINE, *Dictionnaire de Biographie Française*, ed. by J. BALTEAU, M. BARROUX, & M. PRÉVOST, Paris, v. 1, 1933, cols. 1397–1416; a long list of portraits of D'ALEMBERT and a bibliography of his life, cols. 1414–1416. J. L. F. BERTRAND, *Éloges Académiques*, nouv. série, Paris, 1902, "D'Alembert et Lagrange" pp. 291–311. J. L. F. BERTRAND, *Les Grands Écrivains: D'Alembert.* Paris, 1889, 206 pp. ARCHIBALD [**57**], pp. 19–21. STRUIK [**7**], pp. 182–184. SMITH [**4**], v. 1, pp. 479–480.

274. ARCHIBALD [**163**], pp. 21–22. E. W. SCRIPTURE, "Arithmetical prodigies," *Amer. Jn. Psych.*, v. 4, 1891, pp. 18–20, 40–41, etc. F. D. MITCHELL, "Mathematical prodigies," *Amer. Jn. Psych.*, v. 18, 1907, pp. 75–77, etc. R. C. ARCHIBALD, "Mathematicians, and poetry and drama," *Science*, n.s., v. 89, 1939, p. 46.

275. BELL [**53**], "The day of glory—Poncelet," pp. 206–217. *Encycl. Brit.*, eleventh ed., v. 22. CAJORI [**2**], pp. 287–288. SMITH [**150**], "Poncelet on projective geometry" ed. by V. SANFORD, pp. 315–323; "Brianchon and Poncelet on the nine-point circle theorem," ed. by M. M. SLOTNICK, pp. 337–338. J. L. F. BERTRAND, "Éloge historique de J. V. P.," Acad. d. Sci., Paris, *Mémoires*, v. 41, pt. 2, 1879, pp. i–xxv and *Éloges Académiques*, 1890, pp. 105–129. COOLIDGE [**245**], pp. 92–95. E. KÖTTER, *Die Entwickelung der synthetischen Geometrie.* Leipzig, 1901; "Poncelet," very numerous references in the index.

276. "Steiner"; *Encycl. Brit.*, eleventh ed., v. 25, 1911. CAJORI [**2**], pp. 290–292. L. KOLLROS,

Jakob Steiner. (*Kurze Mathematiker-Biographien*, no. 2). *Suppl. Revue de Mathématiques Élémentaires*, Basel, Dec. 1947, 24 pp. There are 3 illustrs. (2 portraits and a facsimile of a page of Steiner's Ms.). COOLIDGE **[245]**, pp. 96–97. KÖTTER **[275]**, many references in the index. J. LANGE, *Jacob Steiners Lebensjahre in Berlin, 1821–1863, nach seinen Personalakten dargestellt.* Berlin, 1899, 70 pp., portr. E. LAMPE, "Zur Biographie von J. S.," *Bibl. Math.*, s. 3, v. 1, pp. 129–141, 1900. W. LOREY, *Das Studium der Mathematik an den deutschen Universitäten seit Anfang des 19. Jahrhunderts*, Leipzig, 1916, Steiner, pp. 46–49, *etc.* J. STEINER, *Gesammelte Werke*, ed. K. WEIERSTRASS, Berlin 2, v., 1881–1882.

In 1803 G. F. MALFATTI of the Univ. of Ferrara proposed the problem to cut three cylindrical holes out of a triangular prism in such a way that the cylinders and the prism have the same altitude and that the volumes of the cylinders be a maximum. This problem was reduced to another, now generally known as MALFATTI'S problem: to inscribe three circles in a triangle so that each circle is tangent to two sides of the triangle and to the other two circles. MALFATTI gave an analytical solution but STEINER published without proof a simple construction by synthetic geometry, remarking that there were 32 solutions, and generalized the problem by replacing the three lines by three circles, and solved the analogous generalized problem for three dimensions: To determine the three sections of a surface of the second order, each of them touching the other two and also two of three given sections of the surface of the second order. This general problem was solved analytically by CAYLEY and CLEBSCH, for example, with the aid of the addition theorem of elliptic functions. The literature of MALFATTI'S problem is very large. See A. WITTSTEIN, *Geschichte des Malfatti'schen Problems.* Diss. Erlangen. Munich, 1871, 39 pp.+4 folding plates. Supplement, Nördlingen, 1878, 27 pp.+2 plates. Also R. C. ARCHIBALD, "Malfatti's problem," *Scripta Mathematica*, v. 1, pp. 170–171, 1932.

PAPPUS stated the theorem that of all segments of a circle whose boundaries are of equal length the half circle has the greatest area. The corresponding result for the sphere was stated by ARCHIMEDES. In 1841 STEINER gave what he believed to be complete proofs that: Of all plane figures of equal boundary length, the circle has the greatest area; and of all figures of equal area the circle has the least boundary. But the proof was faulty since he did not prove that among all closed plane curves of given length there exists one whose area is a maximum, and similarly for the sphere. F. EDLER was the first (1882) to complete STEINER'S proof for the circle. The maximum property for the sphere was first rigorously proved by H. A. SCHWARZ (1884), using WEIERSTRASSIAN results in the field of calculus of variations. The best elementary discussion of the problems here mentioned is in W. BLASCHKE, *Kreis und Kugel.* Leipzig, 1916, x, 169 pp. See also M. ZACHARIAS, *Encykl. d. math. Wissen.*, v. 3_1, 1921, pp. 1118–1137; and R. STURM, *Maxima und Minima in der elementaren Geometrie.* Leipzig u. Berlin, 1910, vi, 138 pp.

277. PONCELET, *Traité des Propriétés Projectives des Figures.* Paris, 1922, pp. 187–190. Also second ed., v. 1, 1865, pp. 181–184, 413–414. In particular all points determined with ruler and compasses can also be found by a carpenter's square or a ruler with parallel edges. Since a parallelogram may be drawn with a double edged ruler we recall the problem of LAMBERT (1774), often discussed later: Given a parallelogram, construct, with ruler only, a parallel to a given line. With two carpenter squares it is possible to solve the problems of trisection of an angle or duplication of a cube. See ARCHIBALD, "Constructions with a double edged ruler," *Amer. Math. Monthly*, v. 25, 1918, pp. 357–359. What PONCELET suggested as possible with ruler alone was carried out in great detail in a famous little book of STEINER: *Die geometrischen Konstruktionen, ausgeführt mittelst der geraden Linie und eines festen Kreises.* Berlin, 1833, 110 pp.+2 plates. There have been Polish, two Russian, and a French (partial) translations; an English edition by MARION E. STARK, with Introduction and Notes by R. C. ARCHIBALD is in the press. Through the efforts of JACOBI the Univ. of Königsberg in 1833 conferred the degree of Ph.D. on STEINER and in 1834 through the influence of JACOBI and the brothers ALEXANDER and WILLIAM VON HUMBOLDT a new chair of geometry was founded for him at the University of Berlin and at the same time he was elected a fellow of the Prussian Academy of Sciences. If, instead of the circle in the PONCELET-STEINER constructions be substituted a given central conic and one of its foci, then every point constructed

with ruler and compasses may be found with ruler alone. Also every problem of the third and fourth degrees could then be solved with ruler and compasses. This was first proved by H. J. S. SMITH, "Mémoires sur quelques problèmes cubiques et biquadratiques," *Annali di Matem.*, s. 2, v. 3, 1869, pp. 112–165, 218–242, or *Coll. Math. Papers*, v. 2, Oxford, 1894, pp. 1–66. For this, and a paper by KORTUM, the Prussian Academy of Sciences made a joint award of its STEINER Prize for 1868.

STEINER became acquainted with the Norwegian genius ABEL when he was in Berlin, and also with A. L. CRELLE who in 1826 founded the famous mathematical *Journal f. d. reine u. angew. Mathematik.* Four of STEINER's papers and one of ABEL's appeared in the first v. of this periodical.

278. SMITH [**150**], "Brianchon's theorem," ed. N. A. COURT, pp. 331–336. See also [**275**].

279. L. WANTZEL, "Recherches sur les moyens de reconnaître si un Problème de Géométrie peut se résoudre avec la règle et le compas," *Jn. de Math.* (LIOUVILLE), v. 2, 1837, pp. 366–372, particularly pp. 369 f. F. CAJORI, "Pierre Laurent Wantzel," Amer. Math. Soc., *Bull.*, v. 24, pp. 339–347, 1918.

279a. APOLLONIUS of Perga in his *Plane Loci* showed that the inverse of a circle is a circle. It is prop. 17, bk. 1, in the restoration, ROBERT SIMSON, *Apollonii Pergaei Locorum Planorum Liber II*, Glasgow, 1749; German translation by J. W. CAMERER, Leipzig, 1796. In effect the proposition is as follows: If from any two points A, B, lines $AC_1=r_1$, $BC_2=r_2$, are drawn, making a constant angle with one another, if $r_1r_2=k^2$, a constant, and if the locus of C_1 is a circle, so also is the locus of C_2, in general. For the more general proposition see PAPPUS, *Collection*, ed. by F. HULTSCH, Berlin, v. 2, 1877, pp. 661f; VER EECKE's French translation, Paris and Bruges, 1933, v. 2, p. 495.

280. PEAUCELLIER, a French army officer, announced his wonderful invention of a compass to draw a straight line, in a letter published in *Nouv. Annales de Math.*, s. 2, v. 3, 1864, p. 414. PEAUCELLIER there merely proposed as a problem to find a "compas" to describe in a continuous manner (1) a right line, (2) a circle, whatever its radius, (3) the conics. The letter concludes "Le mode de construction de ce genre du compas, supprimant tout mouvement de glissement, le tracé des courbes precitées est susceptible d'une extrême précision."

Announcement of PEAUCELLIER's invention of 1864 was made in 1867, without description, by a brother officer of engineers, A. MANNHEIM, at a meeting of the Paris Philomathic Society; but this announcement attracted no particular attention until 1871 when the mechanism was rediscovered by a young man named LIPKIN (Akad. Nauk., St. Petersburg, *Bull.*, v. 16, 1871, pp. 57–60) a pupil of the celebrated CHEBYSHEV (of the Univ. of St. Petersburg) who had been laboring to demonstrate the impossibility of that which his pupil thus achieved. LIPKIN got a substantial reward from the Russian Government for his supposed originality. Thereafter PEAUCELLIER's merit was at last recognized and he was awarded the "Prix MONTYON"—the great mechanical prize of the Institut de France. PEAUCELLIER's first paper on the linkage was published in *Nouv. Annales de Math.*, s. 2, v. 12, 1873, pp. 71–78, and linkages for conic sections, conchoids of a circle, cissoid, were described. SYLVESTER proposed (with interesting comment) the PEAUCELLIER cell as a problem in the *Educ. Times* (*Reprint*, v. 21, 1874, p. 58). PEAUCELLIER's cell had 7 links. It was HARRY HART who discovered the five-bar linkage (*Cambr. Mess. Math.*, v. 4, pp. 82–88, 116–120, 1874).

281. D. H. LEAVENS, "Linkages," *Amer. Math. Monthly*, v. 22, 1915, pp. 330–334; and C. M. HEBBERT, "A cardiograph," *idem*, pp. 12–13. A. EMCH, *An Introduction to Projective Geometry*, New York, 1905, pp. 242–260. V. LIGUINE, "Liste des travaux sur les systèmes articulés" *Bull. d. Sci. Math.*, v. 18, 1883, pp. 145–160; [151 titles published before 1883.] In *Tôhoku Math. Jn.*, v. 37, 1933, pp. 294–319, R. KANAYAMA gave a bibliography of 306 titles (1631–1931) including those of LIGUINE's list. That even this last list is not by any means exhaustive even to the end of 1931, is shown in *Scripta Math.*, v. 2, pp. 293–294, 1934.

282. CAJORI [**2**], pp. 349–350. The proof of RUFFINI appears in his book *Teoria generale delle Equazioni*, Bologna, 1799, and in later articles on the subject. H. BURKHARDT, "Die Anfänge der Gruppentheorie und P. R.," *Abh. z. Gesch. d. mathem. Wissen.*, Heft 6, 1892, pp. 119–159. E. BORTOLOTTI, *Influenza dell'Opera matematica di Paolo Ruffini sullo svolgimento delle teorie algebriche.* Modena, 1902, 57 pp. See also [**215**].

283. W. A. J. ARCHBOLD, "W. G. Horner," Dict. Nat. Biog., v. 27, 1891; SMITH **[150]**, "Horner's method," ed. by M. McGUIRE, pp. 232–252. HORNER's first paper on the subject was publ. in R. Soc. London, *Trans.*, v. 109, 1819, pp. 308–335. It was reprinted, with some additional matter in *Ladies Diary*, 1838, pp. 49–72. Two revisions appeared in LEYBOURN's *Math. Repository*, v. 5, 1830, part II, pp. 21–75, and in *The Mathematician*, "On algebraic transformations," v. 1, 1845, pp. 108–116, 136–142, 311–316. F. CAJORI, "Horner's method of approximation anticipated by Ruffini," Amer. Math. Soc., *Bull.*, v. 17, 409–414, 1911. See also **[215]**.

284. CAJORI **[2]**, pp. 363–364. WOLF **[222]**, v. 4, 1862, pp. 375–389. M. BÔCHER, "The published and unpublished work of Charles Sturm on algebraic and differential equations," Amer. Math. Soc., *Bull.*, v. 18, pp. 1–18, 1911. G. LORIA, "Charles Sturm et son oeuvre mathématique," *L'Enseign. Math.*, v. 37, 1939, pp. 249–272+portrait plate **[246]**.

285. If the roots of the equation $f(x)=0$ are under consideration the theorem states that if N is the number of roots in the complex plane within a closed circuit A, not passing through any of the roots, then

$$N=\frac{1}{2\pi i}\int\frac{f'(x)}{f(x)}\,dx,$$

the integral being taken around A. More generally, if $f(x)$ is continuous except for a finite number of poles, N is the difference between the number of poles and the number of zeros of $f(x)$. See CAUCHY, *Bull. Sci. Math.*, v. 16, 1831, pp. 116–128; Paris, École Polytechnique, *Jn.*, cahier 25, v. 15, 1837, pp. 176–193; C. STURM & J. LIOUVILLE. "Démonstration d'un théorème de M. Cauchy, relatif aux racines imaginaires des équations," *Jn. d. Math.* (LIOUVILLE), v. 1, 1836, pp. 278–289, and C. STURM, "Autres démonstrations du même théorème," *idem*, pp. 290–308. A reference may also be given to E. McCLINTOCK, "A method for calculating simultaneously all the roots of an equation," *Amer. Jn. Math.*, v. 17, 1895, pp. 89–110.

286. BELL **[53]**, "Mathematics and windmills—Cauchy," pp. 294–306. PRASAD **[254]**, v. 1, pp. 68–110. SMITH **[54]**. CAJORI **[2]**, pp. 368–370. R. STRUIK & D. J. STRUIK, "Cauchy and Bolzano in Prague," *Isis*, v. 11, pp. 364–366, 1928. J. BERTRAND, *Éloges Académiques*, Nouv. série. Paris, 1902, "Éloge de Augustin Louis Cauchy," pp. 101–120. P. E. B. JOURDAIN, "The origin of Cauchy's conceptions of a definite integral and of the continuity of a function," *Isis*, v. 1, pp. 661–703, 1914. SMITH **[150]**, "Möbius, Cayley, Cauchy, Sylvester, and Clifford on geometry of four or more dimensions," ed. by H. P. MANNING, p. 524; "Cauchy on higher space" ed. by H. P. MANNING, pp. 530–531; "Cauchy on the derivatives and differentials of functions of a single variable," ed. by E. WALKER, pp. 635–637. I. TODHUNTER, *A History of the Progress of the Calculus of Variations in the Nineteenth Century.* Cambridge and London, 1861, pp. 210–228. etc. C. A. VALSON, *La Vie et les Travaux du Baron Cauchy.* Paris, 2 v., 1868, xxiv, 290, xxiv, 178 pp. CAUCHY, *Oeuvres Complètes.* 25 v. Paris, 1882–1938, to the end of 1947.

287. A. SACHSE, "Versuch einer Geschichte der Darstellung willkürlicher Functionen einer Variablen durch trigonometrische Reihen," *Abh. z. Gesch. d. Math.*, Heft 3, 1880, pp. 229–276. BÔCHER **[246]**. LANGER **[246]**.

288. Herewith is a selected list of references on this topic, also indicating its history: HILDA HUDSON, *Ruler and Compasses*, London, 1916, pp. 112–117. J. S. MACKAY, "Geometrography of Euclid's problems," Edinb. Math. Soc., *Proc.*, v. 12, 1894, pp. 2–16. J. L. COOLIDGE, *A Treatise on the Circle and the Sphere.* Oxford, 1916, pp. 166–179; applications to the problem of APOLLONIUS and MALFATTI's problem. É. LEMOINE, "De la mésure de la simplicité dans les constructions géométriques," Acad. d. Sci., Paris, *Comptes Rendus*, v. 107, 1888, pp. 169–171. É. LEMOINE, *Géométrographie ou Art des Constructions Géométriques.* (*Scientia* series, no. 18). Paris, 1902, 87 pp. ROUCHÉ & COMBEROUSSE, *Traité de Géométrie.* Paris, seventh ed., 1900; Note IV: "Sur la géométrographie," by E. L., pp. 517–548. A. ADLER, *Theorie der geometrischen Konstruktionen.* Leipzig, 1906, pp. 277–301. T. VAHLEN, *Konstruktionen und Approximationen in systematischer Darstellung.* Leipzig 1911; "Geometrographie und Fehlertheorie," pp. 121–128. E. PAPPERITZ, "Darstellende Geometrie," *Encykl. d. math. Wissen.*, v. 3_1, part 4, 1910, pp. 529–531, 536. C. ALASIA, *La recente Geometria del Triangolo.* Citta di Castello, 1900, xvi, 339 pp.; "La geometrografia," pp. 29–49, 282–

283. A different system from that of Lemoine is set forth in A. GRÜTTNER, *Die Grundlagen der Geometrographie.* Progr. Breslau. Leipzig, 1912, 54 pp. On pp. 7–8 GRÜTTNER remarks that already in 1833, in his work on *Geometrical Constructions* [277], STEINER formulated the thought which is the basis of geometrography. LEMOINE wrote also on geometrography of three dimensions, *Mathesis*, 1902, pp. 105–107. LEMOINE was really the founder of *L'Intermédiaire d. Mathématiciens*, of which he and his friend C. A. LAISANT were the editors, v. 1–17, 1894–1910. For sketches, each with a portrait of LEMOINE, see D. E. SMITH, *Amer. Math. Monthly*, v. 3, 1896, pp. 29–33, and C. A. LAISANT, *L'Enseignement Math.*, v. 14, 1912, pp. 177–183. Then the v. by L. AUGÉ DE LASSUS, *La Trompette un demisiècle de musique de Chambre*, Paris, 1911, ii, 239 pp., tells of the remarkable half-century of gatherings at LEMOINE'S home of a good part of the scientific, literary, and artistic circles of Paris, for the celebrated musical soirées.

289. For example, there are *Points* associated with such names as: BROCARD, CRELLE, GAUSS, JEŘÁBEK, LEMOINE, NAGEL, STEINER, TARRY; among *Lines:* antiparallel, BROCARD, isogonal, LEMOINE, PHILO, symmedian; among *Triangles:* BROCARD, FUHRMANN, KIEPERT, LIONNET, MORLEY; among *Circles:* APOLLONIUS, JOACHIMSTHAL, LONGCHAMPS, M'CAY, SCHOUTE, SPIEKER, TAYLOR, TUCKER; among *Ellipses:* BROCARD, CESÀRO, STEINER; among *Hyperbolas:* APOLLONIUS, FEUERBACH, KIEPERT; among *Parabolas:* ARTZ, KIEPERT, MANDART, NEUBERG; and among *Theorems:* CATALAN, CEVA, CHAPPLE, LUCAS, MENELAUS, SCHLÖMILCH.

A few selected references in this field are as follows: CAJORI [3], pp. 259–263. C. ALASIA, *La Recente Geometria del Triangolo.* Città di Castello, 1900, xvi, 339 pp.; there are 566 formulae for the triangle, pp. 309–339. C. ALASIA, *Saggio Terminologico-Bibliografico sulla Recente Geometria del Triangolo.* Bergamo, 1902, iv, 43 pp. J. CASEY, (a) *A Sequel to the First Six Books of the Elements of Euclid.* Fifth edition rev. and enl. Dublin and London, 1888, pp. 165–248; not in earlier or later English eds.—French ed. Ghent and Paris, 1890, 80 pp. of the pages 165–248, and of an unpubl. supplement by CASEY. (b) *A Treatise on the Analytical Geometry of the Point, Line, Circle and Conic Sections.* Second ed. rev. and enl., Dublin and London, 1893, pp. 418–461. J. S. MACKAY, (a) History of the nine point circle," pp. 19–57; (b) "Early history of the symmedian point, pp. 92–103, Edinb. Math. Soc., *Proc.*, v. 11, 1893. A. EMMERICH, *Die Brocardschen Gebilde und ihre Beziehungen zu den verwandten merkwürdigen Punkten und Kreisen des Dreiecks.* Berlin, 1891, 154 pp.+1 folding plate. R. A. JOHNSON, *Modern Geometry of the Triangle and the Circle.* Boston, 1929. xiv, 319 pp. W. J. M'CLELLAND, *A Treatise on the Geometry of the Circle. . . .* London, 1891, pp. 60–120, 207–217, etc. J. J. MILNE, *Companion to the Weekly Problem Papers.* London, 1888, pp. 99–191, by T. C. SIMMONS. R. C. J. NIXON, *Euclid Revised.* Third ed. Oxford, 1899; "The modern geometry of the triangle," pp. 378–403.

290. E. B. WILSON, "Reminiscences of Gibbs by a student and colleague," Amer. Math. Soc., *Bull.*, v. 37, pp. 401–416, 1931. R. E. LANGER, "Josiah Willard Gibbs," *Amer. Math. Monthly*, v. 46, 1939, pp. 75–84. E. B. WILSON, *Dict. Amer. Biog.*, "Gibbs," v. 7, 1931, pp. 248–251. H. A. BUMSTEAD, biographical sketch and bibliography in (a) *Amer. Jn. Sci.*, s. 4, v. 16, 1903, pp. 187–202; (b) *The Collected Works of J. Willard Gibbs*, 2 v., v. 1, New York, 1928, pp. xiii–xxviii. *A Commentary on the Scientific Writings of J. Willard Gibbs.* Edited by F. G. DONNAN and A. HAAS. 2 v., New Haven, 1936; each of these v. deals with the corresponding v. of the *Collected Works.* MURIEL RUKEYSER, *Willard Gibbs*, New York, 1942, xi, 465 pp.; an interesting (though far from competent) biography written by a poet. In 1923 the American Mathematical Society established the JOSIAH WILLARD GIBBS [honorary] Lectureship, for annual semi-popular lectures on various aspects of mathematics and its applications.

291. H. APPLEYARD, *Dict. Nat. Biog. 1922–1930*, 1937. A. RUSSELL, "Mr. Oliver Heaviside, F. R. S.," *Nature*, v. 115, 1925, pp. 237–238. E. T. WHITTAKER, "Oliver Heaviside," Calcutta Math. Soc., *Bull.*, v. 20, 1930, pp. 199–220.

292. At three years of age HAMILTON was a superior reader of English and considerably advanced in arithmetic; at four a good geographer; at five able to read and translate Latin, Greek and Hebrew and liked to recite DRYDEN, COLLINS, MILTON, and HOMER; at eight a reader of Italian and French and giving vent to his feelings in extemporized Latin; at ten a student of Arabic and Sanscrit. His career at Trinity College, Dublin, was extraordinary in that he achieved the

previously unheard of distinction of winning the highest possible marks, both in mathematical physics and in Greek. *A Collection of Papers in memory of Sir William Rowan Hamilton* (*Scripta Mathematica Studies*, no. 2). New York, 1945, 82 pp.+2 plates. [Partial contents: Facsimile pages from HAMILTON'S Notebooks, pp. 12, 36. J. L. SYNGE, "The life and early work of W. R. H.," pp. 13–24. C. C. MACDUFFEE, "Algebra's debt to Hamilton," pp. 25–35; F. D. MURNAGHAN, "An elementary presentation of the theory of quaternions," pp. 37–49; "Poems by William Rowan Hamilton," p. 50. H. BATEMAN, "Hamilton's work in dynamics and its influence on modern thought," pp. 51–63. "The Hamilton postage stamp" pp. 81–82.] A. MACFARLANE, *Lectures on Ten British Mathematicians of the Nineteenth Century*. New York, 1916, 148 pp. W. R. H., pp. 34–39; [the other sketches are of G. PEACOCK, A. DEMORGAN, G. BOOLE, A. CAYLEY, W. K. CLIFFORD, H. J. S. SMITH, J. J. SYLVESTER, T. P. KIRKMAN, I. TODHUNTER]. SMITH [54]. BELL [53], "An Irish tragedy—Hamilton," pp. 340–361. "Quaternion centenary celebration [8 Nov. 1943]," R. Irish Acad., *Proc.*, v. 50A., pp. 69–122+2 plates, 1945. [Partial contents: Opening remarks by the president; messages from J. L. SYNGE and G. D. BIRKHOFF; "The Dublin mathematical school in the first half of the nineteenth century" by A. J. MCCONNELL, pp. 75–88; "Quaternions" HAMILTON ms. 16 Oct., 1843, pp. 89–90; "The sequence of ideas in the discovery of quaternions," pp. 93–98.] H. T. H. PIAGGIO, "The significance and development of Hamilton's quaternions," *Nature*, v. 152, 1943, pp. 553–555. R. P. GRAVES, *Life of Sir William Rowan Hamilton . . . including Selections from his Poems, Correspondence and Miscellaneous Writings*. Dublin, v. 1, 1882, 698 pp.+plate; v. 2, 1885, xvi, 719 pp.+plates; v. 3, 1889, xxxvi, 673 pp.+plates; Addendum, 1891, 17 pp. with particular reference to corrections of R. E. ANDERSON, *Dict. Nat. Biog.*, W. R. H., v. 24, 1890. R. C. ARCHIBALD, "Mathematicians, and poetry and drama," *Science*, v. 89, 1939, pp. 19–26, 46–50; W. R. H., pp. 25–26. Two v. of *The Mathematical Papers of Sir William Rowan Hamilton* have been published: v. 1, *Geometrical Optics*, ed. by A. W. CONWAY & J. L. SYNGE. Cambridge, 1931, xxx, 534 pp.; Éloge by CHARLES GRAVES, pp. xvii–xxviii; v. 2, *Dynamics*, ed. by A. W. CONWAY & A. J. MCCONNELL, 1940, xviii, 656 pp.

Having referred to GIBBS, HEAVISIDE, and HAMILTON, I should also, perhaps, have mentioned a German genius, HERMANN GÜNTHER GRASSMANN (1809–77) and his two "Ausdehnungslehren" (1844, 1862), forming the basis of vector analysis. The first work in English on this subject was *The Directional Calculus Based upon the Methods of Hermann Grassmann* by E. W. Hyde, Boston, 1890, xii, 247 pp. SMITH [150], "Grassmann on the Ausdehnungslehre" ed. by M. KORMES, pp. 684–696. Commenting on GRASSMANN'S many-sided make-up and enormous industry (Amer. Math. Soc., *Bull.*, v. 22, 1916, pp. 150), E. B. WILSON noted that he started as a theologian, invented his analysis of extension, wrote on physics, composed class texts for the study of German, Latin, and mathematics, edited a political paper and a missionary paper, investigated phonetic laws, published a dictionary of the Rig-Veda and a translation of it into verse, harmonized folk-songs in three voices—in addition to carrying on successfully his regular work as a teacher and an administrator, and to bringing up nine of his eleven children.

293. *Mathematical Tables and Other Aids to Computation*, nos. 1–24 (Jan. 1943–Oct. 1948).

294. D. H. LEHMER, Washington, D. C., 1941, xiv, 177 pp.

295. R. C. ARCHIBALD, "The New York Mathematical Tables Project," *Science*, n. s., v. 96, 25 Sept. 1942, pp. 294–296.

296. J. W. L. GLAISHER, 175 pp. See also GLAISHER'S articles in *Encycl. Britannica*, eleventh ed., 1911, art. "Logarithm," v. 16, pp. 868–877, and "Table mathematical," v. 26, pp. 325–336. There are also valuable older articles of A. DE MORGAN, in (a) *Penny Cycl.*, art. "Table," v. 23, 1842, pp. 496–501; and *Suppl.*, v. 2, 1846, pp. 595–605. (b) *Engl. Cycl.*, *Arts and Sci. Sect.*, v. 7, 1861, cols. 976–1016. See also reports of the Tables' Comm. in BAAS, *Report* 1875, 32 pp.; 1878, 6 pp.+1 pl.; 1879, 12 pp.+1 pl.; 1880, 9 pp.+1 pl.; 1881, 6 pp.; 1883, 8 pp. See also annual Committee reports 1930–1939; and the history, 1871–1948, *MTAC*, Jan. 1949.

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Completus. Leipzig, 1794. See also *Math. Tables and Other Aids to Computation*, v. 2, pp. 161-165, 311–312.

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300. D. N. LEHMER, (a) *Factor Table for the First Ten Millions. . . .* Washington, 1909, 488 pp.; (b) *List of Prime Numbers from* 1 to 10,006,721. Washington, 1914, 143 pp.

301. J. C. BURCKHARDT, (a) *Table des Diviseurs pour tous les Nombres des 1er, 2e, et 3e Million. Ou plus exactement, depuis 1 à 3 036000.* Paris, 1817, xii, 114, iv, 112, iv, 112 pp. There was a facsimile edition of this v. published at Berlin in 1909.

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302a. A. J. C. CUNNINGHAM, *A Binary Canon.* 1900, viii, 172 pp. See ARCHIBALD **[163]**, pp. 18–21, and *Math. Tables and Other Aids to Computation*, v. 1, p. 109; v. 3, pp. 143–144.

303. L. WOODRUFF, ed., *The Development of the Sciences.* New Haven, 1923; Mathematics by E. W. BROWN, pp. 1–42; Physics by H. A. BUMSTEAD, pp. 43–73; Astronomy by F. SCHLESINGER, pp. 126–167. Second series, New Haven, 1941; lectures on the history of science organized by the Yale chapter of the Gamma Alpha Graduate Scientific Fraternity. [Mathematics by O. ORE, pp. 1–51; Physics by H. MARGENAU, pp. 91–120; Astronomy by F. SCHLESINGER, pp. 53–89.]

304. F. KLEIN, *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert*, 2 v., Berlin, 1926–1927, xiv, 386, x, 208 pp.

305. R. COURANT, "Riemann und die Mathematik der letzten hundert Jahre," *Naturwissen.*, v. 14, 1926, pp. 813-818.

306. R. S. WOODWARD, "The century's progress in applied mathematics," Amer. Math. Soc., *Bull.*, v. 6, pp. 133–163, 1900; retiring presidential address, 1899.

307. New York, 1938, viii, 315 pp. [Contents: E. T. BELL, "Fifty years of algebra in America," pp. 1–34; J. F. RITT, "Algebraic aspects of the theory of differential equations," pp. 35–55; N. WIENER, "The historical background of harmonic analysis," pp. 56–68; E. J. MCSHANE, "Recent developments in the calculus of variations," pp. 69–97; T. Y. THOMAS, "Recent trends in geometry," pp. 98–135; R. L. WILDER, "The sphere in topology," pp. 136–184; G. C. EVANS, "Dirichlet problems," pp. 185–226; J. L. SYNGE, "Hydrodynamical stability," pp. 227–269; G. D. BIRKHOFF, "Fifty years of American mathematics," pp. 270–315.]

308. R. C. ARCHIBALD, *A Semicentennial History of the American Mathematical Society 1888–1938 with Biographies and Bibliographies of the Past Presidents.* New York, 1938, pp. 107–244.

309. D. E. SMITH & J. GINSBURG, *A History of Mathematics in America before 1900.* Chicago, 1934, x, 209 pp.

310. F. CAJORI, Washington, D. C., 1890, 400 pp.

311. Ed. by H. M. TORY. Toronto, 1939, vi, 152 pp.; Partial Contents: "The history of astronomy in Canada" by W. E. HARPER, pp. 87–99; "An outline of the progress of mathematics in Canada," by S. BEATTY, pp. 100–119; "The advance of physics in Canada," by A. N. SHAW, pp. 120–152.

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314. *The Pneumatics of Hero of Alexandria from the original Greek.* Translated and edited by BENNET WOODCROFT. London, 1851, xx, 118 p. A. G. DRACHMANN, *Ktesibios, Philon and Heron. A Study in Ancient Pneumatics. Publications of the University Library*, Copenhagen, v. 4; *Scientific and Medical Department.* Copenhagen, 1948, xii, 197 p. Water-clocks are discussed, p. 16–41; see **[39]**.

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Numbers refer to pages. Such a symbol as **[42]**, following a name, refers to Note **42** of the Literature List and Notes (328) where the name occurs. Dates of flourishing, or of death, or of birth and death are added in connection with most of the names.

"Thou hast made me known to friends whom I knew not. Thou hast given me seats in homes not my own. Thou hast brought the distant near and made me a brother of the stranger."

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NEWTON AS AN ORIGINATOR OF POLAR COÖRDINATES

C. B. BOYER, Brooklyn College

The name of Newton, indissolubly linked with the calculus, seldom is associated with analytic geometry, a field to which he nevertheless made important contributions. Newton's use of polar coördinates, for example, seems to have been overlooked completely in the historiography of mathematics. The polar coördinate system is ascribed generally [1] to Jacques Bernoulli in 1691 and 1694, although it has been attributed [2] to others as late as Fontana in 1784. It is the purpose here to call attention to an application of polar coördinates made by Newton probably a score of years before the earliest publication of Bernoulli's work.

In the Horsley edition of the *Opera* of Newton there appears a treatise entitled *Artis analyticae specimina vel Geometria analytica* [3] which is essentially the same as the Newtonian *Method of fluxions*, published in 1736 by Colson. The discrepancy in titles—*Geometria analytica* or *Method of fluxions*—conveniently indicates that the work treats of coördinate geometry as well as the calculus. In fact, its analytic form stands in marked contrast to the synthetic style of the *Principia*, which also contained some elements of the calculus. The *Method of fluxions* makes systematic use of coördinates in problems on tangents, curvature, and rectification. Moreover, Newton did not limit himself, as had his predecessors, to a single type of coördinate system. Having shown how to use fluxions in finding tangents to curves given in terms of Cartesian coördinates, oblique as well as rectangular, Newton included some examples illustrating other types. In connection with these he gave, informally, the equivalent of equations of transformation for polar and rectangular coördinates, $xx+yy=tt$ and $tv=y$, where t is the radius vector and v is a line representing the sine of the vectorial angle associated with the point (x, y). Following these exercises, Newton proceeded to give a more definitive account of non-Cartesian systems: "However it may not be foreign from the purpose, if I also shew how the problem may be perform'd, when the curves are refer'd to right lines, after any other manner whatever; so that having the choice of several methods, the easiest and most simple may always be used" [4]. To illustrate his point, Newton suggested eight new types of coördinate system, made up of various combinations of pairs of distances measured radially from given points, or obliquely to given fixed lines, or curvilinearly along arcs of circles. One of the new systems—Newton refers to it as the "Seventh Manner; For Spirals"—is essentially that now known as polar coördinates. Let A be the center and AB a radius of the circle BG (Figure 1), and let D be any point on the curve ADd. Then, designating BG by x and AD by y, the curve ADd is determined by a relationship between x and y. Newton suggested $x^3-ax^2+axy-y^3=0$ as an illustration, and for this curve he determined, from the proportion $\dot{y}:\dot{x}::AD:At$, the polar subtangent AT for a point D. Similarly Newton found the polar subtangents of $y=ax/b$, "which is the equation to the spiral of Archimedes," and the curve $by=xx$; and, he concluded, "thus tangents may be easily drawn to any spirals whatever" [5].

Following the calculation of the radius of curvature for rectangular Cartesian coördinates x and y, $r=\overline{1+zz}\sqrt{\overline{1+zz}}/\dot{z}$ where $z=\dot{y}$ and fluxions of independent variables are taken as unity, Newton again turned to the corresponding problem in polar coördinates. Using a diagram and a notation similar to those applied in connection with tangent problems—but with the radius AB of the reference circle taken as unity—he derived the result

$$r \sin \psi = \frac{y + yzz}{1 + zz - \dot{z}},$$

where $z=\dot{y}/y$ and ψ is the angle between the tangent and the radius vector

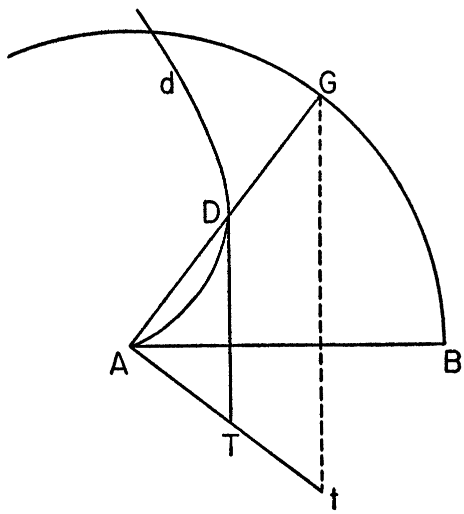

FIG. 1

(fluxions of independent variables again being taken as unity). Newton applied this formula, virtually the same as the modern equivalent, to the spiral of Archimedes and to the curves $ax^2=y^3$ and $ax^2-xy=y^3$. In conclusion he added, "And thus you will easily determine the curvature of any other spirals; or invent rules for any other kinds of curves." That he realized the significance of his use of polar coördinates seems to be implied by his further comment that he here had "made use of a method which is pretty different from the common ways of operation" [6].

The comparison of the parabola with the spiral was a favorite topic of the seventeenth century, and in his treatment of this question, in the *Method of fluxions*, Newton made use of a polar coördinate system yet a third time. Here, however, his scheme differed from that previously presented. The notation, too, was modified, but this may have been done in order to avoid confusion in the simultaneous use of polar and Cartesian coördinates. If D is any point on a curve

ADd, Newton took the coördinates of D as z and v, where z is the radius vector AD and v is the circular arc BD (Figure 2). That is, whereas his earlier coördinates were, in modern notation, $(r, a\theta)$, Newton this time used $(r, r\theta)$. Then if the relation between z and v is given "by means of any equation"; and if a new curve AHh, given in rectangular coördinates $AB=z$ and $BH=y$, is so determined that, for all corresponding positions of D and H, the arc AD is equal to the arc AH; then Newton showed that $\dot{y}=\dot{v}-v\dot{z}/z$, or, if $\dot{z}$ is taken as unity, $\dot{y}=\dot{v}-v/z$. In particular, "if $zz/a=v$ is given as the spiral of Archimedes," then $\dot{v}=2z/a$, and hence $z/a=\dot{y}$ and $zz/(2a)=y$. The lengths of the spirals $z^3=av^2$ and $z\sqrt{a+z}=v\sqrt{c}$ are shown in like manner [7] to correspond respectively to lengths measured along the semi-cubical parabola $z^{3/2}=3a^{1/2}y$ and the curve $(z-2a)\sqrt{ac+cz}=3cy$.

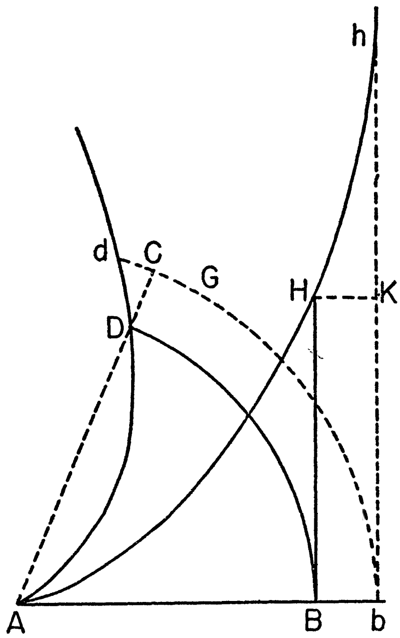

FIG. 2

Evidence indicates [8] that the *Method of fluxions* was composed by 1671, at which time Jacques Bernoulli was in his teens; and there seems to be no reason for suspecting the sections on polar coördinates as later interpolations. The three pertinent passages would appear to be a natural part of the whole; and Horsley, after his editorial examination of three different manuscript copies of the work, apparently saw no reason to question the date or authenticity of this material. It is surprising therefore that this contribution to coördinate geometry should have gone unnoticed so completely that the use of polar coördinates invariably is attributed to others of later periods. Newton is not entitled to priority of publication, for the work appeared posthumously in 1736; but evi-

dence indicates that he was the first one to adopt a system of polar coördinates in strictly analytic form [9]. Moreover, his work in this connection is superior, in flexibility and generality, to any similar proposal to appear during his lifetime.

Priority in the publication of polar coördinates seems to go to Jacques Bernoulli who in the *Acta Eruditorum* of 1691 proposed measuring abscissas along the arc of a fixed circle and ordinates radially along the normals. Three years later, however, he presented in the same journal a system identical, both in conception and notation, with that first proposed by Newton. He used the coördinates y and x, where y is the length of the radius vector of the point and x is the arc cut off by the sides of the vectorial angle on a circle of radius a described about the pole as center. That is, Bernoulli too adopted the coördinates $(r, a\theta)$, whereas in his earlier work he had used a less convenient system equivalent to $(a-r, a\theta)$. Bernoulli, like Newton, was interested primarily in applications of his system to the calculus; and so he also derived a formula for radius of curvature in polar coördinates, [10] and applied it to the spiral of Archimedes, $y=ax:c$.

The polar coördinates of Newton and Bernoulli in 1704 were applied by Varignon [11] to a comparison of the higher parabolas and spirals of Fermat, but no reference was made to Newton's work. Varignon ascribed the idea to Jean Bernoulli and gave to Jacques Bernoulli only the credit for priority of publication. His information in this connection was perhaps not unbiased; and his treatise is tedious and unimaginative in comparison with the work of Newton, at that time still unpublished.

In 1729, two years after Newton's death, Hermann approached polar coordinates from a new point of view. He did not concern himself with spirals, as had Newton, Bernoulli, and Varignon, nor was he chiefly interested in the calculus. He proposed the study of loci "through the relationship which vectorial radii bear to the sine or cosine of the angles of projection, from the consideration of which the properties of curves flow just as elegantly as they are brought out in the usual manner" [12]. That is, Hermann seems first to have thought of polar coördinates as a part of analytic geometry proper. He gave equations for transforming from Cartesian to polar coördinates, and he applied his new system to a number of algebraic curves, including the conics. It should be noted, however, that he did not express his equations specifically in modern form, but wrote them in terms of z, m, and n, where z is the radius vector and m and n are the sine and cosine respectively of the vectorial angle. Moreover, where his predecessors had applied the polar system to spirals alone, Hermann inversely used the scheme exclusively for algebraic curves.

Euler in 1748 seems to have been the first one to combine the points of view of Newton and Hermann. In the influential *Introductio in analysin infinitorum* he devoted a large portion of each of two chapters to polar coördinates, one dealing with algebraic curves and the other with spirals. In the first case [13]

he gave the equations of transformation $x=z \cos \phi$, $y=z \sin \phi$, introducing modern trigonometric symbolism into polar coördinates. He gave general consideration to z as a function of $\sin \phi$ and $\cos \phi$, and he noted in more detail the limaçons $z=b \cos \phi \pm c$ and the conchoids $z=b/\cos \phi \pm c$. In the treatment of transcendental curves Euler adopted a slightly different notion and notation for the independent variable in polar coördinates [14]. Here he studied curves of the form $z=f(s)$, where the argument s is the arc of a unit circle which measures the angle ϕ, feeling, apparently, that coördinates must of necessity denote lengths. In connection with the spiral curves which he drew, Euler made use of the general angle, allowing s to increase indefinitely, both positively and negatively. The spiral of Archimedes therefore appeared, perhaps for the first time, in its dual form [15]. The work of Euler is so thorough and systematic that polar coördinates frequently are attributed to him [16]. Certainly no one after him deserves credit as the inventor of the system. Fontana in 1784 did perhaps supply the name "polar equation" of a curve [17], and he may have been first [18] in studying analytically curves of the form $r=f(\theta, \sin \theta, \cos \theta)$; but one gets the very definite impression that his ideas and manner of treatment were inspired by Euler. It is probably not too much to say that although Newton probably originated polar coördinates, it was the work of Euler which was the decisive factor in making the system a traditional part of elementary analytic geometry. Polar coördinates gradually achieved greater prominence until in 1857 there appeared an entire volume devoted to the analytic geometry of this system in the plane and in space [19]. In 1874 the system was generalized to include elliptic polar coördinates and hyperbolic polar coördinates [20].

It may not be inappropriate to point out here that bipolar coördinates, recently ascribed [21] to Cournot in 1847, also were proposed by Newton. Such a system appeared in the *Method of fluxions* as the "Third Manner" of determining a curve. Here Newton considered [22] the "ellipses of the second order," now known as "ovals of Descartes." In *La gêométrie* [23] Descartes had proposed these curves in connection with problems in refraction, but he handled them, as Newton remarked, "in a very prolix manner," without the application of coördinates. Newton therefore seems to have been the originator of bipolar coördinates in the strict sense. Representing by x and y the "subtenses" (or distances) of a variable point from two fixed points (or poles), Newton wrote "their relation" for the ovals as $a+ex/d-y=0$. From this equation he found the ratio of the fluxions, and hence the tangent line. Newton pointed out further that for $a-ex/d-y=0$, a contrary sense is indicated in the construction; and he noted that if $d=e$, the curve becomes a conic section. He closes this topic with the remark that "it would be easy . . . to give more Examples."

Newton's generalizations of the coördinate idea may not be among his greatest contributions to mathematics, but they do entitle him to a larger place in the history of analytic geometry. In this field, as well as in infinitesimal analysis, one may appropriately declare, *Ex ungue leonem.*

References

1. See, *e.g.*, J. L. Coolidge, *A history of geometrical methods* (Oxford, 1940), p. 171. Cf. Florian Cajori, *History of mathematics* 2nd ed., New York, 1931, pp. 221, 224.

2. See, *e.g.*, D. E. Smith, *History of mathematics* (2 vols., New York, c. 1923–1925), II, 324. Cf. Moritz Cantor, *Vorlesungen über Geschichte der Mathematik* (4 vols., Leipzig, 1880–1908), IV, 513.

3. *Opera quae exstant omnia* (5 vols., Londini, 1779–1785), I, 389–518. This appears also (with another title) in Newton's *Opuscula* (3 vols., Lausannae, 1744), I, 29–200.

4. Sir Isaac Newton, *The method of fluxions and infinite series* (transl. by John Colson, London, 1736), p. 51. Cf. *Opera*, I, 435.

5. *Method of fluxions*, p. 56, *Opera*, I, 440.

6. *Method of fluxions*, p. 70, *Opera*, I, 453.

7. *Method of fluxions*, pp. 132–134, *Opera*, I, 511–512.

8. See, *e.g.*, H. G. Zeuthen, *Geschichte der Mathematik im XVI. und XVII. Jahrhundert* (Leipzig, 1903), p. 374; and Coolidge, *op. cit.*, p. 320. Cf. also Newton, *La méthode des fluxions* (Paris, 1740), Buffon's Preface.

9. More or less vague adumbrations of the idea of polar coördinates can, of course, be found in earlier work going back as far as the time of Archimedes' spiral. In early works on perspective the use of concentric circles and radiating lines in problems relating to "deformations" or "anamorphoses" represents a non-analytic application of the polar coordinate idea, much as ideas of latitude and longitude were forerunners of Cartesian coördinates. Some work of James Gregory in 1668 also represents to some extent an anticipation of such a system. See James Gregory, *Tercentenary memorial volume* (ed. by H. W. Turnbull, London, 1939), p. 493.

10. See Jacques Bernoulli, *Opera* (2 vols., Genevae, 1744), I, 432, 578–580. Cf. also a note by Eneström in *Bibliotheca Mathematica* (3), XIII (1912–1913), 76–77.

11. Pierre Varignon, "Nouvelle formation de spirales," *Académie des Sciences, Mémoires*, 1704, pp. 69–131. Cf. *Académie des Sciences, Histoire*, 1704, pp. 47–57. Generalized spirals in polar coordinates appeared also in C. R. Reyneau, *Usage de l'analyse* (vol. II, Paris, 1708), p. 593 and in J. B. Caraccioli, De lineis curvis (Pisis, 1740), p. 161.

12. Jacob Hermann, "Consideratio curvarum in punctum positione datum projectarum, et de affectionibus earum inde pendentibus," *Commentarii Academiae Petropolitanae*, IV (1729), 37–46.

13. Leonhard Euler, *Introductio in analysin infinitorum* (2 vols., Lausannae, 1748), II, 212 ff.

14. *Ibid.*, II, 284 ff.

15. Gino Loria, "Perfectionnements, évolution, metamorphoses du concept de 'coordonnées,'" *Mathematica*, XVIII (1942), 125–145; XX (1944), 1–22, incorrectly ascribes this dual form to Cournot a century later.

16. See, *e.g.*, E. Müller, "Die verschiedenen Koordinatensysteme," *Encyklopädie der mathematischen Wissenschaften*, III (1), 596–770, especially, pp. 656–657. Cf. *Encyclopédie des sciences mathématiques*, III (3), 1, p. 47. See also Loria, *loc. cit.*

17. Gregorio Fontana, "Sopra l'equazione d'una curva," *Memorie di matematica e fisica della società italiana*, II (part 1, 1784), 123–141. See especially p. 128.

18. Gregorio Fontana, *Disquisitiones physico-mathematicae* (Papiae, 1780), pp. 184–185. Fontana used y instead of r, and for θ he took x, where x is the arc of a unit circle measuring the angle θ.

19. J. A. Grunert, *Analytische Geometrie der Ebene und des Raumes für polare Koordinaten* (Greifswald and Leipzig, 1857).

20. C. A. Laisant, "*Essai sur les fonctions hyperboliques*" (Paris, 1874), pp. 71–83.

21. Loria, *op. cit.*, p. 138. Loria here overlooks also the use of bipolar coördinates by L. N. M. Carnot, *Géométrie de position*, (Paris, 1803), p. 469, and J. D. Gergonne, *Annales de mathematiques*, IV, 1813–14, p. 42 f.

22. *Method of fluxions*, p. 54 f, *Opera*, I, 437 f.

23. See *The geometry of Descartes* (translated by D. E. Smith and M. L. Latham, Chicago and London, 1925), p. 114 ff.

THE MOTION OF A SLIDING HORIZONTAL HOOP

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1. Introduction. Two individuals are playing a game with a circular hoop. The rules of the game are as follows:

(a) The circular hoop is to be thrown or launched so that each of its points is always in contact with a horizontal rough floor.*

(b) The hoop must be launched so that its center has a prescribed initial speed $u_0 > 0$.

The objective of the game is to launch the hoop so that it will travel the farthest, distances being measured from the center of the hoop. We shall prove that:

(A) The center of the hoop travels in a straight line, and if the hoop is given a non-zero initial spin, it will stop spinning at the instant the center comes to rest.

(B) The person who throws last can always win, assuming that he has the physical ability to apply the required knowledge.

This mechanical problem owes much of its interest to the fact that although the differential equations governing the motion cannot be solved in terms of elementary functions, interesting qualitative results can be obtained by using methods on a level with a course in Advanced Calculus.

In order to formulate the problem precisely, the following definitions and assumptions are made.

(*i*) Let μ be the coefficient of friction corresponding to the floor and the hoop. The frictional force per unit mass F (a vector) is defined as follows: If a point P of the hoop is moving, then F at P has a direction opposite to that of the velocity of P relative to the floor, and $|F| = \mu g$. If a point P of the hoop is at rest at a time t_1 then the force $F=0$ at P for the time t_1. The component of the total frictional force on the hoop in a given direction is obtained by the usual process of integration.

2. First results. Let P_0 be the center of the hoop of radius r, and designate the speed of P_0 relative to a fixed coördinate system (ξ, η) by u. Choose an (x, y) coördinate system with center at P_0, so that the positive x-axis has the same direction as that in which P_0 is moving. This is illustrated in the figure, and the (ξ, η) axes are omitted so as not to prejudice the argument. The rotation is assumed to be counterclockwise.

Consider any point P in the hoop with coördinates (x, y), and let θ be the angle between the positive x-axis and the directed line segment P_0P (see figure). The point P relative to P_0 has a velocity V which is perpendicular to P_0P and $|V| = |r\omega|$ where ω is the angular speed of the hoop about P_0. Letting U be the vector velocity of P_0, let ϕ be the angle between U and $U+V$ (see figure), and let $v = r\omega$. Due to assumption (*i*), the force field per unit mass $F(u, v, \theta)$ has the following X and Y components:

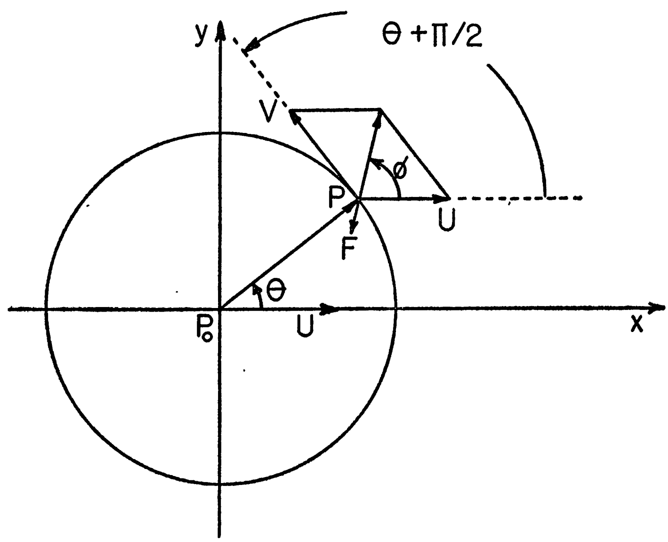
* A circular hoop is a homogeneous one-dimensional circular mass.

$$(1)\quad \begin{cases} X = Y = 0, & \text{for } (u, v, \theta) = (\delta, \delta, \pi/2) \text{ or for } (u, v, \theta) = (0, 0, \theta), \\ X = -\mu g \cos\phi, \quad Y = -\mu g \sin\phi, & \text{(otherwise)}, \end{cases}$$

where δ is an arbitrary real number, where $0 \leqq \theta \leqq 2\pi$, and where $u \geqq 0$, $v \geqq 0$.

Clearly when $(u, v, \theta) \neq (\delta, \delta, \pi/2)$ and when $(u, v) \neq (0, 0)$, by considering the components of U and V, we see that

$$(2)\quad \cos\phi = \frac{u - v\sin\theta}{\sqrt{u^2 + v^2 - 2uv\sin\theta}}, \qquad \sin\phi = \frac{v\cos\theta}{\sqrt{u^2 + v^2 - 2uv\sin\theta}}.$$



Henceforth we use the notation

$$Q = u^2 + v^2 - 2uv\sin\theta.$$

It is important to note that X and Y in equations (1) are bounded for all (u, v, θ), and that they are continuous in these variables except at $(u, v, \theta) = (\delta, \delta, \pi/2)$ and at $(u, v, \theta) = (0, 0, \theta)$. Since U is tangent to the path of P_0, the normal component a_n of the acceleration of P_0 satisfies the equation

$$2\pi r\sigma a_n = 2\pi r\sigma u^2 K = \int_0^{2\pi} Y\sigma r d\theta, \qquad u \neq 0,$$

where K is the curvature of the path. Since Y is continuous except possibly at $\theta = \pi/2$ when $u \neq 0$, and since $|Y| \leqq \mu g$, equations (1) and (2) imply

$$\int_0^{2\pi} Y d\theta = -\mu g \lim_{\epsilon \to 0}\left[\int_0^{\pi/2-\epsilon} \frac{v\cos\theta}{\sqrt{Q}} d\theta + \int_{\pi/2+\epsilon}^{2\pi} \frac{v\cos\theta}{\sqrt{Q}} d\theta\right] = 0,$$

as can be verified readily by means of antiderivatives. Thus when $u \neq 0$, the curvature $K=0$. *Hence the center of the hoop moves on a straight line.* This fact can also be seen by considering the forces $F(u, v, \theta)$ and $F(u, v, \pi-\theta)$, $(0 \leqq \theta \leqq 2\pi)$. The Y components of these two forces are equal except for sign, so that the above integral is zero. Hence the x-axis can be chosen parallel to the ξ-axis of the fixed system.

We next derive the differential equations governing the motion of the hoop. By Newton's second law of motion, and by the principle of the moment of momentum, we have

$$
\begin{aligned}
2\pi r\sigma \frac{du}{dt} &= \int_0^{2\pi} X\sigma r d\theta, \\
I \frac{d\omega}{dt} &= \int_0^{2\pi} (xY - yX)\sigma r d\theta,
\end{aligned} \tag{3}
$$

where $I = 2\pi r^3 \sigma$, and where $\sigma =$ density of the hoop. Since by (1) when $(u, v) \neq (\delta, \delta)$, $\delta \geqq 0$, we have $xY - yX = -\mu g r(\cos\theta \sin\phi - \sin\theta \cos\phi)$, equations (2) imply that

$$
xY - yX = \frac{-\mu g r(v - u \sin\theta)}{\sqrt{Q}}. \tag{4}
$$

Consequently when $(u, v) \neq (\delta, \delta)$, $\delta \geqq 0$, equations (2), (3) and (4) imply that

$$
\begin{aligned}
\frac{du}{dt} &= -a^2 \int_0^{2\pi} \frac{u - v\sin\theta}{\sqrt{Q}} d\theta \equiv f(u, v), \\
\frac{dv}{dt} &= -a^2 \int_0^{2\pi} \frac{v - u\sin\theta}{\sqrt{Q}} d\theta \equiv g(u, v),
\end{aligned} \tag{5}
$$

where $a^2 = \mu g/2\pi$. In the future we shall use (5) to represent the motion also when $(u, v, \theta) = (\delta, \delta, \theta)$, $\delta > 0$. However, in this case when $(u, v, \theta) = (\delta, \delta, \pi/2)$ the integrands in (5) are understood to be zero at $\theta = \pi/2$, in accordance with equations (1). In this case we should note that for $(u, v) \neq (0, 0)$

$$
\begin{aligned}
&\int_0^{2\pi} X r\sigma d\theta \\
&= -\mu g\sigma r \lim_{\epsilon \to 0} \left[\int_0^{\pi/2-\epsilon} \frac{u - v\sin\theta}{\sqrt{Q}} d\theta + \int_{\pi/2+\epsilon}^{2\pi} \frac{u - v\sin\theta}{\sqrt{Q}} d\theta \right].
\end{aligned} \tag{6}
$$

A similar statement holds for the second of equations (5).

THEOREM 1. *For any solution* $u(t), v(t)$, *of equations* (5) *for which* $u(0)=u_0>0$, $v(0)=v_0>0$, *there exists a constant* $T(0<T\leqq 2\sqrt{u_0^2+v_0^2}/\mu g)$ *such that*

$$u(t) > 0, \qquad v(t) > 0, \qquad 0 \leqq t < T, \tag{7}$$

$$\lim_{t\to T} u(t) = 0, \qquad \lim_{t\to T} v(t) = 0. \tag{8}$$

Proof. The functions $f(u, v)$, $g(u, v)$ in (5) are continuous functions of u and v since the integrands in these integrals can have at most a single finite discontinuity at $\theta=\pi/2$. Furthermore, since $|f(u, v)|\leqq 2\pi a^2$, $|g(u, v)|\leqq 2\pi a^2$ for all $u>0$, $v>0$, standard existence theorems* imply that through a point (t^*, u^*, v^*), $u^*>0$, $v^*>0$, there passes a solution defined at least on the interval

$$0 \leqq t \leqq \frac{N}{2\pi a^2}, \qquad N = \min\left(\frac{u^*}{2}, \frac{v^*}{2}\right), \tag{9}$$

so that we stay in the first octant of (u, v, t). To prove (7) multiply the first and second of equations (5) by $u(t)$ and $v(t)$, respectively, and add the results so as to obtain

$$\frac{d}{dt}(u^2 + v^2) = -2a^2\int_0^{2\pi}\sqrt{Q}\,d\theta. \tag{10}$$

Since for $u\geqq 0$, $v\geqq 0$,

$$\sqrt{Q} \equiv \sqrt{u^2 + v^2 - 2uv\sin\theta} \geqq \sqrt{u^2 + v^2} \quad \text{for} \quad \pi \leqq \theta \leqq 2\pi$$

we obtain

$$\int_0^{2\pi}\sqrt{Q}\,d\theta \geqq \int_\pi^{2\pi}\sqrt{Q}\,d\theta \geqq \int_\pi^{2\pi}\sqrt{u^2 + v^2}\,d\theta = \pi\sqrt{u^2 + v^2}.$$

Equations (10) then imply that

$$\frac{d}{dt}(u^2 + v^2) \leqq -2a^2\pi\sqrt{u^2 + v^2}, \qquad u \geqq 0, v \geqq 0.$$

Hence on any interval $0\leqq t\leqq t_1(t_1>0)$ for which $(u^2+v^2)\neq 0$, $u\geqq 0$ $v\geqq 0$, we obtain by integration that

$$0 < \sqrt{u^2 + v^2} \leqq \sqrt{u_0^2 + v_0^2} - a^2\pi t, \qquad 0 \leqq t \leqq t_1. \tag{11}$$

Since $u(0)>0$, $v(0)>0$, (11) implies that $0<t_1\leqq\sqrt{u_0^2+v_0^2}/\pi a^2$.

Equation (11) implies that a *least upper bound* T of the values t for which $u^2(t)+v^2(t)>0$ must exist. Suppose there exists a value t^* with $0<t^*<T$ for which $u(t^*)>0$, $v(t^*)=0$. In this case through the point $[t^*, u(t^*), v(t^*)]$ equa-

* E. Kamke, Differentialgleichungen Reeller Funktionen, Mathematik und ihre Anwendungen (1930), No. 7, pp. 128–130.

tions (5) would have two solutions, the one in question through $[0, u_0, v_0]$ and the solution

$$v_2(t) \equiv 0, \qquad u_2 = u_2(t),$$

where $u_2(t)$ satisfies the equation $du_2/dt = -\mu g$. But when $u \neq v$, $u \neq 0$, $f(u, v)$ and $g(u, v)$ in (5) satisfy the Lipschitz condition* in a neighborhood of (t^*, u^*, v^*), and the solution through this point must be unique. Due to symmetry in (5), we similarly cannot have $u(t^*) = 0$, $v(t^*) > 0$. Hence conditions (7) hold on $0 \leqq t < T$. This same uniqueness theorem implies that we cannot have either

$$\lim_{t \to T} u(t) > 0, \qquad \lim_{t \to T} v(t) = 0, \tag{12}$$

or

$$\lim_{t \to T} u(t) = 0, \qquad \lim_{t \to T} v(t) > 0. \tag{13}$$

To prove (8) in Theorem 1, suppose (8) does not hold. Then the preceding discussion including (12) and (13) together with the fact $|f(u, v)| \leqq 2\pi a^2$, $|g(u, v)| \leqq 2\pi a^2$, $u > 0$, $v > 0$ implies that there exists a constant $M > 0$ such that

$$u(t) > M, \qquad v(t) > M, \qquad 0 \leqq t < T. \tag{14}$$

Choose $t^* = T - M/8\pi a^2$. Then through the point $[t^*, u(t^*), v(t^*)]$ by (9) the solution $u(t)$, $v(t)$ can be extended to be defined on the interval

$$t^* \leqq t \leqq t^* + \frac{N}{2\pi a^2}, \qquad N = \min\left(\frac{u(t^*)}{2}, \frac{v(t^*)}{2}\right).$$

But since by (14), $u(t^*) > M$, $v(t^*) > M$, we have $N \geqq M/2$. Hence $u(t)$, $v(t)$ are defined, and satisfy the relations $u(t) \geqq 0$, $v(t) \geqq 0$, $u^2 + v^2 \neq 0$ on the interval

$$0 \leqq t \leqq t^* + \frac{M}{4\pi a^2} = T - \frac{M}{8\pi a^2} + \frac{M}{4\pi a^2} = T + \frac{M}{8\pi a^2},$$

whence T cannot be the least upper bound of t for which $u^2(t) + v^2(t) > 0$. Consequently the assumption that (8) was false is incorrect, and hence Theorem 1 is proved. Theorem 1 implies the last part of statement (A).

3. A theorem on differential equations. In order to prove statement (B) the following elementary theorem is helpful. Consider the differential equations

$$\frac{du}{dt} = f(u, v, t), \qquad \frac{dv}{dt} = g(u, v, t) \tag{15}$$

where $f(u, v, t)$ and $g(u, v, t)$ are defined in a region $\mathcal{R}(u, v, t)$.

* *Loc. cit.*, p. 141. Since $u(t^*) > 0$, $v(t^*) = 0$ there exists a square (u, v) neighborhood of this point in which $u \neq v$. In this neighborhood the functions $f(u, v)$, $g(u, v)$ have continuous partial derivatives, and hence satisfy the Lipschitz condition.

THEOREM 2. *Suppose the functions* $f(u, v, t)$, $g(u, v, t)$ *are continuous in* $\mathfrak{R}(u, v, t)$. *Furthermore suppose*

$$(16)\qquad \begin{aligned} f(u, v_2, t) &> f(u, v_1, t) &&\text{for } v_2 > v_1,\\ g(u_2, v, t) &> g(u_1, v, t) &&\text{for } u_2 > u_1, \end{aligned}$$

for all (u, v, t) *and* (u_1, v_1), (u_2, v_2) *in* $\mathfrak{R}$. *Let* $u_1(t)$, $v_1(t)$ *and* $u_2(t)$, $v_2(t)$ *be two solutions of* (15) *which are defined for* $0 \leqq t \leqq T$ *in* $\mathfrak{R}$, *and which have no point* (t^*, u^*, v^*) *in common where* $0 \leqq t^* \leqq T$.

Then if

$$(17)\qquad u_2(0) = u_1(0), \qquad v_2(0) > v_1(0)$$

holds, it is true that

$$(18)\qquad u_2(t) \geqq u_1(t), \qquad v_2(t) \geqq v_1(t), \qquad 0 \leqq t \leqq T.$$

The same conclusion holds if (17) *is replaced by*

$$(19)\qquad u_2(0) > u_1(0), \qquad v_2(0) = v_1(0).$$

Proof. Let C be the first connected interval on $0 \leqq t \leqq T$ for which conditions (18) hold. Clearly C is not empty, since $t=0$ is in C. Let t^* be the least upper bound of this bounded connected set. We will prove that $t^* = T$. Suppose $t^* < T$. Since any solution of (15) is continuous, if $u_2(t^*) > u_1(t^*)$, $v_2(t^*) > v_1(t^*)$ then these conditions would still hold on some interval $t^* \leqq t \leqq t^* + \delta$, $\delta > 0$. In this case t^* would not be the least upper bound described. Hence, since, by hypothesis, both equalities in (18) cannot hold on $0 \leqq t < T$ we have either

$$(20)\qquad u_2(t^*) = u_1(t^*), \qquad v_2(t^*) > v_1(t^*)$$

or

$$(21)\qquad u_2(t^*) > u_1(t^*), \qquad v_2(t^*) = v_1(t^*).$$

Suppose (20) holds. Then since $f(u, v, t)$ is continuous in (u, v, t) and since $u_i(t)$, $v_i(t)$, $(i=1, 2)$ are continuous, equations (16) and (20) imply that there exists a $\delta > 0$ such that

$$(22)\qquad f[u_2(t), v_2(t), t] > f[u_1(t), v_1(t), t], \qquad t^* \leqq t \leqq t^* + \delta.$$

Then by equations (15) we have $du_2(t)/dt > du_1(t)/dt$ for $t^* \leqq t \leqq t^* + \delta$. Hence since $u_2(t^*) = u_1(t^*)$, we have

$$u_2(t) \geqq u_1(t), \qquad t^* \leqq t \leqq t^* + \delta.$$

Also since $v_2(t^*) > v_1(t^*)$, from the continuity of these functions there exists a δ_1, $0 < \delta_1 \leqq \delta$ such that $v_2(t) > v_1(t)$ $(t^* \leqq t \leqq t^* + \delta_1)$. Hence

$$u_2(t) \geqq u_1(t), \qquad v_2(t) \geqq v_1(t), \qquad 0 \leqq t \leqq t^* + \delta_1,$$

and both equalities cannot hold for $0 \leqq t < t^* + \delta_1$. Thus t^* is not the least upper

bound as described. A similar result holds if condition (21) holds instead of (20). Consequently we must have $t^* = T$, and Theorem 2 is proved.

4. Proof of statement (B). Statement (B) is restated as follows:

THEOREM 3. *The second man can always win by giving the hoop an initial angular speed absolutely greater than that given by his opponent.*

Proof. Identify $u_1(t)$, $v_1(t)$, $(0 \leqq t \leqq T_1)$, with the first man, and $u_2(t)$, $v_2(t)$, $(0 \leqq t \leqq T_2)$, with the second man. Due to symmetry we can assume without loss of generality that all initial angular speeds are non-negative, so that $v_1(0) \geqq 0$, and so that by hypothesis conditions (17) hold. Note that for the integrands in (5), when

$$(u, v, \theta) \neq (\delta, \delta, \pi/2), \qquad (u, v) \neq (0, 0), \qquad u > 0, v > 0$$

we have

$$(23) \qquad \frac{\partial}{\partial v}\left(\frac{u - v \sin \theta}{\sqrt{Q}}\right) = \frac{\partial}{\partial u}\left(\frac{v - u \sin \theta}{\sqrt{Q}}\right) = \frac{-uv \cos^2 \theta}{Q^{3/2}} \equiv -h(u, v, \theta) \leqq 0.$$

Since these partial derivatives exist for $0 \leqq \theta < \pi/2$, $\pi/2 < \theta \leqq 2\pi$, applying the theorem of the mean, the first of equations (5) and (23) yield the result,

$$f(u, v_2) - f(u, v_1) = a^2(v_2 - v_1)\left[\int_0^{\pi/2-\epsilon} h[u, \bar{v}(\theta), \theta]d\theta + \int_{\pi/2+\epsilon}^{2\pi} h[u, \bar{v}(\theta), \theta]d\theta\right]$$
$$+ \frac{a^2}{\mu g}\int_{\pi/2-\epsilon}^{\pi/2+\epsilon} [X(u, v_2, \theta) - X(u, v_1, \theta)]d\theta, \qquad v_2 > v_1,$$

where $v_2 > \bar{v}(\theta) > v_1$, and where $h(u, \bar{v}(\theta), \theta)$ is a continuous function of θ for $\theta \neq \pi/2$. Since $h(u, \bar{v}(0), 0) > 0$ $(u > 0)$, and since $|X(u, v, \theta)| \leqq \mu g$ we see that (23) implies that the right side of the above equality is positive when $v_2 > v_1$. Hence the first of conditions (16) is satisfied. Similarly $g(u, v)$ in (5) satisfies the second of conditions (16).

Now let $T = \min [T_1, T_2]$, and let C be the first connected interval of $0 \leqq t \leqq T$ for which $[u_1(t), v_1(t)] \neq [u_2(t), v_2(t)]$. If we denote the least upper bound of C by t^*, condition (17) implies that $t^* > 0$. Since the hypotheses of Theorem 2 are satisfied on $0 \leqq t \leqq t^*$, we must have

$$(24) \qquad u_2(t) \geqq u_1(t), \qquad v_2(t) \geqq v_1(t), \qquad 0 \leqq t \leqq t^*.$$

If $t^* < T$, then we have

$$(25) \qquad u_2(t^*) = u_1(t^*) \neq 0, \qquad v_2(t^*) = v_1(t^*) \neq 0.$$

If the solution of equations (5) through $[u_2(t^*), v_2(t^*), t^*]$ is unique, then (25) is impossible. It is important that the usual Lipschitz condition for equations (5) does not hold, since the partial derivatives in (23) are not bounded for

$0 \leqq \theta < \pi/2$, $\pi/2 < \theta \leqq 2\pi$ when $u = v$, nor do the improper integrals of these functions converge on $0 \leqq \theta \leqq 2\pi$, when $u = v$. We can however prove uniqueness easily as follows.

Equation (10) implies that

$$u^2 + v^2 = u^2(t^*) + v^2(t^*) + 2a^2 \int_t^{t^*} \int_0^{2\pi} \sqrt{Q}\, d\theta dt, \qquad 0 \leqq t \leqq t^*. \tag{26}$$

Hence substituting $u_2(t)$, $v_2(t)$ and $u_1(t)$, $v_1(t)$ in (26), respectively, and subtracting, we obtain

$$(u_2 - u_1)(u_2 + u_1) + (v_2 - v_1)(v_2 + v_1) \leqq \left| 2a^2 \int_t^{t^*} \int_0^{2\pi} \left| \sqrt{Q_2} - \sqrt{Q_1} \right| d\theta dt \right|$$

where $Q_i = u_i^2 + v_i^2\, 2u_i v_i \sin\theta$, and $u_i = u_i(t)$, $v_i = v_i(t)$, $(i = 1, 2)$, $(0 \leqq t \leqq t^*)$. Observe that when $\theta = \pi/2$, $\sqrt{Q} = |u - v|$. Hence when $\theta = \pi/2$, $|\sqrt{Q_2} - \sqrt{Q_1}| = \big| |u_2 - v_2| - |u_1 - v_1| \big| \leqq |u_2 - v_2 - u_1 + v_1| \leqq |u_2 - u_1| + |v_2 - v_1|$. Furthermore since $\sqrt{Q}$ has bounded derivatives for all $(u, v) \neq (0, 0)$ and for $0 \leqq \theta < \pi/2$, $\pi/2 < \theta \leqq 2\pi$, the theorem of the mean implies that $\sqrt{Q}$ satisfied the Lipschitz condition for all $\theta \neq \pi/2$. Thus $\sqrt{Q}$ satisfies a Lipschitz condition on $0 \leqq \theta \leqq 2\pi$, and hence a constant $K > 0$ exists such that

$$\begin{aligned} &(u_2 - u_1)(u_2 + u_1) + (v_2 - v_1)(v_2 + v_1) \\ &\qquad \leqq \left| 2a^2 \int_t^{t^*} \int_0^{2\pi} K\{ |u_2 - u_1| + |v_2 - v_1| \} d\theta dt \right| \end{aligned} \tag{27}$$

where $0 \leqq t \leqq t^*$. Conditions (7), (24) and (25) imply that

$$u_2(t) \geqq u_1(t) \geqq N > 0, \qquad v_2(t) \geqq v_1(t) \geqq N > 0 \qquad \text{for } 0 \leqq t \leqq t^*.$$

Hence (27) implies that

$$\begin{aligned} 0 &\leqq u_2 - u_1 \leqq \frac{2\pi a^2 K}{N} \left| \int_t^{t^*} \{ |u_2 - u_1| + |v_2 - v_1| \} dt \right|, \\ 0 &\leqq v_2 - v_1 \leqq \frac{2\pi a^2 K}{N} \left| \int_t^{t^*} \{ |u_2 - u_1| + |v_2 - v_1| \} dt \right|, \end{aligned} \tag{28}$$

where $u_i = u_i(t)$, $v_i = v_i(t)$ $(0 \leqq t \leqq t^*, i = 1, 2)$.

By iteration* equations (28) imply that $u_2(t) \equiv u_1(t)$, $v_2(t) \equiv v_1(t)$ $(0 \leqq t \leqq t^*)$. Since this contradicts the hypothesis $v_2(0) > v_1(0) \geqq 0$, the assumption that $t^* < T$ is false, so that (24) holds on $0 \leqq t \leqq T$. If $T_2 < T_1$, then by (24) we would have $u_2(T_2) \geqq u_1(T_2) > 0$, $v_2(T_2) \geqq v_1(T_2) \geqq 0$. Since this contradicts the hypothesis $u_2(T_2) = 0$, $v_2(T_2) = 0$, we have $T_2 \geqq T_1$, so that $T \equiv \min [T_1, T_2] = T_1$.

Finally if $ds_2/dt \equiv u_2(t)$ and $ds_1/dt \equiv u_1(t)$, equations (24) imply that

$$\frac{ds_2}{dt} \geqq \frac{ds_1}{dt}, \qquad 0 \leqq t \leqq T_1. \tag{29}$$

* *Loc. cit.*, p. 141.

Since from (22) we see that the equality in (29) cannot hold at all points of the interval, and since $s_2(0)=s_1(0)$, we get $s_2(T_1)>s_1(T_1)$. Since $s_2(T_2)\geqq s_2(T_1)$, we have $s_2(T_2)>s_1(T_1)$, and Theorem 3 is proved.

5. A special case. It should be observed that if $u_2(0)=v_2(0)>0$ then

$$\frac{du_2}{dt}=\frac{dv_2}{dt}=\frac{-\mu g}{2\sqrt{2}\pi}\int_0^{2\pi}\sqrt{1-\sin\theta}\,d\theta=-\frac{2\mu g}{\pi}.$$

It is interesting to compare the distance travelled in this case to the case of pure translation where $u_1(0)=u_2(0)$, $v_1(0)=0$. In the latter case

$$\frac{du_1}{dt}=-\mu g,\qquad v_1(t)\equiv 0.$$

Hence since $u_2(0)=u_1(0)$, we get

$$s_2(T_2)=\frac{u_2^2(0)\pi}{4\mu g},\qquad s_1(T_1)=\frac{u_2^2(0)}{2\mu g}\,.$$

Thus $s_2(T_2)=(\pi/2)s_1(T_1)=(1.57+)s_1(T_1)$, which illustrates the diminishing effect of friction due to rotation of the hoop.

ON BOSE NUMBERS

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N. C. Bose Majumdar has given a method* of writing the repetend of the recurring decimal equivalent to the fraction $1/n$, n being a positive integer prime to 10. A brief description of this method is presented before an explanation of it is given.

Taking those fractions whose numerator is 1 and whose denominator does not have a factor 2 or 5, he divided them into four groups according as they have for the last digit in the denominator 1, 3, 7, or 9. He defined the End Number e as the smallest positive integer which when multiplied by the denominator gives a number ending in 9. For each group of such fractions he gave a rule for writing down the "Bose Number b." In general, however, his Bose Number is given by the equation $n\cdot e=10b-1$.

Bose's method consisted in writing the End Number e, multiplying it by the Bose Number b, and placing the last digit of the product before e, calling it e_2 and carrying over the remaining digits, multiplying e_2 by b and adding the

* Ravenshavian, A magazine of Ravenshaw College, Cuttack, Orissa, India; vol. XXVI, no. 2, April, 1942.

number carried over and writing the last digit of this sum as e_3, and so on, until the digits recur.

EXAMPLE. For $1/21$, n is 21; $21\cdot 9$ is $189=19\cdot 10-1$, so that e is 9 and b is 19.

Write down the End Number 9 first, multiply this 9 by the Bose Number 19 and obtain 171. Place 1 before 9 and carry 17; multiply this 1 by the Bose Number 19 and add 17, obtaining 36. Place 6 before 1 and carry 3; multiply this 6 by 19 and adding 3, we get 117. Place 7 before 6 and carry 11, and so on. Continue the process until the numbers recur. For $1/21$ we arrive at $.\dot{0}4761\dot{9}$.

Bose Majumdar did not give any mathematical explanation for his method. It is the purpose of this paper to give a proof for the soundness of this method of writing down the repetend of the recurring decimal.

The relation between n, e, and b is given by the equation

$$n\cdot e = 10b - 1. \tag{1}$$

Suppose r is the repetend of k digits of the recurring decimal for $1/n$, so that,

$$r = e_1 + 10e_2 + 10^2e_3 + \cdots + 10^{k-1}e_k. \tag{2}$$

$$1/n = r/(10^k - 1). \tag{3}$$

Then Bose's method indicates that the successive digits of r (counting from the right toward the left) may be obtained as follows;

The first digit e_1 is e.

The first two digits of e_1+10be_1 give the first two digits of r.

The first three digits of $e_1+10be_1+10^2be_2$ give the first three digits of r; and, in general, the first i digits of $e_1+10be_1+10^2be_2+10^3be_3+\cdots+10^{i-1}be_{i-1}$ are the first i digits of r, $i=2, 3, \cdots, k$.

EXAMPLE. $n=21$, $21\cdot 9=189=10\cdot 19-1$; $e=e_1=9$, $b=19$. Then

$$\begin{array}{lrl}
10\cdot 19\cdot 9 \text{ is} & 1710 & \\
 & 9 & \\
\hline
 & 1719 & \text{showing } e_2 \text{ is } 1.\\
 & & \\
10^2\cdot 19\cdot 1 \text{ is} & 1900 & \\
 & 1710 & \\
 & 9 & \\
\hline
 & 3619 & \text{showing } e_3 \text{ is } 6.\\
 & & \\
10^3\cdot 19\cdot 6 \text{ is} & 114000 & \\
 & 1900 & \\
 & 1710 & \\
 & 9 & \\
\hline
 & 117619 & \text{showing } e_4 \text{ is } 7.
\end{array}$$

This can be condensed into the form

$$\begin{array}{r}
7600000 = 10^5 \cdot 19 \cdot 4 \\
1330000 = 10^4 \cdot 19 \cdot 7 \\
114000 = 10^3 \cdot 19 \cdot 6 \\
1900 = 10^2 \cdot 19 \cdot 1 \\
1710 = 10 \cdot 19 \cdot 9 \\
9 \\
\hline
047619 = r, \text{ so } 1/21 \text{ is } .\dot{0}4761\dot{9}.
\end{array}$$

To prove the soundness of the procedure, we must prove* that

(4) $\quad e_1 + 10be_1 + 10^2be_2 + \cdots + 10^{i-1}be_{i-1} \equiv r, \bmod 10^i \quad$ for $i = 2, 3, \cdots, k$.

Proof. Eliminating n from (1) and (3) with $e_1 = e$, gives

(5) $$(10b - 1)r = (10^k - 1)e_1.$$

Then, using (2), we have

$$10be_1 + 10^2be_2 + \cdots + 10^kbe_k - r = 10^ke_1 - e_1.$$

Thus,

$$e_1 + 10be_1 + 10^2be_2 + \cdots + 10^{i-1}be_{i-1} - r = 10^ke_1 - 10^ibe_i - \cdots - 10^kbe_k;$$

or

$$e_1 + 10be_1 + 10^2be_2 + \cdots + 10^{i-1}be_{i-1} \equiv r, \bmod 10^i; \qquad \text{for } i = 2, 3, \cdots, k.$$

In cases where the numerator is not unity, but is less than the denominator, the End Number is first multiplied by the numerator to get e_1 and then the procedure just outlined is followed.

EXAMPLE. For $3/7$, $7 \cdot 7 = 49 = 10 \cdot 5 - 1$ so that e is 7 and b is 5. The product of 7 and 3 is 21, so that the End Number for 3/7 is written as 1, carrying 2; this 1 is multiplied by the Bose Number 5, and 2 is added, thereby obtaining 7 which is e_2. The method that is used in the case where the numerator is 1 then gives $.\dot{4}2857\dot{1}$ for 3/7.

If the numerator is greater than the denominator, it is first broken down into the whole number plus a fraction (less than 1) which is then used in writing the recurring decimal. In cases where the denominator has a factor 2 or 5, it is written as $1/10^m$ times a fraction whose denominator does not have a factor 2 or 5, for which the equivalent decimal is written in the same manner as above.

EXAMPLE $131/52 = 131/4 \cdot 13 = (25 \cdot 131)/(10^2 \cdot 13)$ or $(1/10^2) \cdot (3275/13) = (1/10)^2 \cdot (251 + 12/13)$. Now, $13 \cdot 3$ is 39 or $10 \cdot 4 - 1$. Thus e is 3, and b is 4. For 12/13 we get $.\dot{9}2307\dot{6}$. So 131/52 is $(1/10)^2 \cdot (251 \cdot \dot{9}2307\dot{6})$ or $2.51\dot{9}2307\dot{6}$.

* The author acknowledges some suggestions that he received from Professor W. B. Carver in connection with this proof.

A PROGRAM OF INFORMATION FOR PROSPECTIVE COLLEGE STUDENTS

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Members of the Michigan Section of the Association have been concerned for some years over the increasing number of students coming to college and wishing to enroll in courses and curricula for which they did not have the necessary prerequisites in high school mathematics. This concern was intensified by the adoption of the Michigan College agreement, which under certain conditions allows high school graduates to be admitted to college without regard to the pattern of courses taken in the high school.

Early in 1947, a committee consisting of Professor H. W. Alexander (Adrian College), Professor P. S. Jones (University of Michigan), and Professor C. C. Richtmeyer (Central Michigan College), was appointed to consider the problem and to make specific recommendations to the Association at the Spring meeting. At this meeting, the committee proposed the following recommendations which were approved by the Association:

1. That the Association authorize the preparation of a pamphlet pointing out to high school counselors and students the mathematical prerequisites necessary for admission to various college courses and curricula, and the difficulties resulting from the deferment of such mathematical preparation until the time for college entrance. In compiling the information for this pamphlet, each college or school of college grade in Michigan should be asked to cooperate by furnishing a list of curricula and courses together with the mathematical requirements for each.

   Another possible means of disseminating information would be the preparation of an attractive poster depicting the areas of collegiate study requiring previous mathematical study.

2. That the officers of this Association or someone designated by them recommend to all college and university registrars, or officers in charge of preparation of college catalogues, that each curriculum specifically state the mathematical prerequisites necessary for entrance upon that curriculum, and that the general statements of entrance requirements carry a reference to these prerequisites. It is further recommended that all collegiate heads of departments of mathematics be urged to work for the insertion of such a statement in their respective college catalogues.

3. That we seek the cooperation of other mathematical organizations in carrying out this project and that such publications as the *Michigan Education Journal* be asked to cooperate in the dissemination of the information.

4. That the Association express its willingness to appropriate sufficient funds to carry out such of the foregoing recommendations as it approves.

Under the direction of Professor Jones, a survey was made of all Michigan Colleges to find out what courses and curricula require high school mathematics

as prerequisite. Professor Alexander undertook the preparation of a chart or poster which would depict the areas of collegiate study requiring high school mathematics. The writer undertook the project of urging the colleges to include in their respective catalogues specific statements regarding mathematical prerequisites for the various courses and curricula. In connection with the latter project, letters were sent to all persons in charge of editing college catalogues urging them to include specific references to mathematical prerequisites in the next edition of their catalogue. Letters were also sent to each college mathematics department head, asking him to cooperate in seeing that this was done. Resolutions regarding this item were presented to the Michigan College Association and adopted by that body.

The results of Professor Jones' survey were incorporated into a pamphlet entitled, *A Mathematics Student—To Be or Not To Be*, which has been mailed recently to all high school principals and all college mathematics department heads in the state. The chart prepared under the direction of Professor Alexander was incorporated as the frontispiece of the pamphlet, and was also printed separately to be used as a bulletin board poster. Copies of the poster were also sent to all high schools.

The pamphlet and chart have created considerable interest both in Michigan and outside the state. It is hoped that they may serve a real guidance function for prospective college entrants.

MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, University of California and Institute for Numerical Analysis of the National Bureau of Standards

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NOTES ON THE GRAEFFE METHOD OF ROOT SQUARING

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1. Introduction. In this paper a method is explained for removing some of the defects of the Graeffe method [1, 2, 3, 4] of root squaring for the determination of the zeros of a polynomial. The Graeffe method as outlined in [1], for example, becomes awkward [4] when two or more zeros have identical or nearly identical moduli. By the procedure of [1] the process of root squaring is first carried out m times, say, yielding the polynomials

$$\sum_{i=0}^{n} a_{im}x^{n-i} = 0,$$

the zeros of which are the (2^m)th powers of those of the original poly-

nomial. Then, if all zeros are real and single, the values $A_{im}=a_{im}/a_{i-1,m}$, $(i=1, 2, 3 \cdots, n)$, are computed, that is, each coefficient is divided by that preceding it, and the (2^m)th roots of these ratios taken, yielding the roots of the original equation.

If double zeros occur, however, it is necessary to skip a coefficient in this dividing process; that is, one should divide by the coefficient before the one immediately preceeding [4]. The (2×2^m)th root is then taken of the resulting A. In general for an s-fold root, $s-1$ coefficients should be skipped and the $(s\times 2^m)$th root taken. Hence it is imperative to know the degree of multiplicity of each root before computing the A's.

2. Auxiliary Table. A simple way of determining the multiplicity of the roots and also determining whether they are real or complex is to construct an auxiliary table composed of the elements

$$\rho_{im} = \frac{a_{i,m-1}}{a_{im}} \left| a_{i,m-1} \right|.$$

The A's can then be determined by considering only those coefficients for which the corresponding ρ's are unity. Dividing each such coefficient by the preceding such coefficient yields the A's from which the zeros may then be obtained, remembering that if $s-1$ coefficients are skipped when computing A_{im}, so that $A_{im}=a_{im}/a_{i-s,m}$, then the $(s\times 2^m)$th root of A_{im} should be taken.

If a zero is real and s-fold the ρ's in the auxiliary table will converge to the binomial coefficients of $(1+1)^s$, it being easy thereby to recognize the presence of real roots. Also if s real roots are very nearly identical, then the corresponding ρ's will differ but slightly from the above binomial coefficients. If s complex roots (i.e., $s/2$ pairs) have identical moduli then the $s-1$ corresponding ρ's between those converging to unity will not converge to any value—it being possible by this means to detect complex roots.

The above can be easily proved by methods similar to those used in [1]. Briefly, letting the negatives of the roots (or Encke roots) of a polynomial be a, b, c, and so on, and assuming that $a \gg b \gg c$, and so on, then it is possible to write

$$x^n + [a]x^{n-1} + [ab]x^{n-2} + [abc]x^{n-3} + \cdots = 0, \tag{1}$$

where $[a]$ denotes $a+b+c+ \cdots$, and $[ab]$ denotes $ab+bc+ac+ \cdots$, and so on.

Because of the large differences in size, (1) can be written

$$x^n + ax^{n-1} + abx^{n-2} + abcx^{n-3} + \cdots = 0. \tag{2}$$

Suppose now that a is s-fold. Equation (2) becomes

$$x^n + C_1^s a x^{n-1} + C_2^s a^2 x^{n-2} + C_3^s a^3 x^{n-3} + \cdots = 0. \tag{3}$$

The squaring process then gives:

$$x^n + C_1^s a^2 x^{n-1} + C_2^s a^4 x^{n-2} + \cdots = 0, \tag{4}$$

the corresponding ρ's becoming $\rho_i = C_i^s a^i / C_i^s a^{2i} \left| C_i^s a^i \right| = C_i^s$. We would proceed similarly if b were s-fold, and so on.

3. Complex Roots. A useful application of Descartes rule of signs occurs with the Graeffe method. With the first squaring yielding the polynomial p_2, say, all real roots of the original polynomial p_1 go over into real positive roots in in p_2. Hence if k is equal to the number of changes in sign in p_2, then $n-k$ sets a lower limit to the number of complex roots present in p_1. Once the moduli of these roots are determined the corresponding angles can be obtained by the method of [5].

4. Illustrative Example. Given the polynomial

$$x^5 - 3x^4 + 10x^3 - 68x^2 + 168x - 128 = 0,$$

determine the moduli r and the multiplicity s of all roots. Computations are shown in Table I. In the notation used, 123^7 is written for 1.23×10^7. The auxiliary table is computed only for the last three squarings since early convergence of the ρ's is unlikely. After the second squaring, with $m=1$, it can be observed that since p_2 has 3 changes in sign there cannot be fewer than $5-3=2$ complex zeros. With regard to the method, an easy way to compute the successive sets of coefficients is to write those last obtained backward on a card spacing them as in the forward arrangement but with every other sign after the first altered. By matching coefficients in the forward and backward arrangements, we compute the new coefficient in any line by accumulating the products of figures which are then opposite, noting that such products are symmetrical about the middle squared term. By this procedure the signs of alternate sets of coefficients will differ, if n is odd, from these obtained from the formulas of [1]. This does not affect the value of the moduli, hence is not significant. This method is used in the following table.

TABLE I

m	a_{0m}	a_{1m}	a_{2m}	a_{3m}	a_{4m}	a_{5m}
0	1	-3^{0}	$+10^{1}$	-68^{1}	$+168^{2}$	-128^{2}
1	1	$+11^{1}$	$+28^{1}$	-2032^{3}	$+10816^{4}$	-16384^{4}
2	1	-65^{1}	$+6712^{4}$	-316288^{6}	$+5040128^{7}$	-26843546^{8}
3	1	$+130015^{5}$	$+41947226^{9}$	-32728386^{12}	$+84223073^{14}$	-72057596^{16}
4	1	-85144550^{9}	$+18448418^{19}$	-36268868^{24}	$+23768684^{29}$	-51922971^{33}
5	1	-35599108^{19}	$+34028236^{38}$	-43845039^{48}	$+18831286^{58}$	-26959949^{67}
6	1	-58673177^{38}	$+11579208^{77}$	-64079656^{96}	$+11820533^{116}$	-72683885^{134}
s			2 (complex)			3
A			11579208^{77}			62771033^{57}
$\log A$			77.0636786			57.7977592
$s2^m$			128			192
$\log r$			.60205999			.301029996
r			4.0			2.0

AUXILIARY TABLE

3:4	1	−1.985318	0.953778	2.953352	2.984400	1.0
4:5	1	+2.036454	1.000182	3.000182	3.000062	1.0
5:6	1	+2.159925	1.0	2.999996	3.000011	1.0

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NOTE ON THE APPROXIMATE SOLUTION OF AN OBLIQUE TRIANGLE WITHOUT TABLES

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If the three sides a, b, c of an oblique triangle ABC are given, the smallest angle A may be obtained to a very good approximation by the formula

$$A = \frac{6\sqrt{(s-b)(s-c)}}{2\sqrt{bc}+\sqrt{s(s-a)}}, \quad \text{where} \quad s = \frac{a+b+c}{2}. \tag{1}$$

Formula (1) may be derived by proceeding in a manner similar to that used for obtaining an analogous formula for the right triangle.* It may be noted that

$$2\csc\frac{A}{2} + \cot\frac{A}{2} = \frac{6}{A} + \frac{A^3}{480} + \frac{A^7}{16,128} + \cdots. \tag{2}$$

As an approximation, for small values of A, we may neglect all but the first term of the right hand member of (2). We have then

$$\frac{A}{2} \cong \frac{3}{\dfrac{2}{\sin\dfrac{A}{2}} + \dfrac{1}{\tan\dfrac{A}{2}}}. \tag{3}$$

The right hand member of (3) is the harmonic mean of $\sin A/2$ and $\tan A/2$, weighted two to one.

Using the functions of the half angles for oblique triangles, we obtain

* Solving a Right Triangle Without Tables. J. S. Frame, this MONTHLY, Vol. 50, 1943, pp. 622. See also article by R. A. Johnson, this MONTHLY, Vol. 27, 1920, p. 365, Bibliography, p. 366.

$$\frac{6}{A} \cong 2\sqrt{\frac{bc}{(s-b)(s-c)}} + \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

or

$$A \cong \frac{6\sqrt{(s-b)(s-c)}}{2\sqrt{bc} + \sqrt{s(s-a)}}\,.$$

The value of an angle obtained by formula (1) is correct to 5 decimal places for angles less than 30° The following table shows the error, E, for given angles.

A (degrees)	0°	10°	20°	30°	40°	50°	60°
E (minutes)	0′	.0′	.0′	.0′	.2′	1.0′	1.5′

where the error is given by the formula

$$\text{(5)}\quad \text{Error (radians)} = A - \frac{6\sin\dfrac{A}{2}}{2+\cos\dfrac{A}{2}} = \frac{A^5}{2880} + \frac{A^7}{96,768} + \cdots \cong .00035A^5.$$

This error is about 1/9 the corresponding error if the arithmetic mean had been used instead of the harmonic mean.

As an example, we may solve the triangle with sides 7, 9 and 14. By use of formula (1)

$$A = \frac{6\sqrt{6}}{6\sqrt{14} + 2\sqrt{30}} = .43997 \text{ radians or } 25°12.5'.$$

Likewise, $B = .57948$ radians or 33°12.1′.

For a better approximation for C, replace C (obtuse) by $\pi - C$ in formula (1). Hence

$$\text{(4)}\qquad \pi - C = \frac{6\sqrt{s(s-c)}}{2\sqrt{ab} + \sqrt{(s-a)(s-b)}}\,.$$

Substituting in formula (4) we obtain

$$\pi - C = 1.01908 \text{ radians or } 58°23.3'$$

so that

$$C = 121°36.7'.$$

Formula (1) gives answers for A and B which are as accurate as those obtained by half-angle formulas and a five place logarithm table. Formula (4) gives an answer for C which is as accurate as if it were obtained by a four place logarithm table. Formulas (1) or (4) should be valuable for long hand computation without tables.

NOTE ON A PERIODIC CONTINUED FRACTION

H. S. WALL, University of Texas

If the complex number a is not a real number less than $-\frac{1}{4}$, then the continued fraction

$$1+\cfrac{a}{1+\cfrac{a}{1+\cdots}} \tag{1}$$

converges to the numerically larger root of the equation $x^2-x-a=0$.* We observe that if Newton's formula

$$x_{p+1} = x_p - f(x_p)/f'(x_p), \qquad p = 0, 1, 2, \cdots, \tag{2}$$

is used to compute a root of $f(x)=x^2-x-a$, starting with x_0 equal to one of the approximants $f_1=1$, $f_2=1+a$, $f_3=(1+2a)/(1+a), \cdots$ of (1), say with f_n, then $x_1=f_{2n}$, $x_2=f_{4n}$, $x_3=f_{8n}$, $\cdots$. The same phenomenon occurs if, instead of (2), we use Frame's† modification of Newton's formula, namely,

$$x_{p+1} = x_p - \cfrac{f(x_p)}{f'(x_p) - \cfrac{f(x_p)f''(x_p)}{2f'(x_p)}}, \qquad p = 0, 1, 2, \cdots, \tag{3}$$

except that here $x_1=f_{3n}$, $x_2=f_{9n}$, $x_3=f_{27n}$, $\cdots$. The following theorem includes both these facts.

THEOREM. *Let* $U_1(x)$, $U_2(x)$, $U_3(x)$, $\cdots$ *denote the sequence of approximants of the periodic continued fraction*

$$x - \cfrac{x^2-x-a}{2x-1-\cfrac{x^2-x-a}{2x-1-\cdots}}, \tag{4}$$

and let $f_1, f_2, f_3, \cdots$ *be the sequence of approximants of* (1). *Then*

$$U_m(f_n) = f_{mn}, \qquad m, n = 1, 2, 3, \cdots. \tag{5}$$

Proof. Inasmuch as $f_n=U_n(1)$, (5) is a consequence of the following more general formula:

$$U_m[U_n(x)] = U_{mn}(x), \qquad m, n = 1, 2, 3, \cdots. \tag{6}$$

In the special case $a=-1/4$, we readily find that

* See, for instance, H. S. Wall, *Analytic Theory of Continued Fractions*, D. Van Nostrand Co., 1948, p. 39.

† This MONTHLY, vol. 51, 1944, pp. 36–38.

$$U_n(x) = \frac{x}{n} + \frac{n-1}{2n},$$

and therefore (6) holds.

If $4a+1 \neq 0$, then the roots r and s of the equation $x^2-x-a=0$ are distinct. We make the transformation

$$\zeta = \frac{x-r}{x-s}, \qquad x = \frac{s\zeta - r}{\zeta - 1}, \tag{7}$$

throwing r to the origin and s to ∞. Under this substitution, the continued fraction (4) becomes

$$\frac{s\zeta - r}{\zeta - 1} - \frac{s-r}{\zeta - 1} \cdot \cfrac{\zeta}{\zeta + 1 - \cfrac{\zeta}{\zeta + 1 - \cfrac{\zeta}{\zeta + 1 - \cdots}}},$$

from which we find that

$$U_n(x) = \frac{s\zeta - r}{\zeta - 1} - \frac{s-r}{\zeta - 1} \cdot \frac{\zeta(\zeta^{n-1} - 1)}{\zeta^n - 1},$$

or

$$U_n(x) = \frac{s\zeta^n - r}{\zeta^n - 1}, \qquad \zeta^n = \frac{U_n(x) - r}{U_n(x) - s}. \tag{8}$$

Consequently, by (7),

$$U_m[U_n(x)] = U_m\left[U_n\left(\frac{s\zeta - r}{\zeta - 1}\right)\right] = U_m\left(\frac{s\zeta^n - r}{\zeta^n - 1}\right) = \frac{s\zeta^{mn} - r}{\zeta^{mn} - 1} = U_{mn}(x),$$

and the theorem is established.

Let the notation be chosen so that $|r| \geqq |s|$, and suppose that $|r| > |s|$. Then

$$\left|\frac{s}{r}\right| = \left|\frac{1-r}{r}\right| < 1, \qquad R(r) > \frac{1}{2}.$$

By (8), $U_n(x) \to r$ or $U_n(x) \to s$, as $n \to \infty$, according as $|\zeta| < 1$, or $|\zeta| > 1$, respectively, i.e., according as $|x-r| < |x-s|$ or $|x-r| > |x-s|$. Thus, if L is the perpendicular bisector of the line-segment from r to s, and H_r and H_s are the open half-planes bounded by L, containing r and s, respectively, then $U_n(x)$ converges, for $n \to \infty$, to r or to s, according as x is in H_r or in H_s, respectively. We note that L passes through the point 1/2. In particular, $U_n(1) \to r$, $U_n(0) \to s$, i.e., $f_n \to r$, $-a/f_n \to s$.

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College and Institute for Advanced Study

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CONTINUED FRACTIONS AND MATRICES

J. S. FRAME, Michigan State College

1. Introduction. The closest rational approximation P/Q (P, Q integers) to a given positive real number x, subject to the condition that the denominator Q shall not exceed a given positive integer N, is most easily found by means of continued fractions. For example, successive approximations to π obtained by this method are 3, 22/7, 333/106, 355/113, $\cdots$. Continued fractions may also be used to facilitate the solution of diophantine equations.

It is our purpose to show how the principal theorems in the theory of continued fractions, and the actual computation of successive convergents to a continued fraction can be presented quite simply by the use of two-rowed matrices and their determinants. To a student familiar with the multiplication of two-rowed matrices a good introduction to continued fractions can be presented in an hour's lecture. For a student who has just been introduced to matrices, their use in connection with continued fractions provides an easy application at the elementary level.

2. Simple continued fractions. For a given positive real x let the integral part be a_1 and the remainder r_1, with $0 \leqq r_1 < 1$. We define successively the positive integers $a_2, a_3, \cdots, a_n$, (called partial denominators) and the remainders $r_2, r_3, \cdots r_n$ so that

$$(1) \qquad x = a_1 + r_1, \qquad \frac{1}{r_1} = a_2 + r_2, \cdots, \frac{1}{r_{n-1}} = a_n + r_n, \qquad 0 \leqq r_k < 1.$$

and we write

$$(2) \qquad x = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots \, \cfrac{}{a_{n-1} + \cfrac{1}{a_n + r_n}}}}}$$

or

$$(2') \qquad x = a_1 + \frac{1}{a_2 +} \quad \frac{1}{a_3 +} \quad \cdots \frac{1}{a_{n-1} +} \quad \frac{1}{a_n + r_n}.$$

After simplifying the complex fraction (2) and collecting coefficients of r_n, we have

$$x = \frac{P_n + r_n S_n}{Q_n + r_n T_n}, \qquad \text{where} \quad M_n = \begin{pmatrix} P_n & S_n \\ Q_n & T_n \end{pmatrix} \tag{3}$$

is an integral matrix, and where, for the case $n=1$, we have from (1)

$$M_1 = \begin{pmatrix} P_1 & S_1 \\ Q_1 & T_1 \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}.$$

It will be convenient to denote the nth denominator in (3) by D_n;

$$D_n = Q_n + r_n T_n \tag{4}$$

To obtain a recursion formula for the P's and Q's, we replace n by $n-1$ in equation (3), divide numerator and denominator by r_{n-1}, and apply (1).

$$x = \frac{P_{n-1} + r_{n-1}S_{n-1}}{Q_{n-1} + r_{n-1}T_{n-1}} = \frac{P_{n-1}(a_n + r_n) + S_{n-1}}{Q_{n-1}(a_n + r_n) + T_{n-1}} = \frac{(a_nP_{n-1} + S_{n-1}) + r_nP_{n-1}}{(a_nQ_{n-1} + T_{n-1}) + r_nQ_{n-1}}. \tag{5}$$

On comparing coefficients of r_n in (3) and (5) we have

$$\begin{aligned} S_n &= P_{n-1}, \qquad & P_n &= a_nP_{n-1} + S_{n-1} \\ T_n &= Q_{n-1}, \qquad & Q_n &= a_nQ_{n-1} + T_{n-1}. \end{aligned} \tag{6}$$

Hence,

$$\begin{aligned} P_n &= a_nP_{n-1} + P_{n-2} \\ Q_n &= a_nQ_{n-1} + Q_{n-2}. \end{aligned} \tag{7}$$

Also from (4) and (5) the ratio D_{n-1}/D_n is seen to be r_{n-1}, and $D_1=1$, so

$$1/D_n = r_1r_2 \cdots r_{n-1}, \tag{8}$$

provided that none of the r's are 0. The set of equations (7) can most easily be written in matrix form

$$M_n = \begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} = \begin{pmatrix} P_{n-1} & P_{n-2} \\ Q_{n-1} & Q_{n-2} \end{pmatrix}\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} = M_{n-1}\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}. \tag{9}$$

By induction we then obtain the fundamental relation

$$M_n = \begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{n-1} & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}. \tag{10}$$

3. Theorems on continued fractions. Since the determinant of the matrix M_n is the product of the determinants of its factors, we obtain from (10) the important relations

$$\begin{vmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{vmatrix} = (-1)^n \tag{11}$$

$$\frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}} = \frac{(-1)^n}{Q_nQ_{n-1}} \tag{12}$$

$$x - \frac{P_n}{Q_n} = \frac{P_n + r_nP_{n-1}}{Q_n + r_nQ_{n-1}} - \frac{P_n}{Q_n} = \frac{r_n(-1)^{n-1}}{Q_nD_n} = \frac{(-1)^{n-1}}{Q_nD_{n+1}}. \tag{13}$$

THEOREM I. *The rational fractions P_n/Q_n, called the "convergents" of x, are alternately less and greater than x, and the difference of successive convergents is the reciprocal of the product of their denominators. Since this product becomes infinite with n, the differences defined by* (12) *approach zero and alternate in sign as n increases, so the sequence P_n/Q_n converges. Its limit is x.*

The proof of these statements follows directly from (13) and (12).

Equation (3) may be replaced by the matrix equation

$$\begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix}\begin{pmatrix} 1/D_n \\ 1/D_{n+1} \end{pmatrix}. \tag{14}$$

Solving for r_n, which is the ratio D_n/D_{n+1}, we have

$$r_n = -\frac{P_n - Q_nx}{P_{n-1} - Q_{n-1}x} \tag{15}$$

$$a_{n+1} + r_{n+1} = \frac{Q_{n-1}x - P_{n-1}}{-Q_nx + P_n}. \tag{16}$$

Equation (16) makes it possible to compute a_{n+1} directly if the matrix M_n is known. The next matrix M_{n+1} is then obtained from (10). For example, in the continued fraction expansion for π the partial denominator a_5 is the largest integer in $(106\pi-333)/(-113\pi+355)=292.+$

THEOREM II. *A periodic continued fraction represents a root of a quadratic equation.*

Proof. If two different remainders are equal, equate them, using (15), and solve for x.

THEOREM III. *The rational fraction P_n/Q_n defined by* (1) *and* (10) *differs from x by not more than $1/Q_nQ_{n+1}$, and it approximates the real number x more closely than does any other rational fraction with a denominator not exceeding Q_n.*

Proof. The difference (13) is equal in absolute value to $1/Q_nD_{n+1}$, which by (4) is certainly less than $1/Q_nQ_{n+1}$. Suppose there were two integers A and B, with $0<B<Q_n$, such that the fraction A/B were closer to x than P_n/Q_n. Then we should have

$$-\frac{1}{Q_nD_{n+1}} < (-1)^n\left(x-\frac{A}{B}\right) < \frac{1}{Q_nD_{n+1}}. \tag{17}$$

Adding $1/Q_nD_{n+1}$ to each term and replacing x by its value in (13) gives

$$0 < (-1)^n\left(\frac{P_n}{Q_n}-\frac{A}{B}\right) < \frac{2}{Q_nD_{n+1}} < \frac{2}{Q_nQ_{n+1}}$$

or

$$0 < (-1)^n(BP_n - AQ_n) < 2B/Q_{n+1}. \tag{18}$$

It is easily shown that the inequality (18) cannot be satisfied by integers A, B with $B<Q_n$. For then the right member is less than 2, and the integer in the middle member could only be 1. By (11) the latter condition is satisfied by taking $A=P_{n-1}$, $B=Q_{n-1}$, and this is the only choice for which $0<B<Q_n$, as we shall see in §4 below. However, with $B=Q_{n-1}$, we are led from (18) to the inequality $1<2Q_{n-1}/Q_{n+1}$, which is impossible by (7).

4. Continued fractions for rational numbers. If at some stage in (1) we have $r_n=0$, then x reduces by (3) to the rational number P_n/Q_n. Conversely, every rational number is represented by a terminating continued fraction. The next to the last convergent is important in solving the linear diophantine equation (19).

Given two integers P_n and Q_n without common factor, to find all pairs of integers u and v which satisfy the equation

$$P_nu - Q_nv = 1. \tag{19}$$

The solution is given by

$$\begin{aligned} v &= (-1)^nP_{n-1} + NP_n \\ u &= (-1)^nQ_{n-1} + NQ_n \end{aligned} \qquad N \text{ any integer,} \tag{20}$$

where P_{n-1} and Q_{n-1} are obtained from the continued fraction expansion of P_n/Q_n.

5. General continued fractions. If the simple continued fraction (2) is replaced by the more general form

$$x = a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{b_4}{\ddots\, \cfrac{}{a_{n-1} + \cfrac{b_n}{a_n + r_n}}}}} \tag{21}$$

in which r_n is not necessarily less than 1, and a_k and b_k are not necessarily positive integers, then it is readily shown that equations (7) become

$$P_n = a_n P_{n-1} + b_n P_{n-2}$$
$$Q_n = a_n Q_{n-1} + b_n Q_{n-2} \tag{22}$$

and the fundamental relation (10) may be written

$$M_n = \begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ b_1 & 0 \end{pmatrix}\begin{pmatrix} a_2 & 1 \\ b_2 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ b_n & 0 \end{pmatrix}, \tag{23}$$

where we define $b_1 = Q_1 = P_0 = 1$, $Q_0 = 0$.

By taking determinants in (23), the difference between successive convergents to x is seen to equal $(-1)^n b_1 b_2 \cdots b_n / Q_n Q_{n-1}$. Since the sum of these terms may not always converge, the question of convergence in this case is more complicated than that for the simple continued fractions, and we shall not discuss it at this time.

It may be of interest to include without proof the expansion

$$\frac{x}{\tan^{-1} x} = 1 + \cfrac{x^2}{3 + \cfrac{(2x)^2}{5 + \cfrac{(3x)^2}{7 + \cdots}}} \tag{24}$$

which is typical of many continued fraction expansions for analytic functions obtained from hypergeometric series.

Setting $x = 1$ in (24) we see that the value of π can be expressed as a continued fraction in which the coefficients are given by a simple law

$$\frac{4}{\pi} = 1 + \cfrac{1}{3 + \cfrac{4}{5 + \cfrac{9}{7 + \cdots}}}. \tag{25}$$

This expansion is of interest because of its regularity, but it does not enjoy the rapidity of convergence which characterizes the simple continued fractions. Successive convergents in (25) are given by

$$\begin{pmatrix} P_{n+1} & P_n \\ Q_{n+1} & Q_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 5 & 1 \\ 4 & 0 \end{pmatrix}\begin{pmatrix} 7 & 1 \\ 9 & 0 \end{pmatrix}\begin{pmatrix} 9 & 1 \\ 16 & 0 \end{pmatrix} \cdots \begin{pmatrix} 2n+1 & 1 \\ n^2 & 0 \end{pmatrix}. \tag{26}$$

We may derive from (26) an alternative expansion for the convergents to $4/\pi$, in which the partial numerators b_i are all 1 but the partial denominators are not all integers. This is as follows:

$$(27)\quad \begin{pmatrix} P_{n+1} & P_n \\ Q_{n+1} & Q_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 5/4 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 28/9 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 81/64 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix}$$

where

$$(28)\quad \begin{aligned} a_{2n} &= (4n-1)\left(\frac{2\cdot 4 \cdots (2n-2)}{3\cdot 5 \cdots (2n-1)}\right)^2, \\ a_{2n+1} &= (4n+1)\left(\frac{1\cdot 3 \cdots (2n-1)}{2\cdot 4 \cdots 2n}\right)^2 \\ \lim_{n=\infty} a_{2n} &= \pi \\ \lim a_{2n+1} &= \frac{4}{\pi}\cdot \end{aligned}$$

Since each matrix in (27) has the determinant (-1), formulas (12) and (13) are valid in this case. The limits in (28) are easily obtained from the Wallis product formula for $\pi/2$. A close estimate for the error of P_n/Q_n is $(\sqrt{2}-1)^{2n-1}$.

DERIVATION OF THE TANGENT HALF-ANGLE FORMULA

F. E. Wood, University of Oregon

The following derivation of the formula for $\tan \theta/2$ appears to be an improvement over standard derivations, for it gives the result directly without a complicated discussion of the appropriate algebraic sign.

From the equation

$$\sin\frac{\theta}{2} = \sin\left(\theta - \frac{\theta}{2}\right) = \sin\theta\cos\frac{\theta}{2} - \cos\theta\sin\frac{\theta}{2}$$

one obtains

$$(1+\cos\theta)\sin\frac{\theta}{2} = \sin\theta\cos\frac{\theta}{2}\cdot$$

Consequently:

$$\tan\frac{\theta}{2} = \frac{\sin\theta/2}{\cos\theta/2} = \frac{\sin\theta}{1+\cos\theta}\cdot$$

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 851. *Proposed by Joseph Rosenbaum, Hartford, Connecticut*

The area of a quadrilateral which has both a circumcircle and an incircle is equal to the square root of the product of its sides.

E 852. *Proposed by Roy Dubisch, Fresno State College*

A theorem due to Lamé states that the number of divisions D in the Euclidean Algorithm required to find the g.c.d. of two numbers a and b $(a>b)$ is never greater than $5p$, where p is the number of digits in b. While this result has been strengthened for special pairs a, b, show, by a counter-example with b as small as possible and corresponding a as small as possible, that the statement $D<5p$ for all a, b is false.

E 853. *Proposed by C. S. Ogilvy, Trinity College*

If $y_1=x$, $y_2=x^{y_1}$, $\cdots$, $y_n=x^{y_{n-1}}$, what is the maximum x for which $\lim_{n\to\infty} y_n$ exists, and what is this limit?

E 854. *Proposed by Jerome C. R. Li, Oregon State College*

Show that $\pi=\sum_{n=0}^{\infty}(n!)^2 2^{n+1}/(2n+1)!$.

E 855. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Planes through the orthocenter of an orthocentric tetrahedron perpendicular to four concurrent cevians cut the spheres described on these cevians in four cospherical circles.

SOLUTIONS

A Combinatorial Identity

E 799 [1948, 502]. Correction to solution II [1948, 503–504].

1. On p. 504, first line and also eq. (3);

$$\text{for } \left(\frac{x-1}{2}\right)^r \text{ read } \left(\frac{x-1}{2}\right)^{n-r}.$$

2. Replace eq. (4) by

$$\sum_{r=0}^{n}\binom{n}{r}\left[\left(\frac{x+1}{2}\right)^{n-r}\left(\frac{x-1}{2}\right)^{r}\binom{n}{r}-\binom{2n-r}{n}\left(\frac{x-1}{2}\right)^{n-r}\right]=0.$$

These changes do not affect the particular application made.

Battle of Digits

E 816 [1948, 317]. *Proposed by P. L. Chessin, New York, N. Y.*

A announces a two digit number from 01 to 99. B reverses the digits of this number and adds to it the sum of its digits and then announces *his* result. A continues in the same pattern. All numbers are reduced modulo 100, so that only two digit numbers are announced. What choices has A for the initial number in order to insure that B will at some time announce 00?

Solution by Eleanor Rankin, University of North Carolina. If A calls 00, B must answer 00. For either A or B to obtain 00 he must receive a number $10a+b$ such that $11b+2a=100$. This linear Diophantine equation, subject to the restriction $0\leqq a,b\leqq 9$, has the unique solution $a=6$, $b=8$. Continuing backwards we obtain as possible initial announcements for A the numbers of the unique chain 68, 16, 80, 76, 56, 64. There are no other possible numbers since the equation $11b+2a=64$ has no solution in admissible a and b.

If B is to be the first one to call 00, then A's initial announcement is limited to the three numbers 68, 80, 56.

Also solved by Murray Barbour, Walter Breen, Paul Brock, R. C. Buck, W. E. Byrne, Monte Dernham, Roy Dubisch, J. C. Eaves, W. J. Graves, Martha Grogan, B. A. Hausmann, O. H. Hoke, B. R. Leeds, H. R. Leifer, Roger Lessard, Nathaniel Macon, D. W. Matlack, J. R. McDonough, W. D. Peeples, Jr., T. L. Reynolds, F. W. Saunders, C. W. Trigg, and the proposer.

Trigg pointed out that the initial announcement of an odd number will insure that all subsequent calls will be odd. He then gave the following groupings into which the even numbers fall in direct order of announcement. An apostrophe (') indicates the beginning of a repetitive group, and an asterisk (*) precedes a number which has been reduced modulo 100.

64, 56, 76, 80, 16, 68, *00
'54, 54
'36, 72, 36
'18, 90, 18
'12, 24, 48, 96, 84, 60, 12
20, '04, 44, 52, 32, 28, 92, 40, 08, 88, *04
86, 82, 38, 94, 62, 34, 50, 10, '02, 22, 26, 70, 14, 46, 74, 58, 98, *06, 66, 78, *02
42, 30, 06, *etc.*, as in line above.

Dernham indicated a simple algorithm for finding the chain leading to a given even terminal number. With the terminal number as the first dividend and 22 as the divisor, take twice the quotient to form the terminal digit of the second dividend, and half the remainder to form its penultimate digit. Continue this process as long as only single digits are obtained; then stop. The several dividends give the desired chain. The divisor is always 22. For example, here is all we need write to compute the chain leading to 00:

Div	Q	R	$R/2$	$2Q$
100	4	12	6	8
68	3	2	1	6
16	0	16	8	0
80	3	14	7	6
76	3	10	5	6
56	2	12	6	4
64	2	20	10	4

Since 10 is not a single-digit integer, it is inadmissible, and we have to stop. The desired chain is 64, 56, 76, 80, 16, 68, 00.

Byrne stated that the analogous problem involving three digits has for solution only the two numbers 779 and 367.

Flexilinear Curves

E 817 [1948, 317]. *Proposed by E. V. Hofler, Colgate University*

If the graph for a polynomial of the fourth degree has two real points of inflection, then the secant through these two points and the curve will bound three distinct areas. Show that two of these areas are equal and the largest area is equal to the sum of the other two.

I. *Solution by W. E. Byrne, Virginia Military Institute.* By a rigid displacement of the coordinate axes we may take the inflection points as $(-a, 0)$ and $(a, 0)$, $a>0$. We then have

$$y'' = 12k(x^2 - a^2), \qquad k \neq 0,$$

whence we find

$$y = k(x^4 - 6a^2x^2 + 5a^4) = k(x^2 - a^2)(x^2 - 5a^2).$$

Now

$$\int_{-a\sqrt{5}}^{-a} ydx = \int_{a}^{a\sqrt{5}} ydx = -16ka^5/5, \qquad \int_{-a}^{a} ydx = 32ka^5/5.$$

Since $|32ka^5/5| = 2|-16ka^5/5|$, the theorem is proved.

II. *Solution by Alan Wayne, Flushing, N. Y.* The proposition to be proved is a special instance of the following theorem:

If the points of inflection of a polynomial curve are all real and collinear, then the sum of the signed areas bounded by the curve and the line of collinearity is zero. Also, if these areas be denoted from left to right by $A_1, A_2, \cdots, A_{n-1}$, then $A_i = (-1)^n A_{n-i}$, $(i=1, \cdots, n-1)$.

Let the polynomial curve, with points of inflection on the line

$$y = mx + b,$$

be designated by

$$(1) \qquad y = \Sigma c_r x^r, \qquad n > 2, \qquad c_n = 1,$$

$$(8) \qquad b_{n-2k} = \frac{a^{2k}n(n-1)\cdots(n-2k+1)}{(-2)^k k!(2n-3)(2n-5)\cdots(2n-2k-1)}$$

for integers k such that $n \geqq 2k > 0$. These curves will serve as interesting examples in courses in college algebra and calculus.

We might call curves, whose flexes are real and collinear, *flexilinear curves.* The curve $y = \sin x$ is an instance of a transcendental flexilinear curve. The Bernouilli polynomials of the third, fourth, and fifth degrees have graphs which are flexilinear polynomial curves.

Also solved by Paul Brock, F. E. Cothran, Ragnar Dybvik, Walter Fleming B. A. Hausmann, Roger Lessard, G. M. Merriman, Norman Miller, Leo Moser, S. T. Parker, R. W. Rector, J. H. Simester, R. P. Stephens, Sieh Su, C. W. Trigg, and the proposer.

Editorial Note. Not only is the proposed problem a special case of Wayne's general theorem, but so also is the well known fact that the inflection point of a cubic polynomial curve is a center of symmetry for the curve.

It is interesting that reality and collinearity of the inflection points implies symmetry for the associated curve (2)—symmetry in an axis if n is even, and symmetry in a point if n is odd. This fact can be used to give some further geometrical properties of flexilinear polynomial curves. Thus, suppose n even, and let $P_1, \cdots, P_n$ be the intersections, from left to right, of the curve with the line L of collinearity of its inflection points. Let V denote the axis of symmetry of the associated curve (2). Then we have: (A) V bisects the segment joining P_i and P_{n-i+1}. (B) Tangents to the curve at P_i and P_{n-i+1} intersect on V. (C) The bitangent of the curve touching the curve between P_i and P_{i+1} and between P_{n-i} and P_{n-i+1} is parallel to L. (D) The tangent to the curve at the point where it cuts V is parallel to L.

Other general theorems can be given for the case where n is odd.

Trigg gave twenty-five theorems and seven corollaries concerning the quartic polynomial curve with real inflections. For some properties of this curve see J. S. Frame, *Tangent triangles to a biquadratic curve*, this MONTHLY, Oct. 1944, pp. 445–452.

Four Fractions

E 818 [1948, 317]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find four different fractions, each of the form $m/(m+1)$, such that their sum is an integer. (For example: $1/2+2/3+6/7+41/42=3=1/2+2/3+8/9+17/18$.)

Solution by C. W. Trigg, Los Angeles City College. For any one of the fractions, f, we have $1/2 \leqq f < 1$. Therefore, $2 < \Sigma f < 4$, so $\Sigma f = 3$. Let the numerators of the fractions, arranged in ascending order of magnitude, be m, n, p, q, respectively. Now $2/3+3/4+4/5+5/6=183/60>3$, so $m=1$. Also, $1/2+5/6+6/7+7/8=515/168>3$, so $n<5$. The sum of the fractions may be written as

$$1/2 + n/(n+1) + p/(p+1) + q/(q+1) = 3,$$

which may be manipulated into the form

$$q = [(n+3)p + (3n+5)]/[(n-1)p - (n+3)].$$

When $n=2$, $q=5+36/(p-5)$, so $(p, q)=(6, 41)$, $(7, 23)$, $(8, 17)$, or $(9, 14)$. When $n=3$, $q=3+16/(p-3)$, so $(p, q)=(4, 19)$ or $(5, 11)$. When $n=4$, $(p, q)=(3, 19)$. Hence the four solutions additional to those given in the proposal are: $1/2+2/3+7/8+23/24=1/2+2/3+9/10+14/15=1/2+3/4+4/5+19/20=1/2+3/4+5/6+11/12=3$.

Also solved by Murray Barbour, Paul Brock, W. E. Byrne, Roger Lessard, S. T. Parker, P. A. Piza, and Kirk Stewart, and partially solved by W. H. Breen, B. A. Hausmann, Leo Moser, and C. S. Ogilvy.

Several solvers pointed out that this problem is equivalent to finding four distinct integers in the list 2, 3, 4, $\cdots$ the sum of whose reciprocals is unity.

Euler's Constant

E 819 [1948, 317]. *Proposed by H. F. Sandham, Trinity College, Ireland*

If

$$S_n = 1/1 + 1/2 + \cdots + 1/n,$$

prove that

$$\gamma < S_p + S_q - S_{pq} \leqq 1,$$

where γ is Euler's constant.

I. *Solution by S. T. Parker, Kansas State College.* We have

$$\begin{aligned} S_p + S_q - S_{pq} &= (S_p + S_q - S_p) - (S_{2p} - S_p) - (S_{3p} - S_{2p}) - \cdots \\ &\quad - (S_{qp} - S_{(q-1)p}) \\ &\leqq S_q - p(1/2p) - p(1/3p) - \cdots - p(1/qp) = 1. \end{aligned}$$

Also we have

$$\gamma < S_{pq} - \ln pq \leqq S_p - \ln p.$$

Therefore

$$0 \leqq S_p - S_{pq} - \ln p + \ln pq.$$

Add to this last the relation

$$\gamma < S_q - \ln q,$$

and we obtain

$$\gamma < S_p + S_q - S_{pq}.$$

Therefore

$$\gamma < S_p + S_q - S_{pq} \leqq 1.$$

II. *Solution by the Proposer.* If we set

$$(p, q) = S_p + S_q - S_{pq}$$

then

$$(p, q) - (p - 1, q) = 1/p - 1/(pq - q + 1) - 1/(pq - q + 2) - \cdots - 1/pq$$
$$< 1/p - q(1/pq) = 0.$$

Thus (p, q) is a decreasing function of p and therefore also of q. The cases $p \to \infty$, $q \to \infty$ and $p = 1$, $q = 1$ are the two sides of the inequality.

Also solved by Michael Aissen, W. E. Byrne, William Gustin, M. S. Klamkin, Roger Lessard, Norman Miller, Leo Moser, Margaret Olmsted, and J. H. Simester.

Factorial as the Difference of Squares

E 822 [1948, 365]. *Proposed by W. R. Ransom, Tufts College*

Every factorial that can be expressed as the difference of two squares can be so expressed in two different ways.

Solution by N. G. Gunderson, University of Rochester. It is clear from the expressions

$$N! = a^2 - b^2 = (a + b)(a - b)$$

and

$$N! = AB = \{(A + B)/2\}^2 - \{(A - B)/2\}^2$$

that the number of different ways of expressing $N!$ as the difference of two squares is the number of different ways of factoring $N!$ into two factors both even or both odd. But since $N!$ is even for $N > 1$, these factors must be even. Therefore N must be greater than 3 if there are to be such factors.

To express $N!$ as $2^i 3^j 5^k \cdots$ we put

$$i = [N/2] + [N/2^2] + \cdots$$

and similarly for $j, k, \cdots$. The number of divisors, and hence the number of factorizations into two factors, of $3^j 5^k \cdots$ is $(j+1)(k+1) \cdots$. Corresponding to each of these factorizations there are $(i-1)$ factorizations of $2^i 3^j 5^k \cdots$ in which each factor is even. But, since $N!$ is never a square for $N > 1$, each such factorization appears twice, and so the number of different ways of factoring $N!$ for $N \geqq 4$ into even factors is

$$\tfrac{1}{2}(i - 1)(j + 1)(k + 1) \cdots .$$

This number is at least 2, since $i \geqq 3$, $j \geqq 1$ for $N \geqq 4$.

Thus there are two different representations of 4! as the difference of two squares, four for 5!, nine for 6!, *etc.*

Also solved by Michael Aissen, Murray Barbour, Barney Bissinger, Monte Dernham, E. J. Finan, M. S. Klamkin, H. L. Lee, Roger Lessard, Eric Michalup, Martin Milgram, Leo Moser, C. S. Ogilvy, Margaret Olmsted, R. P. Stephens, C. W. Trigg, and the proposer.

As a companion problem Moser proposed to prove that, for $N>2$, $N!$ cannot be expressed as the sum of two squares. This, he stated, follows immediately from two known theorems: (1) n is of the form x^2+y^2 if and only if $n=n_1^2n_2$, where n_2 has no prime factor of the form $4r+3$. (Hardy and Wright, p. 297.) (2) If $x>7/2$ there is at least one prime of the form $4r+3$ satisfying $x<p\leqq 2x$. (R. Breusch "Für Verallgemeinerung des Bertrandschen Postulates dass zwischen x und $2x$ stets primzahlen liegen," Math. Zeit., vol. 34 (1932), pp. 505–526, and P. Erdös "Über die Primzahlen gewisser arithmetischer Reihen," Math. Zeit., vol. 39 (1935), pp. 473–491.)

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results found in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4331. *Proposed by V. F. Ivanoff, San Francisco*

Let the equation, $x^n+p_1x^{n-1}+\cdots+p_n=0$, possess a root x_1 whose modulus exceeds that of every other root of the equation. Prove that

$$\lim \frac{\Delta_k}{\Delta_{k-1}} = -x_1,$$

where Δ_k is the k-rowed determinant $(k>n)$

$$\begin{vmatrix} p_1 & p_2 & \cdot & \cdot & p_n & 0 & \cdot & \cdot & 0 & 0 \\ 1 & p_1 & p_2 & \cdot & \cdot & p_n & \cdot & \cdot & 0 & 0 \\ 0 & 1 & p_1 & \cdot & \cdot & \cdot & p_n & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 1 & p_1 & \cdot & \cdot & \cdot & p_{n-1} & np_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 1 & p_1 & 2p_2 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 & p_1 \end{vmatrix}.$$

4332. *Proposed by Paul Erdös, Syracuse University*

Let $a_1 < a_2 < \cdots$ be an infinite sequence of integers. Prove that from the sequence $a_i + a_j$, $i = 1, 2, \cdots, j = 1, 2, \cdots$, one can always select an infinite subsequence such that no element divides another.

4333. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In every system of numeration whose base B has the form $am^p + 1$, each of the p-digit numbers

$$(am^{p-1}+1)^p,\ (2am^{p-1}+1)^p,\ (3am^{p-1}+1)^p, \cdots,\ [(m-1)am^{p-1}+1]^p,$$

gives rise to an infinite sequence of p-th powers. The law of formation is very simple and is sufficiently clear from the following example:

If $B = 97 = 6\cdot 4^2 + 1 = 12\cdot 2^3 + 1 = 6\cdot 2^4 + 1 = 3\cdot 2^5 + 1$, then we have

$$\overline{25}^2 = 6\ \overline{43},\ \overline{24}\ \overline{25}^2 = 6\ 6\ \overline{42}\ \overline{43},\ \overline{24}\ \overline{24}\ \overline{25}^2 = 6\ 6\ 6\ \overline{42}\ \overline{42}\ \overline{43}, \cdots$$

$$\overline{49}^2 = \overline{24}\ \overline{73},\ \overline{48}\ \overline{49}^2 = \overline{24}\ \overline{24}\ \overline{72}\ \overline{73},$$

$$\overline{48}\ \overline{48}\ \overline{49}^2 = \overline{24}\ \overline{24}\ \overline{24}\ \overline{72}\ \overline{72}\ \overline{73}, \cdots$$

$$\overline{73}^2 = \overline{54}\ \overline{91},\ \overline{72}\ \overline{73}^2 = \overline{54}\ \overline{54}\ \overline{90}\ \overline{91},$$

$$\overline{72}\ \overline{72}\ \overline{73}^2 = \overline{54}\ \overline{54}\ \overline{54}\ \overline{90}\ \overline{90}\ \overline{91}, \cdots$$

$$\overline{49}^3 = \overline{12}\ \overline{48}\ \overline{85},\ \overline{48}\ \overline{49}^3 = \overline{12}\ \overline{12}\ \overline{48}\ \overline{48}\ \overline{84}\ \overline{85},$$

$$\overline{48}\ \overline{48}\ \overline{49}^3 = \overline{12}\ \overline{12}\ \overline{12}\ \overline{48}\ \overline{48}\ \overline{48}\ \overline{84}\ \overline{84}\ \overline{85}, \cdots$$

$$\overline{49}^4 = 6\ \overline{30}\ \overline{66}\ \overline{91},\ \overline{48}\ \overline{49}^4 = 6\ 6\ \overline{30}\ \overline{30}\ \overline{66}\ \overline{66}\ \overline{90}\ \overline{91},$$

$$\overline{48}\ \overline{48}\ \overline{49}^4 = 6\ 6\ 6\ \overline{30}\ \overline{30}\ \overline{30}\ \overline{66}\ \overline{66}\ \overline{66}\ \overline{90}\ \overline{90}\ \overline{91}, \cdots$$

$$\overline{49}^5 = 3\ \overline{18}\ \overline{48}\ \overline{78}\ \overline{94},\ \overline{48}\ \overline{49}^5 = 3\ 3\ \overline{18}\ \overline{18}\ \overline{48}\ \overline{48}\ \overline{78}\ \overline{78}\ \overline{93}\ \overline{94},$$

$$\overline{48}\ \overline{48}\ \overline{49}^5 = 3\ 3\ 3\ \overline{18}\ \overline{18}\ \overline{18}\ \overline{48}\ \overline{48}\ \overline{48}\ \overline{78}\ \overline{78}\ \overline{78}\ \overline{93}\ \overline{93}\ \overline{94}, \cdots.$$

Establish the general rule.

4334. *Proposed by H. F. Sandham, Trinity College, Ireland*

Prove that the feet of the six perpendiculars from the Bennett point on the sides of a complete quadrangle lie on a conic.

4335. *Proposed by D. D. Wall, Cambridge, Massachusetts*

If n is a prime greater than 3, show that

$$\prod_{x=1}^{n}\left(1 + 2\cos\frac{2\pi x}{n}\right) = 3, \tag{1}$$

$$\sum_{x=1}^{n} \sec\frac{2\pi x}{n} = (-1)^{(n-1)/2} n. \tag{2}$$

The restrictions on n can be lessened somewhat in each case.

SOLUTIONS

Isogonic and Isodynamic Centers

4196 [1946, 160]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let A_1, B_1, C_1 be vertices of equilateral triangles constructed exteriorly, or interiorly, on the sides BC, CA, AB of a triangle, and let A_2, B_2, C_2 be the intersections $(BC_1, CB_1)(CA_1, AC_1)$, (AB_1, BA_1). Then, V and W being the first (or second) isogonic and isodynamic centers of ABC, show that $1/VW=1/A_1A_2+1/B_1B_2+1/C_1C_2$.

I. *Solution by O. J. Ramler, Catholic University.* From a figure one can readily see that triangles A_2BC and AB_2C are directly similar, and thus show that triangles CA_1A_2 and CB_2B_1 are also directly similar. Therefore A_1A_2 and B_1B_2 are parallel. In this manner we may show that A_1A_2, B_1B_2, C_1C_2 are parallel.

Since A_1, A_2 and V, W are pairs of isogonal conjugate points for triangle ABC, it follows that A, W, A_2 are collinear. Let AA_1 and AA_2 cut BC in V_1 and W_1 respectively. Then the points A, V, V_1, A_1 on lines AA_1 are in one-to-one isogonal correspondence with the points W_1, W, A, A_2 on line AA_2. The two sets of points are therefore projectively related, and their axis of homology is the line BC. Hence VA_2 and WA_1 intersect on BC. It now follows that triangles B_1VA_2 and B_2WA_1 are coaxal with BC as axis. These triangles are then also copolar, and VW is parallel to A_1A_2 and B_1B_2.

As a consequence of the above we have

$$VW/A_1A_2 = AV/AA_1, \qquad VW/B_1B_2 = BV/BB_1, \qquad VW/C_1C_2 = CV/CC_1.$$

But, since V is an isogonic center of triangle ABC,

$$VA + VB + VC = AA_1 = BB_1 = CC_1. \tag{1}$$

Therefore

$$1/VW = 1/A_1A_2 + 1/B_1B_2 + 1/C_1C_2.$$

II. *Solution by the Proposer.* We shall establish the more general

THEOREM. Given a triangle ABC and an arbitrary point V in its plane, let the lines AV, BV, CV cut the circles BVC, CVA, AVB again in the points A_1, B_1, C_1 and let W, A_2, B_2, C_2 denote the isogonal conjugates, for triangle ABC, of V, A_1, B_1, C_1. Then

$$1/VW = 1/A_1A_2 + 1/B_1B_2 + 1/C_1C_2.$$

Let Γ, Γ_a, Γ_b, Γ_c be the conics inscribed in triangle ABC and having V and W, A_1 and A_2, B_1, and B_2, C_1 and C_2 as foci. Then conic Γ is the envelope of segments PQ subtending at V an angle equal to $180°-BVC$, P and Q being on AB and AC. But conic Γ_a is the envelope of segments P_1Q_1 subtending at A_1 an angle equal to $180°-BVC$. To each tangent PQ of Γ corresponds a parallel tangent P_1Q_1 of Γ_a such that VP, VQ are parallel to A_1P_1, A_1Q_1. Conic Γ_a is thus the transform of Γ under the homothety $(A, AA_1/AV)$. Therefore VW and A_1A_2 are

parallel and $VW/A_1A_2=AV/AA_1$. In this manner, by similarly treating the conics Γ_b and Γ_c, we obtain

$$VW(1/A_1A_2 + 1/B_1B_2 + 1/C_1C_2) = AV/AA_1 + BV/BB_1 + CV/CC_1.$$

But

$$AV/AA_1 + BV/BB_1 + CV/CC_1 = 1, \tag{2}$$

whence

$$1/VW = 1/A_1A_2 + 1/B_1B_2 + 1/C_1C_2.$$

We have the given problem if V is taken as one of the isogonic centers of triangles ABC.

Editorial Note. The proposer pointed out that VW, A_1A_2, B_1B_2, C_1C_2 of solution I have been shown to be parallel to the Euler line of triangle ABC by A. Boutin in the *Journal de Mathêmatiques Êlêmentaires de G. de Longchamps*, 1889.

Relation (1) above is a well known property of the isogonic centers, and may be found, for example, in art. 353 of Johnson's *Modern Geometry.* If V is not inside triangle ABC, then certain signs must accompany VA, VB, VC, and the same signs will be attached to the corresponding fractions $1/A_1A_2$, $1/B_1B_2$, $1/C_1C_2$.

Relation (2) may be established as follows; Invert the figure with respect as ρ^2 as power. Designating the inverted points by corresponding lower case letters, the inverted figure is a triangle abc with three cevians aa_1, bb_1, cc_1 concurrent in V. We have

$$AV = \rho^2/aV, \qquad BV = \rho^2/bV, \qquad CV = \rho^2/cV,$$

$$AA_1 = \rho^2/aV + \rho^2/Va_1, \qquad BB_1 = \rho^2/bV + \rho^2/Vb_1, \qquad CC_1 = \rho^2/cV + \rho^2/Vc_1.$$

Therefore

$$\begin{aligned} AV/AA_1 + BV/BB_1 + CV/CC_1 &= Va_1/aa_1 + Vb_1/bb_1 + Vc_1/cc_1 \\ &= \Delta bVc/\Delta bac + \Delta cVa/\Delta cba + \Delta aVb/\Delta acb = 1. \end{aligned}$$

A Special Tetrahedron

4201 [1946, 225]. *Proposed by Victor Thêbault, Tennie, Sarthe, France*

A tetrahedron is given for which the difference of squares of opposite edges is the same for the three pairs. (1) The three medians are equal and the line joining the circumcenter to the centroid is perpendicular to one of the faces. (2) One of the altitudes passes through the symmetric of the orthocenter of the corresponding face with respect to the circumcenter of the face. (3) The sum, or the difference, of the cosines of two opposite dihedrals of the tangential tetrahedron is the same for the three pairs.

I. *Solution by the Proposer.**

(1) Let $T \equiv ABCD$ be a tetrahedron such that

$$|a^2 - a'^2| = |b^2 - b'^2| = |c^2 - c'^2| \neq 0,$$

where $BC = a$, $CA = b$, $AB = c$, $AD = a'$, $BD = b'$, $CD = c'$. We shall limit our discussion to the case where

$$a^2 - a'^2 = b^2 - b'^2 = c^2 - c'^2,$$

as the three other possibilities give rise to like conclusions.

If we denote by N, Q, S the midpoints of the edges DA, DB, DC, we find that triangles BNC, CQA, ASB are isosceles. For instance, $\overline{BN}^2 = (2c^2 + 2b'^2 - a'^2)/4$, $\overline{CN}^2 = (2b^2 + 2c'^2 - a'^2)/4$. The bimedians MN, PQ, RS are therefore perpendicular respectively to BC, CA, AB. Hence the planes drawn through N, Q, S perpendicular to BC, CA, AB (perpendicular bisectors of BC, CA, AB) intersect in a line Δ through the circumcenter O_d of triangle ABC perpendicular to the plane ABC. As the bimedians meet at the centroid G of T, Δ contains G and $GA = GB = GC$. Likewise the circumcenter O and the Monge point Ω of T are on Δ.

(2) Let G_d and H_d be the centroid and the orthocenter of triangle ABC. Let D' be the orthogonal projection of D on the plane ABC. Since G projects into O_d and $DG = 3GG_d$,

$$D'O_d = 3\,O_dG_d = O_dH_d.$$

Hence D' is the symmetric of H_d with respect to O_d.

(3) Let $T_2 \equiv A_2B_2C_2D_2$ be the tangential tetrahedron of T, with the plane of the face $B_2C_2D_2$ tangent to the circumsphere (O, R) of T at A. If α is the plane angle of $A_2 - B_2C_2 - D_2$, and α' the plane angle of $B_2 - A_2D_2 - C_2$, etc., we have

$$a^2 = 2R^2(1 + \cos\alpha'), \qquad b^2 = 2R^2(1 + \cos\beta'), \qquad c^2 = 2R^2(1 + \cos\gamma'),$$
$$a'^2 = 2R^2(1 + \cos\alpha), \qquad b'^2 = 2R^2(1 + \cos\beta), \qquad c'^2 = 2R^2(1 + \cos\gamma),$$

and

$$\cos\alpha' - \cos\alpha = \cos\beta' - \cos\beta = \cos\gamma' - \cos\gamma,$$

provided that T is interior to T_2.

If, for example, D_2 and O are on opposite sides of the plane $A_2B_2C_2$, i.e. T is exterior to T_2, we have

$$a^2 = 2R^2(1 + \cos\alpha'), \qquad a'^2 = 2R^2(1 - \cos\alpha), \cdots,$$

and

$$\cos\alpha' + \cos\alpha = \cos\beta' + \cos\beta = \cos\gamma' + \cos\gamma.$$

* Translated and checked by W. E. Byrne, Virginia Military Institute.

If $a^2-a'^2=b^2-b'^2=c^2-c'^2=0$, we have an isosceles tetrahedron, which is discussed on pages 94–102 of Court, *Modern Pure Solid Geometry.*

II. *Generalization by R. Blanchard, Le Havre, France.* If, using the notations of the solution by the proposer, $a^2+ka'^2=b^2+kb'^2=c^2+kc'^2=l$, k arbitrary:

(1) D' is on the line joining O_d and H_d, and $O_dD'=O_dH_d/k$.

(2) The line which joins the circumcenter O of T to the point K of the median DG_d such that $KG_d/KD=k/3$ is perpendicular to the plane ABC. The segements AA_1, BB_1, CC_1 which join A, B, C to K and are terminated at the opposite faces of T are of equal length.

(3) The centroid G of T is projected orthogonally on ABC in a point G_d' of O_dG_d such that $O_dG_d'/O_dG_d=3(k+1)/4k$.

(4) $\cos\alpha'+k\cos\alpha=\cos\beta'+k\cos\beta=\cos\gamma'+k\cos\gamma$ if T is interior to T_2.

Let us take the point D_1 on DO_d such that $O_dD_1=kO_dD$ and write $D_1A=a_1'$, $D_1B=b_1'$, $D_1C=c_1'$. We find $k\overline{DA}^2-\overline{D_1A}^2=(k-1)\overline{AO}_d^2+k\overline{DO}_d^2-\overline{D_1O}_d^2$. If R_d denotes the radius of the circumcircle of the triangle ABC, we have

$$ka'^2-a_1'^2=(k-1)R_d^2-k(k-1)\overline{DO}_d^2.$$

Hence, if

$$l-(k-1)R_d^2+k(k-1)\overline{DO}_d^2>0,$$
$$a^2+a_1'^2=l-(k-1)R_d^2+k(k-1)\overline{DO}_d^2=b^2+b_1'^2=c^2+c_1'^2,$$

which shows that the tetrahedron $ABCD_1$ is orthocentric.

To construct a tetrahedron T satisfying the statement of the generalized problem, it is sufficient to start with an orthocentric tetrahedron $ABCD_1$, to take a point D on O_dD_1 such that $O_dD=O_dD_1/k$.

(1) D' is on O_dH_d and $O_dD'/O_dH_d=O_dD/O_dD_1=1/k$.

(2) Let L be the point where the perpendicular at G_d to the face ABC meets DD_1. Since $KG_d/KD=k/3$ and $O_dL/O_dD=O_dD_1/3O_dD=k/3$, KO_d and G_dL are parallel. Hence O_dK is perpendicular to the face ABC and passes through O. Furthermore, as K is the centroid of A, B, C, D with the weights $1, 1, 1, -k$,

$$AA_1/AK=BB_1/BK=CC_1/CK=(3-k)/(2-k)$$

and since $AK=BK=CK$, it follows that $AA_1=BB_1=CC_1$.

(3) We have

$$O_dG_d'=O_dG_d+G_dG_d'=O_dG_d+G_dD'/4.$$

But

$$G_dD'=O_dD'-O_dG_d=O_dG_d(3-k)/k.$$

Hence

$$O_dG_d'=3(n+1)O_dG_d/4k.$$

(4) If T is interior to T_2,

$$a^2 + ka'^2 = 2R^2[1 + \cos\alpha' + k(1 + \cos\alpha)],$$

$$\cos\alpha' + k\cos\alpha = \cos\beta' + k\cos\beta = \cos\gamma' + k\cos\gamma = (l/2R^2) - (k + 1).$$

If $k = -1$, the results are:

(1) $O_dD'/O_dH_d = -1$, D' is the symmetric of H_d with respect to O_d.

(2) $KG_d/KD = -1/3$, K coincides with G. OG is perpendicular to the plane ABC, and the medians AA_1, BB_1, CC_1 are equal.

(3) $O_dG_d' = 0$, which shows again that OG is perpendicular to the face ABC. We find the question proposed by V. Thébault as a special case of our generalization.

If $k = 3$, D' coincides with G_d and conversely if $D' \equiv G_d$, $k = 3$. The necessary and sufficient condition that the orthogonal projection of D on ABC be the centroid of ABC is that $a^2 + 3a'^2 = b^2 + 3b'^2 = c^2 + 3c'^2$.

Likewise, the necessary and sufficient condition that the orthogonal projection of D on ABC be the center of the nine-point circle of ABC is that

$$a^2 + 2a'^2 = b^2 + 2b'^2 = c^2 + 2c'^2.$$

Isogonal Conjugate Points of a Tetrahedron

4208 [1947, 50]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Given an orthocentric tetrahedron. If two isogonal conjugate points are also conjugate with respect to the circumsphere, their pedal sphere is orthogonal to the sphere, belonging to the linear net determined by the circumscribed and conjugate spheres, and whose center is the complementary point of the orthocenter with respect to the tetrahedron.

*Solution by R. Bouvaist, Vincelles, Saône-et-Loire, France.** We choose rectangular axes with the origin at the orthocenter H of the orthocentric tetrahedron $T \equiv ABCD$, and with the circumcenter O of T on the x-axis at $(d, 0, 0)$. The conjugate sphere (H) of T is given by

$$(H) \equiv x^2 + y^2 + z^2 - \rho^2 = 0$$

with $d^2 = R^2 + 3\rho^2$.† The circumsphere (O) of T has the equation

$$(O) \equiv (x - d)^2 + y^2 + z^2 - R^2 = 0,$$

or

$$(O) \equiv x^2 + y^2 + z^2 - 2dx + 3\rho^2 = 0.$$

Let $(\Gamma) \equiv b^2(u^2 + v^2 + w^2) - (ux_1 + vy_1 + wz_1 + s)(ux_2 + vy_2 + wz_2 + s) = 0$ be the tangential equation of a quadric of revolution with the given isogonal conjugate points $F_1(x_1, y_1, z_1)$, $F_2(x_2, y_2, z_2)$ as foci. The condition that (Γ) be inscribed in T is:

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* Translated by W. E. Byrne, Virginia Military Institute.

† Court, Modern Pure Solid Geometry, p. 265.

$$3.^{\prime 2} - x_1x_2 - y_1y_2 - z_1z_2 + \rho^2 = 0. \tag{1}$$

Since F_1, F_2 are conjugate with respect to (O), we have

$$x_1x_2 + y_1y_2 + z_1z_2 - d(x_1 + x_2) + 3\rho^2 = 0. \tag{2}$$

The pedal sphere (S) of F_1 and F_2 is:

$$\left(x - \frac{x_1 + x_2}{2}\right)^2 + \left(y - \frac{y_1 + y_2}{2}\right)^2 + \left(z - \frac{z_1 + z_2}{2}\right)^2$$
$$= b^2 + \left(\frac{x_1 - x_2}{2}\right)^2 + \left(\frac{y_1 - y_2}{2}\right)^2 + \left(\frac{z_1 - z_2}{2}\right)^2,$$

or

$$(S) \equiv x^2 + y^2 + z^2 - (x_1 + x_2)x - (y_1 + y_2)y - (z_1 + z_2)z$$
$$+ x_1x_2 + y_1y_2 + z_1z_2 - b^2 = 0.$$

The condition that the sphere

$$\lambda(H) + (O) = 0 \tag{3}$$

of the coaxal pencil determined by (H) and (O), be orthogonal to (S) is that $\lambda = \frac{1}{2}$. For $\lambda = \frac{1}{2}$, equation (3) reduces to

$$x^2 + y^2 + z^2 - 4dx/3 + 5\rho^2/3 = 0, \tag{4}$$

or

$$(x - 2d/3)^2 + y^2 + z^2 = 4d^2/9 - 5\rho^2/3 = R^2/9 + (a^2 + a'^2)/12,$$

where $a = BC$, $a' = AD$.* The center $\Omega(2d/3, 0, 0)$ of the sphere (4) is the complementary point of H with respect to T since

$$\overrightarrow{G\Omega} = -\overrightarrow{GH}/3.$$

The Rebounding Projectile

4249 [1947, 286]. *Proposed by W. B. Campbell, Philadelphia Textile Institute*

A body is projected from a point O in a plane making an angle A with the horizontal, the direction of projection being in a vertical plane containing a line of greatest slope of the plane, and making an angle B with the upward direction of that line. If the plane be smooth and the body perfectly elastic, derive expressions for t_n, the time consumed in the nth flight, and for x_n, the coördinate of the point of impact at the end of the nth flight. Will it ever strike O again, and will any of its flights be vertical? What is maximum x_n?

Solution by F. G. Fender, Rutgers University. Take OX up the plane and OY normal to it, and let v be the initial velocity. Then

$$y = vt \sin B - \tfrac{1}{2}gt^2 \cos A$$

* For the details see Court, loc. cit., p. 271.

for the first arch y returns to zero at

$$t = 2v \sin B/g \cos A \equiv T$$

and, at the end of the first flight

$$\dot{y} = v \sin B - \tfrac{1}{2}g \cos A(2\cdot 2v \sin B/g \cos A) = -v \sin B.$$

Then, by the perfect elasticity, each new arch has the same initial y-component of velocity, and hence t_n the time of flight in each arch will be $\Delta t = T$ a constant. If x_n is the value of x at the end of the nth arch, x_n is also the value of x at the beginning of the $(n+1)$th arch. Similarly u_n, the velocity at the end of the nth arch, is also the velocity at the beginning of the $(n+1)$th arch. Then

$$\Delta x = x_n - x_{n-1} = u_{n-1}T - \tfrac{1}{2}gT^2 \sin A.$$

Also $u_n = u_{n-1} - gT \sin A$. Since $u_0 = v \cos B$ we have

$$u_n = u_0 - ngT \sin A = v \cos B - 2nv \sin B \tan A$$

or

(1) $\quad u_n = v \sin B \tan A(\cot A \cot B - 2n) = K(L - 2n), \qquad n = 0, 1, 2, \cdots,$

where

$$K \equiv v \sin B \tan A, \qquad L \equiv \cot A \cot B.$$

Then

(2) $\Delta x = K[L - 2(n-1)]T - \tfrac{1}{2}gT^2 \sin A = KT(L - 2n + 1), \; n = 1, 2, \cdots.$

Furthermore we have

$$(3) \qquad x_n = \sum_{j=1}^{n} \Delta x = KT \sum_{j=1}^{n} (L + 1 - 2j) = KTn(L - n).$$

If A and B are assumed acute, then K, L, T are all positive. Then, no matter how large L may be, Δx is negative for all n greater than $\tfrac{1}{2}(L+1)$. Hence the maximum value of x_n occurs when n equals the smallest integer which exceeds $\tfrac{1}{2}(L+1)$. For greater values of n, Δx is always negative and increases indefinitely in magnitude, so that $x_n \to -\infty$ as $n \to \infty$.

If L is any odd positive integer, $2m+1$, then x_m and x_{m+1} are equal, so that the $(m+1)$th flight is vertical. Also $x_{m+k+1} = x_{m-k}$ $(0 \leqq k \leqq m)$. If L is a positive even integer, $2m$, then the mth flight ends normal to the plane, $n = m+1$ gives the maximum x_n, hence also $x_{m+k} = x_{m-k} (0 \leqq k \leqq m)$. From (3) if $x_n = x_{n'}$, n and n' integers, we must have $n = n'$ or $n - n' = L$. Thus the body never strikes twice in the same place except when L is a positive integer.

It is interesting to note, and easily shown, that the maximum abscissa, $x_{\max}$ (with positive y) is given by

$$x_{\max} = \tfrac{1}{4}KTL^2,$$

which exceeds maximum x_n in all cases except when L is a positive even integer.

Also solved by N. J. Fine and the Proposer.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Solid Analytic Geometry. By J. M. H. Olmsted. New York, D. Appleton-Century Company, Inc., 1947. 13+257 pages. $4.00.

That solid analytic geometry constitutes a recognizable subject matter seems generally agreed. In one view the parts are loosely assembled from other disciplines to which they properly belong. In another, those same parts cohere in a reasonably well defined discipline. The author has produced a comprehensive text in which the latter view preponderates.

The purpose of the book, as declared in the preface, is to meet the needs of any of several courses: a brief course emphasizing planes and lines and treating only the simpler equations of second degree; a standard first course which would include the notion of rank of a matrix, coordinate transformations, and a more complete study of equations of second degree; and an extensive course enlarging the study of planes and lines and employing matrix methods in the treatment of transformations and the general equation of second degree. Basing himself squarely on coordinates, the author carries through this program. To provide the required flexibility he seeks to postpone difficulties as much as possible and to combine simplicity of presentation with authenticity.

Beginning with the properties of Euclidean space, coordinates are introduced and the necessary preliminaries disposed of. There follows a treatment of planes and lines. In the succeeding chapter facts about determinants are recalled and matrix and rank are defined. (An outline proof of the invariance of rank under elementary transformations is included.) Systems of planes, and of simultaneous linear equations generally, are dealt with on this basis. The chapter concludes with a brief account of finite dimensional real and complex space.

The major concern of the text is the study of surfaces, restricted to be algebraic. Associated with a real polynomial equation $f(x, y, z)=0$ are a graph and an algebraic surface. The algebraic surface consists of irreducible surfaces (graphs), each with a multiplicity. General properties of surfaces (and of curves of intersection) are treated first, as well as such special sorts of surfaces as cylinders and the like. A chapter is then devoted to the seventeen canonical quadric surfaces, identified by their equations. (Only real graphs are analyzed, although two surfaces are regarded as distinct if their complex graphs are distinct.) With these "standard" quadric surfaces at hand, a preliminary analysis of the general equation of second degree is undertaken. This first attack, utilizing matrices associated with the equation, proceeds without transformations and is concentrated upon such matters as centers, tangents, and rulings. The following chapter treats of translations and rotations as transformations either of coordinates

or of points. The ultimate goal is attained in proving that an equation of second degree is reducible under rigid motions (reflections included) to one and only one of the canonical seventeen.

A last and essentially independent chapter develops the elements of matrix algebra.

The author proposes a course of some thirty lessons, more or less. Much additional material suitable for an extended version or for supplementary reading is available. The text abounds with exercises ranging from routine to penetrating. Suggestions and illustrative examples are disposed at strategic places.

The plan of the book entails the avoidance of algebraic complications. For this reason, while the author does not demand it, a familiar acquaintance with parts of algebra is certainly desirable. Determinants are of frequent occurrence and their properties are used. Crucial theorems from the theory of real polynomials are quoted, with references, but their significance may escape the uninitiated. It is stated, for example, that polynomials can be decomposed into irreducible factors essentially uniquely and that a polynomial which vanishes everywhere is the zero polynomial.

Two features of the presentation should be remarked. First, the definition of equivalence for real polynomial equations in terms of their graphs is eschewed. Instead equivalence is based on these transformations of an equation: multiplication of both members by the same nonzero real number and addition to both members of the same polynomial. (By tacit agreement equal polynomials, as xy and yx, for example, may replace each other anywhere.) Second, an algebraic surface is not the graph, real or complex, of a real polynomial equation $f(x, y, z)=0$. To each of the distinct real irreducible factors $g(x, y, z)$ of $f(x, y, z)$ corresponds an irreducible surface, the complex graph of $g(x, y, z)=0$. The totality of these graphs, each bearing the multiplicity of $g(x, y, z)$ as a factor of $f(x, y, z)$, constitutes the algebraic surface. This choice of definitions makes for a certain amount of fussiness which largely evaporates for quadric surfaces. Throughout, the real part of a surface naturally predominates, the nonreal part making only incidental appearances.

The reviewer found no misprints, contrary to custom, but did note these places where misunderstanding is possible. A scrupulous effort to verify the assertion at the top of page 137 that ϕ must vanish under all circumstances requires actual examination of the canonical quadric surfaces. And the figure of section 11 is still misleading in spite of the author's caveat.

A. L. PUTNAM

The Outlook for Women in Mathematics and Statistics. By Mary H. Brilla. Bulletin No. 223–4 of the Women's Bureau, U. S. Department of Labor. Washington, Government Printing Office, 1948. 10+21 pages. $0.10.

This small booklet is part of a larger study of occupational opportunities for women in all scientific fields. The great demand for women with scientific train-

ing during World War II prompted the Women's Bureau to make this study. An attempt is made to tell where these women work, what kind of work they are doing, and what other young women may expect in the way of future opportunities to do work of this type.

Although written primarily for women, much of the material in the booklet is of interest to all who expect to earn a living by following some mathematical profession. Estimates are made of the number of mathematicians in teaching and in other fields. The prewar distribution of mathematicians, the average number of annual additions, wartime changes, present earnings and advancement, and the outlook for the future are other topics treated in detail. The appendix contains a bibliography which should prove useful to all who are concerned with the advisement of students of mathematics.

H. M. GEHMAN

Calculus and its Applications. By R. D. Douglass and S. D. Zeldin. New York, Prentice-Hall, Inc., 1947. 8+568 pages. $5.15.

In the preface the authors state that their purpose in writing this textbook on the Calculus was to cover the material essential for students in engineering and science in a shorter period of time than that now required in most schools and colleges. While it is true that they have omitted some topics normally included in a text of this type and have discussed other topics more briefly than is usual, the book contains ample material for a full year's course and nearly all the topics that are usually included in such a course.

The method of presentation employed throughout the book consists of short discussions or statements of purpose followed by numerous illustrative examples solved in great detail. In fact, one of the excellent features of the book is the many well chosen illustrative examples. There are more than 360 of these and nearly all are treated with unusual care and thoroughness.

An excellent treatment of differentials is given in Chap. IV followed immediately by the introduction to the indefinite integral. Integration is defined as the process of finding a function $F(x)$ which has for its differential a given expression $f(x)dx$. Moreover, answers are verified by finding their differentials so that students should not so readily adopt the annoying habit of treating dx as excess baggage.

The definite integral is postponed until Chap. VI by which time the student has had ample drill in formal integration and the applications of the indefinite integral. This chapter is clearly written and well illustrated by examples.

The first part of Chap. XIII (infinite series) is very well done. This reviewer agrees wholeheartedly with the authors that the integral test, when properly presented, is perhaps the simplest technique for the beginner to understand and should be introduced early.

Other parts of the book not specifically mentioned above contain most of the material normally found in a book of this type, including a chapter on dif-

ferential equations. In addition, there is a chapter on vectors. This final chapter (on vectors) embraces no material that could not be taught to college sophomores. The entire book is carefully written, is developed in a logical fashion, and appears to be very teachable. There are nearly 200 figures the majority of which are excellent. A list of problems is included in nearly every section and a review list is found at the end of each chapter. There are over 2700 problems in the book and the answers to all problems are given.

Naturally there are some criticisms of an adverse nature that this reviewer feels should be made. Parametric equations are not introduced until Chap. VIII, and polar equations are withheld until Chap. IX. These can profitably be used much earlier. Why the authors treat only the indeterminate form 0/0 when some of the others are so easily converted into this form is puzzling. Why a chapter on vectors is included and such standard topics as approximate integration, cylindrical and spherical coordinates, and a fuller discussion of plane curves are omitted is also hard to understand. The chapter on partial differentiation is somewhat out of balance. A lengthy and rather complete discussion of partial and total derivatives of composite functions is followed by a weak and inadequate treatment of maxima and minima.

A few errors were discovered, most of which were either trivial or mildly serious. For example, on page 33 there is a delta t where it should be delta x; on page 74, the answer to example 3 should be verified; on page 146, the lower limit of sigma is missing; and in all about 20 errors of this type were noted. There were a few of a slightly more serious nature, such as stating that the circle of curvature has a radius equal to the radius of curvature with no mention of sign; or in discussing approximating by differentials the statement is made "By means of differentials, however, the calculation is shortened, and even though the resulting value *may be* approximate, the error made *is* negligible." (Italics mine.) An interchange of verbs would improve this statement. In the discussion on fluid pressure on a vertical surface, page 168, the statement is made "if delta h *is small*, the surface of each rectangular element is approximately parallel to the surface of the liquid." This obviously is false as the element remains vertical regardless of the size of delta h.

Finally a word concerning the topography. The paper is a good grade and the printing is clear excepting for fractional exponents. These are nearly always difficult to read and are frequently almost illegible. There are some bad breaks in equations such as $y=3^{1/2}$ on one line and sin x on the next line. The same size type is used for text, illustrative examples, and problems. This makes it difficult to determine where an illustrative example ends and a new discussion begins.

In summary, this reviewer feels that this is a very carefully written text with remarkably few inaccuracies most of which are of a trivial nature. Since the good points of this book far outweigh the weak points, it is a worthy addition to the present list of texts on the Calculus.

P. M. HUMMEL

NEW BOOKS RECEIVED

A Source Book in Greek Science. By M. R. Cohen and I. E. Drabkin. New York, McGraw-Hill Book Co., 1948. 22+597 pages. $9.00.

Mathematics of Finance. By P. M. Hummel and C. L. Seebeck. New York, McGraw-Hill Book Co., 1948. 11+365 pages. $4.00.

A Philosophy of Mathematics. By L. O. Kattsoff. Ames, The Iowa State College Press, 1948. 9+263 pages. $5.00.

The Kelley Statistical Tables. Revised Edition. By T. L. Kelley. Cambridge, The Harvard University Press, 1948. 9+223 pages. $5.00.

Rinehart Mathematical Tables. By H. D. Larsen. New York, Rinehart and Co., 1948. 8+264 pages. $1.50.

Differential- und Integralrechnung im Hinblick auf ihre Anwendungen. By L. Locher-Ernst. Basel, Birkhauser, 1948. 594 pages. 48 Fr.

Spherical Harmonics. Second Revised Edition. By T. M. MacRobert. New York, Dover Publications, 1948. 15+372 pages. $4.59.

Bericht über die Mathematiker-Tagung in Tübingen vom 23 bis 27 September 1946. The Mathematics Institute, Tübingen University. Tübingen, Druck von H. Laupp, Jr., 1946. 143 pages.

Leibniz. By J. T. Merz. New York, Hacker Press, 1948. 8+216 pages. Lithoprint Edition. $2.75.

Number Theory and Its History. By Oystein Ore. New York, McGraw-Hill Book Co., 1948. 10+370 pages. $4.50.

Industrial Electric Furnaces and Appliances. Vol. 2. By V. Paschkis. New York, Interscience Publishers, 1948. 14+320 pages. $8.00.

Mendeleyev. By D. Q. Posin. New York, Whittelsey House, McGraw-Hill Book Co., 1948. 12+345 pages. $4.50.

The Mathematical Solution of Engineering Problems. By M. G. Salvadori and K. S. Miller. New York, McGraw-Hill Book Co., 1948. 10+245 pages. $3.50.

Tables of Coefficients for Obtaining the First Derivatives Without Differences. (National Bureau of Standards, Applied Math. Series 2.) By H. E. Salzer. Washington, U. S. Govt. Printing Office, 1948. 20 pages. $0.15.

Mathematics Our Great Heritage. Essays on the Nature and Cultural Significance of Mathematics. By W. L. Schaaf. New York, Harper and Brothers, 1948. 11+291 pages. $3.50.

The Mathematical Basis of the Arts (Parts 1, 2, 3). By J. Schillinger. New York, Philosophical Library, 1948. 10+696 pages. $12.00.

Tables of the Bessel Functions of the First Kind of Orders Twenty-eight Through Thirty-nine. (Annals of the Computation Laboratory of Harvard University, No. 10.) Cambridge, The Harvard University Press, 1948. 694 pages. $10.00.

Lezioni de Analisi Matematica. Sixth Edition. By F. Tricomi. Padova, Cedam, 1948. 12+335 pages.

Theory of Equations. By J. V. Uspensky. New York, McGraw-Hill Book Co., 1948. 7+353 pages. $4.50.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

EDITOR'S NOTE. Many of the papers presented by undergraduates at mathematics club meetings merit publication. It has been called to our attention that *The Pentagon*, the official publication of Kappa Mu Epsilon, will accept some worthy papers written by undergraduates and submitted by a responsible member of the faculty.

It is not necessary, at present, for the student to be a member of Kappa Mu Epsilon in order for his paper to be accepted. Manuscripts should be sent to the editor of *The Pentagon*, Prof. Harold D. Larsen, Albion College, Albion, Michigan.

CLUB REPORTS, 1947–48

Mathematics Club, Brown University

The following papers were presented during the thirty-fourth annual program of the *Mathematics Club* of Brown University:

Numbers, by Dr. R. P. Boas, Jr., Executive Editor of the *Mathematical Reviews*

Some geometric extremal problems, by Anthony Behr

End-point maxima and minima, by Edward Fisher

An isoperimetric problem, by Ann Patterson

The snow plow problem, by Donald Haas

On the relation between mathematics and physics, by Prof. R. B. Lindsay, Professor of Physics at Brown University

Napoleon's problem, by Beverly Woonton

The game of Nim, by Reginald Flanders

Descartes, by Ann Kernan

Mechanical integrators, by Loren Wood.

Each of the regular meetings concluded with a social and discussion period at which light refreshments were served. The Mathematics Club presented a 15-minute program over station WBRU of Brown University on the history, program and purposes of the organization.

The Committee on Program and Arrangements for the year 1947–48 consisted of: L. W. Peckham, Jr., chairman; T. C. Andrews; P. J. Bray; K. M. Crowe; P. C. Curtis, Jr.; E. T. Mansfield; D. M. Nolan; H. G. Smith, Treasurer; Prof. D. W. Western, faculty representative.

The members of the Committee for the year 1948–49 are: Philip Curtis, Jr., chairman; Joseph Coudele, Eleanor Mansfield, Alexander Marshall, Ann Patterson, Adeline Petke, Rita Sisson, Chester Thomas, and Loren Wood.

Kappa Mu Epsilon, College of St. Francis

The *Illinois Delta* Chapter of *Kappa Mu Epsilon* records the following papers for 1947–48:

Fermagoric triangles, by Sister Mary Claudia O.S.F.
Attacks on mathematics and how to meet them, by Sister Mary Elizabeth O.S.F.
Macheroni's constructions, by Sister M. Rita Clare, O.S.F.
Calculating prodigies, by Mary Jean Lafond
Cryptographs and ciphers, by Patricia Young
Three fallacies, by Marie Crawley
History and trancendence of pi, by Betty Lanaue
Comparisons of astronomy and the calendar, by Anne Hutchings
The Pythagorean theorem and its extensions, by Lillian Rafac
Kirkman's "School-girl" problem, by Mary Lou Hodor
Gamma function, $0!=1$, by Jane Rourke
Pedal triangles, by Eleanor O'Connor.

The books, *Schoolmasters Assistant*, by Thomas Dilworth and *Mathematics and Religion*, by Sister Noel Marie were reviewed by Betty Frieburg and Sister Mary Hilary O.S.F.

The officers for 1947–48 were: President, Eleanor O'Connor; Vice-President, Elizabeth Lanoue; Secretary, Jane Rourke; Treasurer, Mary Jean Lafond; Faculty Advisor; Sister M. Rita Clare, O.S.F.

The officers for 1948–49 are: President, Jane Rourke; Vice-President, Mary Jean Lafond; Secretary, Lillian Rafac; Treasurer, Rita Groban.

Graduate Mathematics Club, Indiana University

During the second year of its existence, the *Graduate Mathematics Club* heard a series of talks by members of the mathematics department and two speakers from other departments. The topics were:

The arithmetic progression theorem, by Dr. G. W. Whaples
Bracketing and the ping-pong problem, by Dr. W. S. Gustin
Applications of linear algebra in quantum mechanics, by Dr. E. Konopinski
The nature of mathematical proof, by Dr. L. M. Graves
Bounded functions, by Dr. M. A. Zorn
Some statistical problems in bacteriophage work, by Dr. R. Dulbecco.

The Executive Committee for the year 1947–48 was: Mrs. Lee Pruett, I. N. Herstein, and Gordon Overholtzer.

The new Executive Committee is: Mrs. Lee Pruett, Joseph Sullivan, and David van Tijn.

Pi Mu Epsilon, Bucknell University

The program for the year 1947–48 of *Pennsylvania Beta* Chapter of *Pi Mu Epsilon* was:

The mathematics of jet propulsion, by H. I. Holman
The toss of a coin, by F. S. McFeely, Jr.
Ocean waves, by Prof. H. V. Flinsch

The history and computation of pi, by Mrs. W. E. Elze

Number, the language of science, by Marguerite Muller.

Officers for 1947–48 were: Director, Prof. D. P. Souders; Vice-Director, J. W. Sprout; Secretary, Mrs. W. E. Elze; Treasurer, Marguerite Muller.

Officers for 1948–49 are: Director, Prof. L. M. Swartz; Vice-Director, J. A. Bortner; Secretary, M. F. Nightingale; Treasurer F. S. McFeely, Jr.

Pi Mu Epsilon, Lehigh University

The *Pennsylvania Gamma* Chapter of *Pi Mu Epsilon* was reactivated in November 1947. Among the papers to be read at meetings were:

Certain aspects of statistics, by Prof. Garrison

Nomography, by Layton Butts

Topics in the theory of relativity, by Prof. K. W. Lamson

Some mathematical problems, by Prof. V. F. Cowling, a series of problems challenging the ability of the members.

Steps were taken to cooperate more closely with the *Newtonian Mathematics Society* for increasing the interest and scholarship in mathematics at Lehigh University.

Nine undergraduates and one faculty member were initiated in April at which occasion Prof. J. B. Reynolds, the retiring head of the department, was guest of honor. He gave an account of the progress and amusing events he has seen take place in the more than forty-one years he has been associated with Lehigh University.

The officers serving during 1947–48 were: Director, Prof. V. F. Cowling; President, Valerio Hunt; Secretary, J. H. Vogelsong; Treasurer, J. S. Adam.

The officers elected are: Director, Prof. R. R. Stoll; President, J. F. Ahern; Secretary, K. E. Ferree; Treasurer, L. E. Butts.

Mathematics Club, Ball State Teachers College

The following papers were presented to the *Mathematics Club* during 1947–48:

Topics from algebra, by James Swinford

Opportunities in statistics, by Merle Guthrie

Topics in trigonometry, by Kenneth Poucher

Paradox lost, by Kenneth Conkling

Applications of complex numbers in alternating currents, by Dr. Hummel

Vector analysis, by Elmo Purlee

Non-Euclidean geometry, by Dwain Small

Computational short-cuts, by Robert Dillon

Mathematics for the imagination, by Lewis Ward, Jr.

The annual picnic was held in May.

Officers who served during 1947–48 were: President, Elmo Purlee; Vice-President, Charles Galancy; Program Chairman, William Flora; Secretary, Edward Shreve; Treasurer, Nell Young.

Officers elected for 1948–49: President, John Pruden; Vice-President, Richard Moore; Secretary, Doris Mooney; Treasurer, Gale Brown.

Kappa Mu Epsilon, Hofstra College

The papers presented to the *New York Alpha* Chapter of *Kappa Mu Epsilon* for the year 1947–48 include:

The four-color problem, by Prof. L. F. Ollmann

Our nearest star, by Francis Wilson

A new series and some useless limits, by G. B. Charlesworth

Some properties of primes, by W. L. Marshall

A comparison of Oxford with American Universities, by Prof. Banesh Hoffmann, of Queens College

A short history of Kappa Mu Epsilon, by E. Marie Hove.

Following the second program the club visited the Hayden Planetarium. A Square Dance was held just before the Christmas vacation. The address by Dr. Hoffmann was given at the annual Initiation Banquet when 21 new members were initiated.

A prize was given to Fred Bloschies as the best student of Freshman Mathematics.

Officers for the year 1947–48 were: President, W. L. Marshall; Vice-President, Samuel Williams; Secretary, Gertrude Decker; Treasurer, J. G. Lutz; Corresponding Secretary, Frank Hawthorne; Faculty Sponsor, E. Marie Hove.

The officers elected for 1948–49 are: President, David Jordan; Vice-President, Richard Krause; Secretary, Gladys Meyer; Treasurer, Salvatore Cannizzaro; Corresponding Secretary, Beatrice Rohr; Faculty Sponsor, E. Marie Hove.

Pi Mu Epsilon, Michigan State College

During the year student and faculty speakers were alternated. The following topics were presented:

Mathematical problems in geography, by Joyce Clark

An envelope of pedal lines of a triangle, by Dr. Stewart

A two-variable maximum problem, by Joyce Deisch and Eugene Parker

Using I. B. M. *cards to find prime numbers*, by Prof. Frame

A theorem on partitions, by Charles Kraft

A theorem in continuous functions, by Cliff Gray

Radar equations, by Dr. Bell

A statistics program, by Dale Hekhuis and Robert Zabel.

Eleven members were initiated at the spring term and twenty at the fall term initiation. Since some members graduated, the total active membership is now thirty-four students.

In addition to the regular meetings, picnics were held in the fall and spring terms. At the annual banquet the L. C. Plant awards for deserving mathematics students were won by James Powell (first prize of $50) and Richard Zindler (second prize of $40). Prof. C. C. MacDuffee of the University of Wisconsin spoke on the topic *Is mathematics important?*

Officers for the year 1947–48 were: President, Eugene Parker; Vice-Presi-

dent, Nancy Waldo; Secretary, Joyce Clark; Treasurer, James Powell; Faculty Advisor, Dr. C. P. Wells.

Mathematics Club, Carleton College

The *Mathematics Club* of Carleton College has completed its second year of monthly meetings. Social activities included a picnic and a Christmas party with a program of mathematical recreations. The following talks were given during the year 1948–48:

Symbolic logic, by Dr. K. May

The mathematics of atomic collisions, by Dr. F. Vergrugge

Combinatorial topology, by Dr. G. K. Kalisch

Counting by dozens, by Prof. C. S. Carlson

Pythagorean numbers, by Mr. W. Crum.

Officers for 1947–48 were: President, Joan Shapper; Vice-President, Bob Henderson; Secretary, Jean Marie Baldwin; Faculty Advisor, Professor Kenneth May.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

ARMY RESERVE COMMISSIONS FOR MATHEMATICIANS AND STATISTICS SPECIALISTS

Reserve commissions in the Army of the United States are now available to qualified mathematicians and statistics specialists who meet the requirements of age, education, experience and physical condition. The commissions range from those of Second Lieutenant up to and including full Colonel, depending upon the qualifications and experience the mathematician or statistical specialist possesses in his specified field. Applicants must be at least twenty-one (21) years of age for appointment in the Reserve Corps,and applications will be considered from individuals up to fifty-five (55) years. All must be citizens of the United States. No previous military experience is required.

A bachelor's degree in mathematics or statistics, or in a field based heavily on mathematics, such as physics, engineering, economics, banking and finance, and accounting, qualifies for appointment into the Officers' Reserve Corps. Accepted applicants will be appointed in the grade of Second Lieutenant for service as mathematicians and statistics specialists in the Army Security Reserve; Medical Service Corps Reserve; and the Military Intelligence Reserve.

For appointments in grades of First Lieutenant through and including Colonel, applicants, in addition to the educational degree specified above, must have had qualifying education or progressive, professional experience in any one or any combination of the areas shown below. The Reserve sections for which

appointments in the grade of First Lieutenant and above will be made are: Army Security Reserve; Chemical Corps Reserve; Finance Department Reserve; Medical Service Corps Reserve; Military Intelligence Reserve; Quartermaster Corps Reserve; Signal Corps Reserve; and Staff and Administrative Reserve.

Six specific areas of experience may be offered by an applicant as meeting the minimum experience qualifications. These areas are: (1) Analysis; (2) Algebra; (3) Geometry; (4) Probability and Statistics; (5) Applied mathematics; (6) Other specified areas such as theory of numbers, methods of computation, combinatorial analysis, biometry, econometrics, and market research.

The above experience may be in any one or any combination of the following activities: (1) Professional practice in a field of applied mathematics; (2) Teaching; (3) Research; (4) Consulting; (5) Technical writing and editing.

Determination of the grade in which an applicant is to be commissioned is the function of the examining board which interviews the applicant, subject to final approval of The Adjutant General. The grade will be based on the total number of years of qualifying college education and/or experience according to the following scale of minimum requirements for each grade. For commission as a Second Lieutenant, the combined qualifying college education and/or experience shall amount to a minimum of four years. Qualifying college education and/or experience for commissioning as First Lieutenant shall equal a minimum of seven years; for Captain, eleven years; for Major, sixteen years; for Lieutenant Colonel, twenty-one years; and for Colonel, twenty-six years.

Full details of the program for commissioning of civilian mathematicians and statistics specialists are provided in Department of the Army Circular No. 210, dated 14 July 1948. Application forms and Circular No. 210 may be obtained from local Reserve Unit Headquarters or from Organized Reserve Unit Instructors, or by writing to Army Headquarters at New York, Baltimore, Atlanta, Chicago, Houston or San Francisco. Inquiries and requests for application forms and the circular may also be addressed to The Adjutant General, Department of the Army, Washington 25, D. C.

PERSONAL ITEMS

Professor I. S. Sokolnikoff of the University of California at Los Angeles has been awarded the Presidential Certificate of Merit for "outstanding contributions in aid of the war effort against the common enemies of the United States and its allies in World War II." Also, Professor Sokolnikoff has received a grant from Research Corporation, New York City, in support of the research project entitled "Two-dimensional problems in the theory of elasticity."

At Marshall College a Mathematics Conference of the elementary, secondary, and college teachers of West Virginia will be held on March 18–19, 1949. The purpose of the conference is to arouse interest in the improvement of the training of teachers.

The University of Akron announces the promotion of Mr. Louis Ross to an

assistant professorship and the appointment of Miss Edith E. Robbins as a part-time instructor.

University of Alabama announces the following appointments to instructorships: Miss Margaret Berger of Wagner College, Mr. Louis Jaffe of Ohio State University, Mrs. A. M. Jones of Mississippi Southern College, Mr. Hassel Palmer, and Mrs. Nell M. Reid.

University of Arizona makes the following announcements: Assistant Professor D. L. Webb has been promoted to an associate professorship; Assistant Ruth A. Fish has been promoted to an instructorship; Miss Phyllis Gray, assistant at University of Wisconsin, Miss Rose A. Grundman, graduate student at Northwestern University and Mr. D. B. Marsh, graduate student of the University of Arizona, have been appointed to instructorships; Miss Shirley A. Samuelson has been appointed to an assistantship. Professor R. F. Graesser, head of the Department of Mathematics, has been elected Chairman of the Southwestern Section of the Association and Professor E. J. Purcell has been elected Vice-Chairman of the Section.

University of California at Berkeley announces the following: Assistant Professor R. H. Sciobereti has been promoted to an associate professorship; Dr. E. W. Barankin has been promoted to an assistant professorship; Dr. T. M. Apostol, Dr. J. L. Hodges, Jr., Mr. T. A. Jeeves, Professor Michel Loeve of the University of London, Dr. S. F. Neustadter, Dr. Stephan Peters, and Miss Elizabeth L. Scott have been appointed Lecturers.

The State College of Washington announces the following appointments to instructorships: Jessie V. Allhands, Tyler Allhands, W. R. Ballard, Adalene M. Bibby, Celia E. Klotz and E. L. Salisbury.

The University of Kentucky makes the following announcements: Associate Professor Tadeusz Leser, Emory and Henry College, has been appointed to an assistant professorship; Mr. R. E. Wheeler has been appointed to an instructorship; Mr. E. C. Steele has received an appointment as graduate assistant; Miss Virginia Baskett and Mr. Wimberly Royster have been promoted to instructorships.

The University of Maine reports: Professor N. R. Bryan has retired with the title of Professor Emeritus; Mr. P. C. Rogers and Miss Joanne M. Springer have been appointed to instructorships.

The University of Massachusetts announces: Professor F. C. Moore, head of the Department of Mathematics, has retired with the title of Professor Emeritus; Assistant Professor A. E. Andersen has been appointed Professor and Head of the Department; Mr. S. I. Allen, formerly principal of the Graham-Eckes School, Palm Beach, and Mr. P. D. Ritegr, who has been Mathematics Master, The Phelps School, Malvern, Pennsylvania have been appointed to instructorships.

At the University of Miami, Mr. F. E. Adams and Mr. Ira Rosenbaum have been appointed to assistant professorships: Assistant Professor Robert Tindall has been appointed Assistant Professor of Engineering.

The University of Michigan announces the following: Professor L. C. Karpinski has retired with the title of Professor Emeritus; Associate Professor J. R. Britton of the University of Colorado has been appointed Visiting Associate Professor; Mr. R. K. Ritt, Dr. W. R. Scott, Dr. R. G. Stanton and Dr. W. R. Utz have been appointed to instructorships.

The Department of Mathematics of the University of Minnesota reports: Dr. B. B. Gelbaum and Dr. E. B. Nering of Princeton University have been appointed to assistant professorships; Dr. R. E. Graves has been promoted to an assistant professorship.

The Department of Mathematics and Mechanics of the University of Minnesota announces: Professor C. A. Herrick has retired with the title of Professor Emeritus; Associate Professors H. A. Doeringsfeld and F. E. Miller have been promoted to professorships; Assistant Professors J. C. Sanderson and R. W. Siler have been promoted to associate professorships; Assistants P. G. Kirmser and O. M. Rye have been promoted to instructorships; Mr. R. R. Johnson, Jr., Mr. L. G. Kelly and Mr. K. M. McMillin have been appointed to instructorships; Mr. G. C. Francis, Mr. A. R. Fredriksen, Mr. J. Joseph and Mr. J. Rynning have been appointed to teaching assistantships.

The University of Mississippi makes the following announcements: Professor T. A. Bickerstaff has been promoted to the chairmanship of the department; Associate Professor G. R. Trott has been promoted to a professorship; Mr. A. H. Samuels has been promoted to an assistant professorship; Mr. N. A. Childress and Mrs. Corrie D. Quarles have been appointed to instructorships. The annual meeting of the Louisiana-Mississippi Section of the Association will be held at the University of Mississippi, as a part of the centennial program of the University on April 8–9, 1949. Professor Saunders MacLane will be guest speaker. Special recognition will be given Dr. Alfred Hume, a charter member of the Association.

At the University of Missouri, Assistant Professor F. O. Duncan has retired; Assistant Professor Choy-tak Taam of Canton, China, has been appointed to an assistant professorship.

The University of Nebraska announces the promotions of Dr. Edwin Halfar and Dr. W. G. Leavitt to assistant professorships and the appointment of Miss F. Marion Clarke, formerly lecturer at the University of Southern California, to an instructorship.

The University of New Hampshire reports: Assistant Professor W. L. Kichline has been promoted to an associate professorship; Miss Elizabeth J. Fraser and Mr. Bernhart Snyder have been appointed to instructorships; Assistant Professor A. R. Harvey is now Research Fellow at California Institute of Technology.

At the University of North Carolina, Professor Archibald Henderson, formerly head of the Department of Mathematics, has retired; Dr. W. M. Whyburn, formerly president of Texas Technological College, has been appointed

Head of the Mathematics Department; Mr. I. R. Hershner, Jr., of the University of Chicago has been appointed Lecturer.

The Woman's College of the University of North Carolina announces the retirement of Professor Cornelia Strong and the appointment of Miss Lila P. Walker to an assistant professorship.

The University of North Dakota reports: Mr. S. C. Simonson has been appointed to an instructorship; Associate Professor S. L. Mason is on sabbatical leave and is studying at the University of Washingon.

The University of Oklahoma announces: Dr. S. W. Reaves has retired with the title Professor Emeritus of Mathematics; Associate Professor Dora McFarland has been promoted to a professorship; Assistant Professors Arthur Bernhart and B. S. Whitney have been promoted to associate professorships; Associate Professor A. A. Grau of the University of Alabama has been appointed to an associate professorship; Assistant Professor W. A. Catenaro of the University of South Carolina and Mr. R. B. Deal, Jr., have been appointed to instructorships.

The University of Oregon announces the following appointments: Assistant Professor F. J. Massey of the University of Maryland to an assistant professorship; Miss Vivienne Odishaw of the University of Pittsburgh and Mrs. Constance Stevens, University of Oregon, to instructorships.

The University of Pennsylvania reports the following: Associate Professor I. J. Schoenberg has been promoted to a professorship; Professor A. S. Besicovitch of Trinity College, Cambridge, England has been appointed Visiting Professor; Dr. R. D. Anderson of the University of Texas and Mr. Russell Remage, University of Delaware, have been appointed to instructorships.

At the University of Rochester, Instructors Frederick Bagemihl and Horace Komm have been promoted to assistant professorships; Mr. Arthur Danese has been appointed to an instructorship; Mr. Eugene Trabka has received an appointment as part-time instructor.

The University of Tennessee announces the following appointments: Dr. K. E. Brown, formerly at Wagner College, to an associate professorship; Mr. E. C. Campbell as lecturer; Mrs. Zoe E. Albert to an instructorship; Mr. Kenneth Carman, Mr. Nickolas Heerema, Mr. Joe Parsons and Mr. Lan Hsing Yieh to graduate assistantships; Mr. Ralph Grimble to a teaching assistantship.

The University of Toledo reports: Professor O. L. Dustheimer of Baldwin-Wallace College has been appointed Professor of Astronomy and Mathematics; Dr. C. J. Blackall, recently at the University of Detroit, has been appointed Associate Professor of Mathematics; Professor H. L. Zeiders of Midland College has been appointed to an assistant professorship.

The University of Utah makes the following announcements: Professor E. W. Pehrson has been made Professor Emeritus; Mrs. Marian Dickman has been appointed Lecturer; Associate Professor C. J. Thorne is on leave for the year 1948–49 and is at the University of California at Los Angeles.

The University of Virginia announces the following appointments to instructorships: R. R. Bernard, H. F. De Francesco, F. T. Dresser, W. H. Johnson, R. H. Kasriel, D. B. Lowdenslager, W. E. Pace, Shirley A. Rubenstein, C. M. Spaulding, and D. C. Walker.

The following appointments and promotions have been made at the University of Wisconsin: Professor Max Dehn, Black Mountain College, has been appointed Visiting Professor; Professor L. C. Young, University of Cape Town, has been appointed to a professorship; Assistant Professor O. G. Owens of the University of Nevada has been appointed to an assistant professorship; Mr. C. C. Hsiung, Michigan State College, Mr. J. B. Kelly, Massachusetts Institute of Technology, and Mr. C. S. Yih, Iowa State College, have been appointed to instructorships; Associate Professor S. C. Kleene has been promoted to a professorship; Mr. R. E. Fullerton has been promoted to an assistant professorship.

Washington University announces the following appointments to graduate assistantships: O. A. Biberstein, P. J. Ceike, G. C. Cree, John Hoelzer, A. D. Martin, R. A. Moore, M. C. Palmer, D. R. Scholz, George Schriro, and J. Isabel Stewart.

At Western Michigan College, Associate Professor C. H. Butler has been promoted to a professorship; Assistant Professor William Halnon has been promoted to an associate professorship.

The following appointments to instructorships have been made at West Virginia University: Mr. Ralph Brown, Mr. Earl Crisler, Mrs. Helen Godfrey, Miss Helen Heater, Miss Helen Pollock and Miss Laurie Wear.

Yeshiva University announces the promotions of Dr. Henry Sisman to an assistant professorship and of Mr. Daniel Block to an instructorship.

Mr. M. W. Aylor of the University of Virginia has been promoted to an assistant professorship in the Department of Engineering Mathematics.

Dr. H. G. Booker, lecturer in mathematics at Cambridge University and supervisor of applied mathematics in Christ's College, has been appointed Professor of Electrical Engineering at Cornell University.

Reverend J. C. Burke, C.S.C., of the University of Notre Dame has been promoted to an assistant professorship.

Associate Professor J. A. Clarkson of the University of Pennsylvania has been appointed Head of the Department of Mathematics at Tufts College.

Mr. R. T. Donnell, University of Tennessee, has been appointed to an instructorship at Union College, Jackson, Tennessee.

Mrs. Rosalie L. Dunham, director of mathematics at Okennegee Junior College, has received an appointment as acting instructor at the University of Tulsa.

Miss Gertrude Foster of the University of South Carolina has retired.

Professor R. E. Gaines of the University of Richmond has retired with the title of Professor Emeritus.

Assistant Professor N. C. Hunsaker of Utah State Agricultural College has been promoted to an associate professorship.

Miss Dorothy Ingram, who has been teaching at New Armijo School, New

Mexico, has been appointed to an instructorship at the University of New Mexico.

Miss Rosamond Jones, University of Wisconsin, is teaching at Ely Junior College, Ely, Minnesota.

Mr. George Lawler of the University of North Dakota has received an appointment at Coalinga Junior College, Coalinga, California.

Dr. A. O. Lindstrum, Northwestern University, has been appointed to an assistant professorship at Knox College.

Mr. Hugo Mandelbaum has been appointed to an instructorship at Wayne University.

Mr. S. S. McNeary of Drexel Institute of Technology has been promoted to an associate professorship.

Dr. L. L. Merrill, formerly consulting mathematician in the research department of Stromberg-Carlson, Rochester, has been appointed Director of Graduate Studies and Professor of Mathematics at Clarkson College of Technology.

Dr. W. H. L. Meyer, Jr. of the College of the University of Chicago has been promoted to an assistant professorship.

Mr. C. R. Newell has been appointed to an instructorship at Niagara University.

Mr. J. J. Rowland, State College of Washington, has accepted a research position in the Engineering Department of the Boeing Airplane Company, Seattle.

Mr. W. J. Scharf has been appointed to an instructorship at John Carroll University.

Dr. J. A. Schouten has been appointed to a professorship at the University of Amsterdam.

At the University of Wyoming, Associate Professor Nathan Schwid has been promoted to permanent tenure.

Associate Professor D. R. Shreve of the University of Tulsa has accepted a position as research mathematician with the Carter Oil Company, Tulsa.

Dr. Mary-Elizabeth L. Solari has been appointed Lecturer at Chelsea Polytechnic, London, England.

Dr. G. L. Tiller of the University of Kentucky has been appointed to an assistant professorship at Utica College of Syracuse University.

Mr. A. C. Baird of Hamilton, Ohio died October 13, 1948. He was a charter member of the Association.

Dr. J. T. Rorer of Temple University died August 13, 1948.

Professor Emeritus C. T. Sullivan of McGill University died September 17, 1948.

Dean R. C. Tolman of the Graduate School, California Institute of Technology, died September 5, 1948.

Associate Professor Emeritus J. W. A. Young of the University of Chicago died October 26, 1948. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Report and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following one hundred thirty-six persons have been elected to membership by the Board of Governors on applications duly certified:

DON ALKIRE, M.A. (South Dakota) Asst. Professor, Mankato State Teachers College, Mankato, Minn.

J. E. ALMAN, M.A. (Claremont) Instructor, College of Liberal Arts, Boston University, Mass.

J. S. ARONOFSKY, M.S. (Stevens Inst.) Research Engineer, Westinghouse Research Labs., Pittsburgh, Pa.

P. R. BEESACK, Student, McMaster University, Hamilton, Ontario, Canada

ALICE K. BELL, M.A. (Michigan) Asso. Professor, Fresno State College, Calif.

LEON BENSON, A.B. (New York) Instructor, Lehigh University, Bethlehem, Pa.

A. F. BENTLEY, Ph.D. (Johns Hopkins) R. 2, Paoli, Ind.

JAMES BERCOS, B.S. (Illinois) Instructor, University of Georgia, Athens, Ga.

R. R. BERNARD, M.A. (Virginia) Instructor, University of Virginia, Charlottesville, Va.

D. W. BLACKETT, M. A. (Princeton) Graduate Student, Princeton University, N. J.

A. P. BOBLETT, B.S. (Western Reserve) Instructor, Kent State University, Ohio

A. B. BOGGS, M.A. (Michigan) Asst. Professor, Michigan College of Mining & Technology, Houghton, Mich.

W. L. BOYD, B.S. (Oklahoma) Graduate Asst., Oklahoma University, Norman, Okla.

HELEN J. BRADLEY, B.A. (Rosemont) Instructor, University of Tennessee, Knoxville, Tenn.

MILDRED J. BRANNON, M.A. (Illinois) Asst. Professor, Michigan State Normal College, Ypsilanti, Mich.

W. C. BROWN, B.S. (Colorado A & M) Graduate Asst., University of Oklahoma, Norman, Okla.

F. E. BRUNTZ, M.A. (Colorado State College) Asst. Professor, University of Denver, Colo.

REV. V. J. BURKE, T.O.R., M.A. (St. Francis) The College of Steubenville, Ohio

MRS. E. R. BUTLER, B.S. (Iowa State) Instructor, Iowa State College, Ames, Iowa

W. H. CLATWORTHY, M.A. (Kentucky) Instructor, Wayne University, Detroit, Mich.

L. A. COLQUITT, M.A. (Ohio State) Asst. Professor, Texas Christian University, Fort Worth, Tex.

ESTHER A. COMPTON, M.A. (Indiana) Instructor, Cumberland College, Williamsburg, Ky.

W. F. CORNELL, M.A. (Ohio State) Asso. Professor, Bowling Green State University, Ohio

JANE S. CRONIN, Ph.D. (Michigan) ONR Fellow, Princeton University, N. J.

A. E. DANESE, A.M. (Harvard) Instructor, University of Rochester, N. Y.

H. A. DOERINGSFELD, C.E. (Wisconsin) Professor, University of Minnesota, Minneapolis, Minn.

F. T. DRESSER, B.S. (Virginia Military Inst.) Part-time Instructor, University of Virginia, Charlottesville, Va.

MRS. ESTHER S. DUNKELBERGER, M.Litt. (Pittsburgh) Instructor, Duquesne University, Pittsburgh, Pa.

O. L. DUSTHEIMER, Ph.D. (Michigan) 4935 East 71st St., Cleveland, Ohio

REV. A. J. EIARDI, S.J., M.S. (Boston College) Asso. Professor, Boston College, Chestnut Hill, Mass.

W. R. EIKELBERGER, M.S. (Fort Hays) Asst. Professor, University of Denver, Colo.

J. L. EMMERT, M.Ed. (Pittsburgh) Asst. Professor, The College of Steubenville, Ohio

J. L. ERICKSEN, B.S. (Washington) Teaching Fellow, Oregon State College, Corvallis, Ore.

G. W. FAIRCHILD, Student, Arizona State College, Tempe, Ariz.

RUTH A. FISH, M.S. (Arizona) Instructor, University of Arizona, Tucson, Ariz.

PEARL L. FORD, A.M. (Michigan) Asst. Professor, Western Michigan College of Education, Kalamazoo, Mich.

W. B. FULKS, M.A. (Arkansas) Instructor, University of Minnesota, Minneapolis, Minn.

J. R. GARRETT, A.M. (Duke) Visiting Instructor, Duke University, Durham, N. C.

R. E. GETTIG, M.S. (Pittsburgh) Instructor, University of Pittsburgh, Pa.

E. W. GOINGS, M.S. (Michigan) Asst. Professor, Michigan State Normal College, Ypsilanti, Mich.

D. A. GORSLINE, B.S. (Hendrix) Graduate Asst., University of Oklahoma, Norman, Okla.

H. E. GOULD, M.A. (New York) Instructor, Rhode Island State College, Kingston, R. I.

D. O. GRAY, M.B.A. (Houston) Instructor, University of Houston, Tex.

L. D. GREGORY, B.A. (Oklahoma) Instructor, University of Oklahoma, Norman, Okla.

MRS. BLANCHE B. GROVER, M.A. (Texas) Asst. Professor, University of Houston, Tex.

ROSE A. GRUNDMAN, M.S. (Northwestern) Instructor, University of Airzona, Tucson, Ariz.

FRANKLIN HAIMO, Ph.D. (Harvard) Asst. Professor, Washington University, St. Louis, Mo.

E. E. HASKINS, Ph.D. (Boston) Asso. Professor, Fenn College, Cleveland, Ohio

MARQUERITE Z. HEDBERG, Ph.D. (Missouri) Adjunct Professor, University of South Carolina, Columbia, S.C.

M. S. HENDRICKSON, Ph.D. (Ohio State) Asso. Professor, University of New Mexico, Albuquerque, N. M.

RUTH E. HENNING, M.A. (Illinois) Instructor, Virginia Junior College, Minn.

REV. J. C. HILFERTY, B.A. (St. Charles) Instructor, St. Thomas More High School, Philadelphia, Pa.

J. R. HODGES, B.S. (Tulane) Teaching Fellow, Tulane University, New Orleans, La.

M. R. HORAK, M.A. (Michigan) Vice Principal, Randolph Central School, N. Y.

C. C. HSIUNG, Ph.D. (Michigan State) Instructor, University of Wisconsin, Madison, Wis.

K. C. HSU, B.S. (Chiao-Tung University) Graduate Student, University of Kansas, Lawrence, Kan.

S. L. HULL, M.S. (Iowa) Instructor, University of Arkansas, Fayetteville, Ark.

W. H. JOHNSON, B.A. (Virginia) Instructor, University of Virginia, Charlottesville, Va.

F. E. JUSTIS, M.S. (West Virginia) Asst. Professor, Geneva College, Beaver Falls, Pa.

W. C. KALINOWSKI, Ph.D. (Saint Louis) Asst. Professor, St. John's University, Collegeville, Minn.

GILBERT KASKEY, M.A. (Delaware) Instructor, University of Delaware, Newark, Del.

R. H. KASRIEL, B.S. (Tampa) Graduate Student, University of Tampa, Fla.

B. C. KENNY, B.S, (Bethany) Graduate Asst., Lehigh University, Bethlehem, Pa.

MAX KRAMER, M.A. (Columbia) Instructor, University of Illinois, Champaign, Ill.

CHARLES KURLAND, B.A. (Buffalo) Student, University of Buffalo, N. Y.

ROBERT LANKTON, M.A. (Wayne) Asst. Professor, Iowa State Teachers College, Cedar Falls, Iowa

M. L. LAWSON, M.S. (Oklahoma A & M) Instructor, University of Oklahoma, Norman, Okla.

MARY B. LIEBERKNECHT, B.S. (Iowa State) Instructor, Iowa State College, Ames, Iowa

R. K. LONGLEY, B.A. (Buffalo) Head of Dept., Canisteo Central School, N. Y.

HUGO MANDELBAUM, Dr. rer. nat. (Hamburg) Instructor, Wayne University, Detroit, Mich.

MARY O. MARQUARDT, M.A. (Illinois) Asst. Professor, Industrial Labor Relations School, Cornell University, Ithaca, N. Y.

D. C. B. MARSH, JR., M.S. (Arizona) Instructor, University of Arizona, Tucson, Ariz.

J. E. MARTIN, M.S. (Vanderbilt) Asst. Professor, Davidson College, N. C.

M. G. MCCUAN, M.A. (West Texas State) Asso. Professor, Amarillo College, Tex.

PAUL MEIER, M.A. (Princeton) Asst. Professor, Lehigh University, Bethlehem, Pa.

D. M. MESNER, B.A. (Nebraska) Graduate Student, Northwestern University, Evanston, Ill.

J. W. METTLER, M.A. (Bucknell) Instructor, Lehigh University, Bethlehem, Pa.

P. T. MIELKE, M.Sc. (Brown) Industrial Research Fellow, Purdue University, West Lafayette, Ind.

ROSEMARY MILKOVITCH, M.A. (Montana) Instructor, Bemidji State Teachers College, Minn.

P. D. MINTON, M.Sc. (Southern Methodist) Instructor, University of North Carolina, Chapel Hill, N. C.

LEONID MIRSKY, M.Sc. (London) Lecturer, University of Sheffield, England

R. D. MORRISON, M.S. (Oklahoma A & M) Asst. Professor, Oklahoma A & M College, Stillwater, Okla.

RUTH E. MUSE, M.A. (Minnesota) Instructor, Lincoln University, Jefferson City, Mo.

SAM NAIDITCH, Ph.D. (Calif. Inst. of Tech.) Asst. Professor, Duquesne University, Pittsburgh, Pa.

A. C. NELSON, JR., M.S. (Delaware) Instructor, University of Delaware, Newark, Del.

MARY NELSON, M.S. (State Univ. of Iowa) Asst. Professor, Utah State Agricultural College, Logan, Utah.

PAUL NESBEDA, Ph.D. (Pisa) Instructor, Catholic University, Washington, D.C.

COL. C. P. NICHOLAS, B.S. (U.S.M.A.) Professor, U. S. Military Academy, West Point, N. Y.

DOROTHY M. OEHMKE, B.S. (Detroit) Graduate Student, University of Detroit, Mich.

R. H. OEHMKE, B.S. (Michigan) Instructor, University of Detroit, Mich.

R. R. OTTER, Ph.D. (Indiana) Asst. Professor, University of Notre Dame, Ind.

O. E. OVERN, Ph.D. (Columbia) State Teachers College, Milwaukee, Wis.

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JEAN M. WYRE, M.A. (Oberlin) Asst. Instructor, Berea College, Ky.

R. P. YIELDING, B.S. (U.S.M.A.) Instructor, University of Oklahoma, Norman, Okla.

J. A. ZILBER, A.M. (Harvard) Lecturer, Columbia University, N. Y.

SPRING MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The spring meeting of the Section was held at the United States Naval Academy, Annapolis, Maryland, on May 8, 1948, with Professor J. B. Scarborough presiding for the Chairman, Professor E. J. Finan, who was unable to attend.

There were one-hundred and fourteen persons attending the meeting including the following sixty-three members of the Association: H. C. Ayres, N. H. Ball, Archie Blake, J. A. Buikstra, R. S. Burington, H. C. Campaigne, S. R. Clements, Abraham Cohen, W. H. Cowles, G. F. Cramer, A. E. Currier, C. H. Denbow, J. A. Duerksen, Anselm Fisher, M. K. Fort, Jr., B. C. Getchell, Michael Goldberg, R. A. Good, J. R. Gorman, E. S. Grable, E. C. Gras, D. W. Hall, J. R. Hammond, B. I. Hart, G. C. Helme, M. A. Hyman, S. B. Jackson, W. Jennings, L. M. Kells, H. L. Kinsolving, W. D. Lambert, O. E. Lancaster, A. E. Landry, D. C. Lewis, G. A. Lyle, M. H. Martin, E. S. Mayer, C. V. McCamman, K. F. McLaughlin, Emanuel Mehr, Joseph Milkman, T. W. Moore, W. K. Morrill, W. H. Norris, J. F. Paydon, J. W. Popow, O. J. Ramler, C. H. Rawlins, Jr., R. W. Rector, J. N. Rice, R. E. Root, J. B. Scarborough, E. D. Schell, V. G. Schult, V. E. Spencer, H. C. Stilwell, O. M. Thomas, P. D. Thomas, J. A. Tierney, M. M. Torrey, W. R. Utz, C. H. Wheeler, H. S. Wilson.

The following papers were presented at the morning session:

1. *Some new algorithms*, by Captain N. A. Draim, United States Navy, introduced by the Secretary.

Algorithms for the extraction of square roots, the determination of the factors of composite numbers, and the solution of equations of the form $n^2+2an+b=\phi^2$ were presented.

2. *Recent advances in the Brun approach to the Goldbach problem*, by Dr. R. C. Rand, United States Naval Academy.

A summary of Buchstab's recent contributions to the Goldbach problem was presented. It was based on lectures of R. D. James at the Canadian Mathematical Congress in 1948 at Toronto. If $F(x, x^{1/\alpha})$ denotes the number of partitions of an even integer x into two positive integers both relatively prime to all primes less than $x^{1/\alpha}$, the Goldbach hypothesis will be established once it is shown that $F>0$ for $\alpha=2$. With the Bruin sieve it can be shown that

$$F(x, x^{1/\alpha}) = \lambda(\alpha)\frac{x}{\log^2 x} + O\left(\frac{x}{\log^3 x}\right)$$

where $\lambda>0$ for $\alpha\geqq 8$. Buchstab has shown that $\lambda(5)>0$, from which it follows that every sufficiently large even integer is the sum of two integers each containing at most four factors.

3. *On the number of solutions of the equation $k\phi(n)=n-1$*, by Dr. G. F. Kramer, and Dr. H. H. Campaigne, of the Navy Department.

According to D. H. Lehmer, if the equation $k\phi(n)=n-1$ (where $k>1$ is an integer and $\phi(n)$ is Euler's totient function) has a solution n which is composite, n must be of the form $n=p_1p_2 \cdots p_r$, where the p_i are distinct odd primes and $r\geqq 7$. The question of the existence or non-existence of composite solutions of this equation was left unanswered, but it was shown that for a given integer $r>1$, there cannot be more than a finite number of composite solutions n having exactly r distinct prime factors when k is a positive integer.

4. *Roots of numerical equations by number sequences*, by Professor John Tyler and J. P. Hoyt of the United States Naval Academy, introduced by the Secretary.

The sequence $f(0), f(1), f(2), \cdots$ was used to find the rational roots and roots of the form $(a\pm\sqrt{b})/c$ of the equation

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$$

in which the coefficients a_i are integers. To each rational root there corresponds a subsequence constructed from the factors of $f(0), f(1), f(2), \cdots$ whose first order differences are constant. Corresponding to a root of the form $(a\pm\sqrt{b})/c$ there is a subsequence whose second differences are constant.

5. *The veteran and the accelerated course in mathematics*, by W. H. Norris of the Veteran's Center, Norfolk, Virginia.

A survey of the problems met in accelerated courses for veterans was followed by an appraisal of accelerated courses in the light of modern pedagogical thought.

6. *American mathematics turns useful*, by M. H. Slud of Catholic University.

The speaker dealt with the difficulties encountered by the engineer in attempting to reduce the problems in metallurgy to mathematical form. The various modern theories of brittleness, hardness, elasticity, and plasticity were summarized and compared.

M. H. MARTIN, *Secretary*

MAY MEETING OF THE SOUTHWESTERN SECTION

The Southwestern Section of the Mathematical Association of America met concurrently with the Southwestern Division of the A. A. A. S. on the campus of New Mexico Highlands University, Las Vegas, New Mexico, on Monday, May 3, 1948. Professor R. F. Graesser of the University of Arizona, Chairman of the Section, presided. Twenty-three persons were in attendance.

Officers elected for 1948–1949 were: Chairman, Earl Walden, of New Mexico College of Agriculture and Mechanical Arts; Vice-Chairman, E. J. Purcell, University of Arizona. Professor A. W. Boldyreff, University of New Mexico, was chosen as visiting lecturer for the year.

The program consisted of the following papers:

1. *Abstract isomorphism*, by Duane Studley, Colorado Springs, Colorado.

The relation of isomorphism was investigated at some length, and after consideration of the usual uses of the term certain generalizations were introduced. An abstract isomorphism was defined so as to contain the usual sense on specification. Some of the aspects of this consideration are new, and therefore some positive contribution was made.

2. *On Fermat's congruence* $A^{n-1}=1$ (mod. n) *for composite values of* n, by A. W. Boldyreff, University of New Mexico, Albuquerque, New Mexico.

3. *Extended analytic geometry applied to simultaneous equations*, by R. S. Underwood, Texas Technological College, Lubbock, Texas.

Given two simultaneous equations in three variables, a single equation in the three variables is found which is satisfied by all common solutions of the original pair, and whose locus is a curve coinciding with or containing the intersecting curve on a 3-axes plane. From this the common integral solutions may be found by a method similar to the one for simultaneous linear equations which was presented to this Section a year ago.

4. *Modern mathematics for everybody*, by Duane Studley, Colorado Springs, Colorado.

Part Six of the little book *Modern Mathematics for Everybody* entitled "Other Types of Set" was read. After opening remarks on the set concept, there were sections dealing with the topics domain, field, ring, algebra, and group. The simple exposition contained definitions and examples (those for groups being given at greatest length), and included the concepts of factor group and lattice. Cosets were clearly defined and the material was developed to the Galois level.

5. *Inverse variation problem*, by Lincoln La Paz, University of New Mexico.

This paper consisted of a systematic formulation and analysis of inverse variation problems, other than those connected with the Hamilton-Caratheodory function, by means of a generalization of the curvilinear congruence theory of differential geometry.

B. D. Roberts, *Secretary*

MAY MEETING OF THE NEBRASKA SECTION

The twenty-fourth meeting of the Nebraska Section of the Mathematical Association of America was held at the University of Nebraska in Lincoln, on Saturday, May 1, 1948. Professor H. M. Cox, Chairman of the Section, presided.

The following eleven members of the Association were present: W. C.

Brenke, C. C. Camp, Helen E. Clarkson, H. M. Cox, J. M. Earl, C. B. Gass, E. H. Hadlock, C. A. Huck, C. R. Perisho, H. L. Rice, and Lulu L. Runge.

At the business meeting the following officers were elected: Chairman, H. L. Rice, University of Omaha; Vice-Chairman, H. M. Cox, University of Nebraska; Secretary, Lulu L. Runge, University of Nebraska; Alternate, W. G. Leavitt, University of Nebraska; Governor, C. C. Camp.

The following papers were presented:

1. *On the decomposition of a space*, by Dr. Edwin Halfar, University of Nebraska, introduced by H. M. Cox.

In a topological space L defined by a derived set operator d satisfying $d(A+B)=dA+dB$, $D^2A=d(dA)\subset dA$, $dO=O$, such that for each $A\subset L$, there is a $DA\subset L$, one defines a maximal self-dense subset of A as the largest subset B of A satisfying $B\subset dB$. It is shown that the frontier of a set A has in general a maximal self-dense subset, and that this subset is decomposable into a sum of orthogonal self-dense subsets arising from the derived sets formed by operating with d on the isolated parts of the successive derived sets of the frontier of A.

2. *On the matrices whose elements are holomorphic functions*, by W. G. Leavitt, University of Nebraska.

This speaker considered similarity transformations (that is, transformations of type $T^{-1}AT$) of a matrix $A(z)$ each of whose elements is a function of the complex variable z. The results apply to a region R of the z-plane such that, in and on the boundary of R, each element of $A(z)$ and each of the roots of its characteristic equation is a holomorphic function. It was shown that there exists a matrix $T(z)$ whose determinant is non-vanishing in R and such that $T^{-1}AT$ is in a normal form with zeros below the main diagonal. It was first shown that the set of all functions holomorphic over R satisfies all the hypotheses of a principal ideal ring. It was therefore possible to use in the course of the proof certain abstract algebraic theorems known for such rings.

3. *On the deductive method*, by L. A. Rife, Philosophy Department, University of Nebraska, introduced by Professor Cox.

It was noted that the deductive method is a powerful tool in the fields of natural science as well as in philosophy. It is sometimes referred to as the methodology of deductive sciences or of mathematics. The value of a clear and concise understanding of the method was emphasized. Reference was made to *The Basis and Structure of Knowledge*, by W. H. Werkmeister. A lucid illustration, using only two axioms and other necessary principles for a model and interpretation of a deductive theory, was taken from Chapter VI of *Introduction to Logic*, by Alfred Tarski.

4. *Mathematics of finance*, by Professor C. A. Huck, Nebraska State Teachers College, Peru.

Professor Huck demonstrated the importance of the subject of mathematics of finance by illustrating some of its applications.

5. *The accuracy of computed results*, by E. Z. Palmer, Bureau of Business Research, University of Nebraska, introduced by Professor Cox.

This paper dealt with the usual case in which the numbers used in computation are inaccurate, the exact values being equally probable over the range between definite limits. The probability functions for the results of the addition, subtraction, multiplication, and division of two numbers were given, also those for the powers and roots of a number. The formulas for the mean and the standard deviation, and in some cases a general formula for the moments of these distributions, were given. It was pointed out that in the results of division, computation of powers, and root

extraction, the mean of all possible results was not identical with the central or usual integral result.

A suggestion for the proper statement of computed results was made, namely that the unit of accuracy of such a result should be taken as the integral power of ten which is immediately less than three times the standard deviation of the probability function of that result.

6. *On p-adic numbers*, by W. T. Lenser, University of Nebraska, introduced by Professor Cox.

This was an expository talk on the field of p-adic numbers as originally developed by Hensel, with particular reference to order relations and the addition and multiplication of certain complex numbers in p-adic form.

7. *On an experiment in teaching essentials of mathematics to deficient students*, by Dr. Edwin Halfar, University of Nebraska, introduced by Professor Cox.

8. *Practical applications of mathematics of certain aspects of naval activities*, Lieut. Commander G. E. Peddicord, USN, University of Nebraska.

Commander Peddicord discussed the problem of hitting an air target, the most difficult of all fire-control problems. The computer used to calculate the corrections for this type of problems does all of its calculating by mechanical basic mechanisms. It transmits and receives most of its signals electrically.

An air target position is established by three coördinates, namely, range, bearing and elevation. The computer must generate continuous values of range as a part of its many calculations. Ship and target motion are resolved into vectors in and across the line of sight by component solvers. The range component, in the line of sight, is due to ship motion and target horizontal and vertical motion. The range components are added in differentials and the sum multiplied by increments of time in a range or disk-type integrator. The product is then added to initial range in another differential, and thus continuous present range to the target is computed in the basic mechanisms of the computer.

9. *Exhibit of manuscript arithmetic handwritten in 1848 by James Riley Cox*, by H. M. Cox, Bureau of Instructional Research, University of Nebraska.

10. *Demonstration of punched card tabulating machines*, by H. M. Cox, University of Nebraska.

LULU L. RUNGE, *Secretary*

MAY MEETING OF THE UPPER NEW YORK STATE SECTION

The annual meeting of the Upper New York State Section of the Mathematical Association of America was held at Union College, Schenectady, New York, on Saturday, May 1, 1948. The Chairman of the Section, Professor D. S. Morse of Union College, presided at the morning session; the Vice-Chairman, Professor E. B. Allen of Rensselaer Polytechnic Institute, presided at the afternoon session. At the conclusion of the afternoon session tea was served in the Faculty Lounge of Hale House.

About one hundred and fifteen persons were present, including the following sixty-three members of the Association: E. B. Allen, H. T. W. Aude, G. B. Banks, R. A. Beaver, W. W. Bessell, J. S. Biggerstaff, Harry Birchenough, W. J. Bruns, F. J. H. Burkett, K. A. Bush, S. S. Cairns, I. S. Carroll, W. B. Carver, F. F. Decker, F. J. Downie, W. C. G. Fraser, H. M. Gehman, B. H. Gere, Lillian Gough, J. B. Freeley, W. L. Greene, Lucille Hetzelt, H. K. Holt, M. E. Jenkins,

A. W. Jones, David Kotler, Reverend B. J. Kuhn, F. W. Lane, Caroline Lester, J. V. Limpert, Ingo Maddaus, C. T. Male, Dis Maly, June McArtney, Janet McDonald, Harriet Montague, Mabel Montgomery, D. S. Morse, C. W. Munshower, D. J. Myatt, E. E. Nash, Abba Newton, Sister Noel Marie, E. F. Ormsby, B. C. Patterson, Hillel Poritsky, C. E. Rhodes, Vera Sanford, Edith Schneckenburger, J. E. Snover, A. D. Snyder, Ellen Stokes, Ruth Stokes, Mary Suffa, Bryant Tuckerman, Nura Turner, A. K. Waltz, J. F. Wardwell, W. M. Warnock, Mary E. Williams, A. M. Wootton, Frances Wright, W. C. Yates.

The following officers were elected: Chairman, E. B. Allen, Rensselaer Polytechnic Institute; Vice-Chairman, W. H. Durfee, Hobart College; Secretary, C. W. Munshower, Colgate University. The 1949 meeting will be held at the University of Buffalo in the spring of 1949.

The following papers were presented:

1. *Auxiliary tricks which make mathematical notions stick*, by Reverend Benjamin Kuhn, Siena College.

The speaker made suggestions with respect to aiding the memories of mathematics students.

2. *Complex geometry*, by Professor B. C. Patterson, Hamilton College

Observations and comments were made on the analytic study of geometries in the complex plane, with particular reference to the representation of points, the representation of curves and other varieties, and the algebraic and differential invariant theories.

3. *Non-planar graphs*, by Dr. Bryant Tuckerman, Cornell University.

Following introductory remarks about non-planar graphs, several possible applications with respect to electrical circuits were indicated.

4. *The history of topics in analytic geometry*, by Professor Vera Sanford, Oneonta State Teachers College.

The speaker commented upon some of the outstanding contributions to analytic geometry commencing with Descartes and Fermat.

5. *Some models useful in the teaching of high school and college mathematics*, by Professor Ruth Stokes, Syracuse University.

Professor Stokes exhibited at least one model illustrating some figure from almost every branch of mathematics, and offered suggestions for the use of such models in teaching. In particular, she exhibited models constructed of plastic materials to illustrate propositions in solid geometry, analytics, and the calculus; a double cone string model for use with a projecting lantern and slides for showing the conic sections; Klein's bottle made of glass, illustrating a one-sided surface; and, for illustrating the fourth and higher dimensions, she showed stick models, of balsa wood, of hyperpyramids and hypercubes.

6. *Oriented elements in analytic geometry*, by Professor W. J. Bruns, Syracuse University.

It was contended that even in teaching the very elements of analytic geometry, three details should not be neglected: (1) any true geometric property should be presented as an invariant under displacements, or, what is the same, under coordinate transformations; (2) only purely analytic processes should be permissible; (3) any proof of a formula or a theorem should be independent of the accidental position of the parts of a figure and of the position of the figure relative

to the coordinate system. These demands are frequently violated in our textbooks. One way to avoid such shortcomings is to use oriented elements such as the oriented line or "spear," the oriented plane, the oriented angle between non-oriented lines as well as between oriented ones, and the oriented polygon. Several illustrations were given by the speaker, who demonstrated in detail how difficulties encountered in the usual treatment of the normal form of the straight line vanished when oriented elements were introduced.

7. *Monotonic transformations*, by Professor Edith Schneckenburger, University of Buffalo.

Topological properties of monotonic transformations were discussed. Emphasis was put upon invariant properties and conditions resulting in topological equivalence. Some suggestions were made regarding the use of this transformation in other geometries.

8. *Applied mathematics*, by Dr. Hillel Poritsky, General Electric Company.

This paper was concerned with the mathematics which is found useful in an industrial concern. A general discussion of applied versus pure mathematics was given, typical problems in various fields of applied mathematics were presented, and methods and tools used in their solution were briefly discussed.

C. W. MUNSHOWER, *Secretary*

MAY MEETING OF THE WISCONSIN SECTION

The sixteenth annual spring meeting of the Wisconsin Section of the Mathematical Association of America was held at Beloit College, Beloit, Wisconsin, on Saturday, May 8, 1948. Professor R. C. Huffer, Chairman of the Section, presided at the morning and afternoon sessions.

There were seventy-three in attendance, including the following thirty members of the Association: K. J. Arnold, R. H. Bardell, Leon Battig, L. J. Berner, A. C. Berry, W. W. Bigelow, R. H. Bing, H. J. Cohen, B. H. Colvin, H. H. Conwell, H. P. Evans, Harold Glander, E. G. Harrell, W. W. Hart, R. C. Huffer, W. R. Jarmain, J. F. Kenney, R. E. Langer, Sister Mary Felice, J. R. Mayor A. C. Moeller, H. P. Pettit, O. W. Rechard, R. D. Specht, Abraham Spitzbart, P. L. Trump, E. C. Varnum, J. I. Vass, R. D. Wagner, and Louise A. Wolf.

At the business meeting held during the afternoon session the following officers were elected for the coming year; Chairman, A. C. Berry, Lawrence College; Secretary, Louise A. Wolf, University of Wisconsin in Milwaukee; Program Committee, Sister Mary Felice, Mount Mary College, Elli Otteson, Eau Claire High School, R. D. Specht, University of Wisconsin. It was announced that R. H. Bardell, University of Wisconsin in Milwaukee, had been elected Sectional Governor from Wisconsin.

The afternoon program included reports by representatives of the various educational institutions on the status of mathematical education in Wisconsin. This was followed by the passing of a resolution that the Chairman appoint a committee to be charged with the duty of seeing to the distribution of the pamphlet on guidance throughout the schools of the state.

The following papers were presented:

1. *Fourier series*, by Professor R. E. Langer, University of Wisconsin.

The paper purported to review the growth and development of the ideas which are the nucleus of the theory of the Fourier series, and to project this development back upon the historical and biographical environment in which it took place. Beginning with the analysis of the vibrating elastic string, the contrasting approaches to the problem, and the conflicting ideas, assumptions, and implications which culminated in the D'Alembert-Euler-Bernoulli controversy (about 1750) were set forth. Euler's determination of the coefficients (about 1775) was explained, and the work of Fourier (soon after 1800) was discussed, both as to its formal character, and as to its novel interpretations and revolutionary consequences.

In conclusion, a brief outline of developments which owed their inceptions to the Fourier series and which proved to be of fundamental significance in the shaping of analysis generally was given. This included references to Dirichlet's distinction between the absolute and conditional convergence of an infinite series, the clarification of the notion of a function, the discovery of uniform convergence, the definition of the Riemann integral, the discovery of the continuous non-differentiable curve, Cantor's conceptions of the theory of point sets and of trans-finite numbers, the theory of integral equations, the theory of differential boundary value problems, etc.

2. *The use of nomograms in industry*, by E. C. Varnum, Mathematician, Barber-Colman Company, Rockford, Illinois.

After a brief review of nomographic theory based on determinantal methods, several useful nomograms constructed in the development department of a large industrial plant were shown, along with their applications to electrical and mechanical problems.

LOUISE A. WOLF, *Secretary*

CALENDAR OF FUTURE MEETINGS

Joint Meeting with American Society for Engineering Education, Troy, New York, June 20–21, 1949.

Thirty-first Summer Meeting, Boulder, Colorado, August 29–30, 1949.

Thirty-third Annual Meeting, New York City, December 30, 1949.

ALLEGHENY MOUNTAIN, West Virginia University, Morgantown, May 7, 1949
ILLINOIS, Bradley University, Peoria, May 13–14, 1949.
INDIANA, University of Notre Dame, May 7, 1949
IOWA, Drake University, Des Moines, April 15–16, 1949
KANSAS, Manhattan, April 2, 1949
KENTUCKY
LOUISIANA-MISSISSIPPI, University of Mississippi, Oxford, April 8–9, 1949
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, University of Virginia, Charlottesville, Spring, 1949
METROPOLITAN NEW YORK, Brooklyn College, April 9, 1949
MICHIGAN, Wayne University, Detroit, April 2, 1949
MINNESOTA, Gustavus Adolphus College, St, Peter, May 7, 1949
MISSOURI, University of Missouri, Columbia April 9, 1949
NEBRASKA, Lincoln, May, 1949
NORTHERN CALIFORNIA
OHIO, Ohio State University, Columbus, April 2, 1949
OKLAHOMA
PACIFIC NORTHWEST, Oregon State College, Corvallis, March 25–26, 1949
PHILADELPHIA, Haverford College, November 26, 1949
ROCKY MOUNTAIN, Colorado School of Mines, Golden, April, 1949
SOUTHEASTERN, University of Alabama, University, March 18–19, 1949
SOUTHERN CALIFORNIA, John Muir Junior College, Pasadena, March 12, 1949
SOUTHWESTERN
TEXAS, Denton, Spring, 1949
UPPER NEW YORK STATE, University of Buffalo, April 30, 1949
WISCONSIN, Lawrence College, Appleton, May 14, 1949

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THE STORY OF THE BINOMIAL THEOREM

J. L. COOLIDGE, Harvard University

1. The early period. The Binomial Theorem, familiar at least in its elementary aspects to every student of algebra, has a long and reasonably plain history. Most people associate it vaguely in their minds with the name of Newton; he either invented it or it was carved on his tomb. In some way or other it was his theorem. Well, as a matter of fact it wasn't, although his work did mark an important advance in the general theory.

We find the first trace of the Binomial Theorem in Euclid II, 4, "If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle of the segments." If the segments are a and b this means in algebraic language

$$(a+b)^2 = a^2 + b^2 + 2ab. \tag{1}$$

The corresponding formula for the square of a difference is found in Euclid II, 7, "If a straight line be cut at random, the square on the whole and that on one of the segments both together, are equal to twice the rectangle contained by the whole and said segment, and the square on the remaining segment."

Here if a represents the whole, and b the first segment, we have

$$a^2 + b^2 = 2ab + (a-b)^2. \tag{2}$$

It would have been perfectly easy for Euclid to go ahead and prove the formula for the cube of a binomial, but that would have broken the thread of the argument. In Books II and X he was prodigiously interested in the squares of binomials, any generalization of these does not seem to have interested him at all. The modern tendency to generalize as far as possible, and stretch each theorem to its most general form, was quite foreign to the thinking of the Greeks in mathematics; clearness and precision were the sovereign qualities which were always sought.

We find a wider mathematical curiosity in Diophantus who cubed various binomials, especially $(n-1)$. Whether he had a general formula, or multiplied out each time is not clear.

It is a curious fact that the first use, beyond Euclid's, for finding binomial power formulae, was to discover the approximate values of roots. We have a significant remark in the commentary of Eutocius on Archimedes' essay on the measurement of the circle:

"Quo modo adpropinquando radix quadratadati numeri invenienda est, dictum est ab Herone in Metricis a Pappo, Theone, compluribus aliis, qui magnum Syntaxin Claudii Ptolemi interpretati sunt" [1].

This suggests a search in Ptolemy's Syntaxis. I have failed to find the passage. Tannery assures us that Pappus followed the general method of Hero of Alexandria [2]. Hero's method is simplicity itself. If we wish to find an ap-

proximation to $\sqrt{A}$, and a_1 is a first value, a closer one will be

$$a_2 = \frac{1}{2}\left[a_1 + \frac{A}{a_1}\right]. \tag{3}$$

As a matter of fact, this is merely a special case of a famous method of approximating to a simple root of any function, which we associate with the name of Newton, for if a_1 is an approximation to a root of $f(x) = x^2 - A$ a better approximation is

$$a_2 = a_1 - \frac{f(a_1)}{f'(a_1)}\cdot$$

We find something much closer to our familiar method of finding square roots in the work of Theon of Alexandria, who uses our technique of adding to a_1 our correction $(A - a^2)/2a_1$ [3]. Of course it is a question merely of order of procedure, for if we add this correction we have Hero's Formula (3).

We pass to cube roots. Heath says on p. 63, "In no extant Greek writer do we find any description of the method of finding cube roots." If we date Theon about A.D. 390 we have to wait more than 100 years for the Hindu Aryabhata; there are various translations of his Aryabhatiya; I follow that of Datta and Singh, p. 174 [4]:

"Divide the second aghana place by thrice the square of the cube root; subtract from the first aghana place the square of the quotient multiplied by thrice the preceding (cube root) and (subtract) the cube (of the quotient) from the aghana place. The quotient put down at the next place (in the line of the root) gives the root."

I think that this shows clearly enough that Aryabhata was familiar with the binomial formula for a cube. Whether the Hindus had the curiosity to raise binomials to higher powers or not I can not say; a power higher than the third may have appeared to them practically useless. Yet someone must have seen the importance of such matters as may be judged from the following quotation, which is highly significant for the whole purpose of this paper. The writer is Omar Khayyam, and in speaking of a work of his own, now most unfortunately lost, he writes [5]:

"Les Indiens possèdent des méthodes pour trouver les côtés des carrés et des cubes. J'ai composé un ouvrage sur la démonstration de l'exactitude de ces méthodes, et j'ai prouvé qu'elles conduisent, en effet, à l'objet cherché. J'ai en outre augmenté les espèces, c'est à dire j'ai enseigné à trouver les côtés du carré-carre, du cubo-cube, du quadratro cube, à une étendue quelconque, ce qu'on n'avait pas fait précédément. Les démonstrations que j'ai donné à cette occasion, ne sont que des démonstrations arithmétiques, fondées sur les parties arithmétiques des Eléments d'Euclide."

This is an extremely interesting paragraph. Tropfke expresses his opinion in no uncertain terms:

"Die letzte Bemerkung kan man offenbar nur auf Benutzung der binomschen Entwickelung für beliebig hohe Exponenten deuten, wodurch dann Alkhayammi als Entdecker des Binomialtheorems für ganzzählige Exponenten anzusehen wäre" [6].

This seems to me eminently true and important, provided we take it literally. It all depends on "une étendue quelconque." If he could find any root by arithmetical means, he presumably used the binomial theorem, but the only actual roots he mentions are quartic, sextic, and ninth, each of these could be found by repeating the processes he knew for quadratic and cube roots. When we reflect on how inferior was the mathematical notation of his time, I think there is some doubt whether he could really extract, let us say, a seventh root.

A cautious note is sounded in a very recent discussion:

"Man hat die den modernen Mathematiker naheliegende Vermutung ausgesprochen; das Omar Haiyami den binomschen Entwickelung für beliebig hohe Exponenten etwa in der Weise arbeitete wie wir im 16 Jahrhundert bei Apian, Stifel, und andere Mathematiker der Renaissance beobachten" [7].

Luckey does not definitely pronounce on the point in question, but he seems inclined to the view that Omar could only find roots that were based on the quadratic and cubic. I am puzzled by his writing in connection with the quadrato cubic, "Quadratokubus (x^5)." Personally I can not avoid the sentimental hope that he really found the general formula.

2. The arithmetical triangle. We are safe in saying that by the year 1300 at least one capable mathematician was familiar with the binomial expansion for positive integral exponents. Some two hundred years after Omar, there lived in the Flowery Kingdom Chu-Shih-Chieh to whom we are indebted for the interesting diagram

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 8 & \cdots & & & & \cdots & 8 & 1 \end{array}$$

Mikami comments, "This is indicated as an old calculation but not his invention. . . . This may perhaps have been borrowed from the Arabs in some way" [8]. The horizontal arrangement of the figures suggests strongly that these numbers were found by the expansion of binomials, but of course we have nothing suggesting a proof. Various mathematicians have suggested that the Chinese could expand binomials to quite high powers.

The first writer to give us something really solid is Michael Stifel who published in 1544 his *Arithmetica Integra.* Here we find the diagram

1				
2				
3	3			
4	6			
5	10	10		
6	15	20		
7	21	35	35	
8	28	56	70	
9	36	84	126	126
10	45	120	210	252

This, of course, can be extracted at once from Chu Shih-Chieh's table, but it is very unlikely that Stifel ever saw the latter. The two significant facts for us are that he was interested in the approximate extraction of roots, and we should like to know the manner in which he explains the construction of his table. In the first column, we have the integers in natural order. Each subsequent column begins two places lower than the preceding one; it starts with the number immediately on its left, and each subsequent number in the column is the sum of the number immediately above and the number to the left of the latter. Now if we write

$$(a+b)^n = (a+b)(a+b)^{n-1}$$

and, if we know the expansion of $(a+b)^{n-1}$, we find the coefficients of the expansion of $(a+b)^n$ by exactly this process. It would seem that Stifel was showing what we should write in modern notation

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}. \tag{4}$$

A third form of this figure we owe to Pascal, whose famous triangle appears in the form

1	1	1	1	1	1	1
1	2	3	4	5	6	
1	3	6	10	15		
1	4	10	20			
1	5	15				
1	6					
1						

Probably Pascal was familiar with Stifel's table; he gives the same rule for the

construction of the triangle, as well as some other identities. He points out that the numbers in a N.E. running diagonal are the binomial coefficients, and shows how we find the number of groups of r things taken from n things. Finally in Pascal we have the general rule which we should write [9]

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r(r-1)(r-2)\cdots 1}. \tag{5}$$

It is important to say that a priority in this has been awarded to others, especially Briggs. Netto writes "Die binomsche Formel findet sich zuerst bei H. Briggs Arithmetica Logarithmica 1620," and Tropfke, that it is on p. 21 of Gellibrand's *Trigonometria Britanica*, a posthumous work of Briggs [10]. This may be. I can only say that I have found no trace. Netto gives no page number and I have seen nothing suggestive of it in the French translation, the only thing available to me; as for Tropfke, all I find on the page in question is a non-triangular form of Pascal's triangle, and there is nothing suggesting Formula (5) in the work.

3. Gregory and Newton. The first writer to approach the binomial expansion of a fractional power was James Gregory, who gave the formula in 1670. His method of approach was curiously indirect, his ostensible desire was to find an antilogarithm. Let us start with two numbers b and d with the logarithms $\log b = e$, $\log (b+d) = e+c$.

To find the number whose log is $(e+a)$,

$$\log b + \frac{a}{c}\left[\log (b+d) - \log b\right] = e + a$$

$$e + a = \log\left[b\left(1+\frac{d}{b}\right)^{a/c}\right].$$

Take the two series,

$$b,\ d,\ \frac{d^2}{b},\ \frac{d^3}{b^2}\cdots$$

$$\frac{a}{c},\ \frac{a-c}{2c},\ \frac{a-2c}{3c},\ \frac{a-3c}{4c}\cdots$$

Combine like this

$$b + \frac{a}{c}d + \frac{a}{c}\cdot\frac{a-c}{2c}\cdot\frac{d^2}{b} + \frac{a}{c}\cdot\frac{a-c}{2c}\cdot\frac{a-2c}{3c}\cdot\frac{d^3}{b^2} = b\left(1+\frac{d}{b}\right)^{a/c}.$$

There is of course no sign of proof [11].

It is time to turn to Sir Isaac Newton to whom we referred somewhat disparagingly in our opening paragraph. The story of his interest in the subject is told at length in a letter to Oldenburg, dated October 1676, and hence six years

after Gregory's letter just mentioned [12]. He tells us that he was interested early in the study of interpolation by Wallis. This admirable mathematician studied curves whose equations were of the type

$$y = (1 - x^2)^{0/2}, \qquad y = (1 - x^2)^{1/2}, \qquad y = (1 - x^2)^{2/2}, \qquad y = (1 - x^2)^{3/2},$$
$$y = (1 - x^2)^{4/2}, \qquad y = (1 - x^2)^{5/2}, \qquad y = (1 - x^2)^{6/2}.$$

If we take the area of the figure bounded by the positive axes, the curve and the ordinate, we have for the cases of the first, third, fifth and seventh curve

$$x, \qquad x - \tfrac{1}{3}x^3, \qquad x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5, \qquad x - \tfrac{3}{3}x^3 + \tfrac{3}{5}x^5 - \tfrac{1}{7}x^7.$$

How did Wallis discover these formulae, without the aid of integration? He studied quotients of the form

$$\frac{0^p + 1^p + 2^p \cdots n^p}{n^p + n^p + n^p \cdots n^p}$$

and noticed that the limit as n increased indefinitely was asymptotically $1/(p+1)$, this worked out at least in several instances [13]. We pass easily from this to finding the area under the curve $y = x^p$. We divide into n parts the segment from the origin to $x = n\Delta x$ on the axis, the points of division being 0, Δx, $2\Delta x$, $\cdots$, $n\Delta x$. We erect rectangles on these, one upper vertex being on the curve. The areas will be

$$\Delta x^p \cdot \Delta x, \ (2\Delta x)^p \cdot \Delta x, \ (3\Delta x)^p \cdot \Delta x, \ \cdots, \ (n\Delta x)^p \cdot \Delta x.$$

The sum will be

$$(\Delta x)^{p+1}(1^p + 2^p + 3^p + \cdots + n^p).$$

Now

$$\lim_{n\to\infty} \frac{(0^p + 1^p + \cdots + n^p)}{(n+1)n^p} = \frac{1}{p+1};$$

$$\lim_{n\to\infty} (\Delta x)^{p+1}[0^p + 1^p + 2^p + \cdots + n^p] = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)\frac{(n\Delta x)^{p+1}}{p+1} = \frac{x^{p+1}}{p+1}.$$

This gives the fundamental formula

$$\int x^p dx = \frac{x^{p+1}}{p+1}. \tag{6}$$

Naturally Wallis did not set things up in anything like this form but such is the essence of his reasoning. Moreover, an equivalent formula for quite a number of cases had been proved by Cavalieri and others [14]. Wallis knew also that the integral of a sum is the sum of the corresponding integrals, so there is no real mystery about his discovery of these areas.

Let us return to Newton. He wished to find the areas under the other curves.

beginning with the second which is the circle $y=(1-x^2)^{1/2}$. Newton notes that in all of Wallis' cases the first term is x and the denominators are 1, 3, 4, 7, $\cdots$. They cause no trouble. They come in through the integration, not through the expansion. The second terms are:

$$-\tfrac{0}{3},\ -\tfrac{1}{3},\ -\tfrac{2}{3},\ -\tfrac{3}{3}\cdots.$$

Now $(1-x^2)^{(k+1)/2}$ is a mean proportional between $(1-x^2)^{k/2}$ and $(1-x^2)^{(k+2)/2}$, and this gives the first numerator in cases 1, 3, 5, $\cdots$. He guesses that this averaging process works in every case, and the other expansions should begin

$$x-\frac{\tfrac{1}{2}x^3}{3},\quad x-\frac{\tfrac{3}{2}}{3}x^3,\quad x-\frac{\tfrac{5}{2}}{3}x^3\cdots.$$

Suppose, then, the expansion begins $z-(m/3)x^3$. How shall the subsequent terms be found? Let us follow his own words [**15**], "Quaerebam itaque quomodo in his seriebus ex datis duobus primus figuris reliquae derivari present. Et inveni quod posita secunda figura, reliquae producerentur per continuarum multiplicationem terminorum hujus serie"

$$\frac{m-0}{\quad}\times\frac{m-1}{\quad}\times\frac{m-2}{\quad}\times\frac{m-3}{\quad}\times\cdots.$$

The essential word here certainly is "inveni." Did Newton work this out for himself or, more likely, did he follow Pascal's and Stifel's formula which holds in the integral case, and guess that it was always correct, and then work out some cases? We shall never know the answer, and on this, largely, I think, depends the amount of credit which he should receive. All that we surely know is that he wrote out

$$(1-x^2)^{1/2}=1-\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^6}{16}\cdots$$

$$(1-x^2)^{3/2}=1-\frac{3x^2}{2}-\frac{3x^4}{8}-\frac{x^6}{16}\cdots$$

$$(1-x^2)^{1/3}=1-\frac{x^2}{3}-\frac{x^4}{9}-\frac{5x^6}{81}\cdots.$$

He squared or cubed the series and reached $(1-x^2)$. In the first case he found the square root by the usual method and reached the series. I cannot see that he did more than this, in which case, brilliant as was his genius in other matters, I do not think he deserves extraordinary credit for his contribution to the binomial theorem.

4. Attempts at proof. What shall we now say about demonstrations of the theorem; have any of these writers really proved it, or have they merely followed the example of all beginners, showing merely that no other solutions are

possible? We have seen that Omar Khayyam had doubts on the point, he says "J'ai prouvé qu'elles conduisent en effet à l'objet cherché." Newton's verifications, as far as they go, show that the series are equal to binomials. Pascal in another connection used mathematical induction. He was one of the first users, but he did not prove our theorem in this way. How did anyone know *a priori* that any non-integral power of a binomial was actually equal to a certain convergent series?

In 1742 there appeared an article by Giovanni Salvemini who lived in Castiglione, for which insufficient reason he was frequently referred to as De Castillon. In the *Philosophical Transactions*, vol. 42, 1742–3, we find his article. He points out that everyone knows Newton's formula, but no one, as far as he knows, has proved it. He distinguishes three cases (a) a positive integral exponent, (b) a positive fractional exponent, (c) a negative exponent. The first case he handles by a method still used today. Let us replace $(p+q)^n$ by the product $(p+q_1)(p+q_2)\cdots(p+q_n)$. The numerical coefficient of the term of the rth order in the q's will be the number of combinations of n things taken r at a time, namely $\binom{r}{n}$. Then we set all the q_i's equal to q. When it comes to expanding $(p+q)^{r/n}$, we are safe in taking the first exponent as r/n, for that is the case when $q=0$; thus

$$(p+q)^{r/n} = Ap^{r/n} + Bp^{r/n-1}q + Cp^{r/n-2}q^2 + \cdots$$
$$(p+q)^r = p^r(A + Bp^{-1}q + Cp^{-2}q^2 + \cdots)^n.$$

He knows how to expand a binomial to any positive integral power, and blithely expects that he can do the same thing with a convergent power series, treating it as a binomial. In expanding, the new coefficients kindly come in one at a time, so that we have

$$1 = A^n; \qquad r = nA^{n-1}B; \qquad \frac{r(r-1)}{1\cdot 2} = nA^{n-1}C + \frac{n(n-1)}{1-2}A^{n-2}B^2$$

$$1 = A; \qquad \frac{r}{n} = B; \qquad \frac{\frac{r}{n}\left(\frac{r}{n}-1\right)}{1\cdot 2} = C.$$

The negative expansion comes by taking the reciprocal of the positive one.

A much quicker method was devised by a far abler mathematician, Colin Maclaurin. Here is what he writes on pp. 607–8 of vol. 2 of his *Fluxions*, Edinburgh, 1742:

"Let it be required to find $\overline{1+x}^n$ where n may represent any integer number or fraction, positive or negative. It is evident from what is shown in the common algebra concerning powers and their roots that the first term of any power of $(1+x)$ is 1, and the subsequent terms involve $x, x^2, x^3, x^4, \cdots$ with invariable coefficients. We suppose therefore

$$\overline{1 + x}^n = 1 + Ax + Bx^2 + Cx^3 + \cdots$$

represents the general formula. By taking the fluxions on both sides

$$n\dot{x}\,\overline{1 + x}^{n-1} = A\dot{x} + 2Bx\dot{x} + 3Cx^2\dot{x}.$$

This is an identity, hence if we take $x=0$ (or because the first term of $\overline{1+x}^n$ must be 1) we must have $A=n$." The other coefficients are quickly found by similar processes and further differentiation.

This demonstration was not essentially new, it appeared five years earlier in the work of Colson [16]. The reasoning is less clear as he uses the the same letter to mean two different things; he writes on succeeding lines $y=\overline{a+x}|^m$ and $y=a^m$. However he comes out all right in the end. But he makes on p. 310 an important remark which seems to have escaped Maclaurin:

"Indeed it can hardly be said that this or any other that is developed from the Method of Fluxions is a strict investigation of this Theorem. Because the Method itself is originally derived from the method of raising Powers, at least integral Powers, and presupposes the knowledge of Unciae or numerical coefficients."

Exactly this same difficulty occurred somewhat later to Euler, who gave two other demonstrations, of which I reproduce the second [17]. We start with the equations

$$\begin{aligned}(1 + x)^n &= 1 + Ax + Bx^2 + Cx^3 + \cdots\\ (1 + x)^{n+1} &= 1 + A'x + B'x^2 + C'x^3 + \cdots\\ &= (1 + x)(1 + x)^n.\end{aligned}$$

Suppose n is an integer. When $n \leqq 0$, all coefficients vanish; when $n \leqq 1$ all after A; when $n \leqq 2$ all after B; and so on. Let us write

$$A = \alpha n, \qquad B = \beta n(n - 1), \qquad C = \gamma n(n - 1)(n - 2), \cdots$$

$$\begin{aligned}(1 + x)^{n+1} &= 1 + \alpha(n + 1)x + \beta(n + 1)nx^2 + \cdots\\ &= (1 + x)(1 + \alpha nx + \beta n(n - 1)x^2 + \cdots .\end{aligned}$$

Subtracting

$$0 \equiv (\alpha - 1)x + (2\beta n - \alpha n)x^2 + \cdots .$$

Dividing out x, and setting $x=0$, we have

$$\alpha = 1, \qquad \beta = \frac{1}{2}, \qquad \gamma = \frac{1}{2\cdot 3},$$

and so on. Euler concludes, "Prorsus superfluum foret hos casus ulterius prosequi, cum jam luce meridianam clarius apparat pro singulis litteris sequentibus eosdem plane valores necessario prodire debere, quos evolutio New-

tonianae docuit, atque haec demonstratio naturae rei tam apprime accomodato videtur, ut illi etiam in primis Analyseos elementis denegeri nequeat. Quin etiam universum ratiocinium qui hic usi sumus, unam vim retinet, etiamso adeo n ut imaginarius spectaretur'' [18].

I must confess that I am not much impressed by this proof. It has the important advantage of being equally applicable to all values of the exponent but the best reason for the assumption as to the form of the coefficients is that it is correct in the integral case; the statement that something is "luce meridiana clarius" is not the same thing as a mathematical demonstration.

5. Convergence. There remains the important problem of the convergence of the series. The early writers were more or less aware of the existence of this question but were unable to handle it completely. The honor for doing this goes to Niels Hendik Abel. His contribution is much too long to be repeated here [19].

LEMMA (1) *Let $\rho_1\rho_2\rho_3 \cdots \rho_n$ be a series of positive quantities such that* $\lim \rho_{m+1}/\rho_m = \alpha > 1$ *and ϵ_m be quantities whose absolute values do not approach* 0 *as a limit as m increases without limit, then the series $\epsilon_0\rho_0+\epsilon_1\rho_1+\epsilon_2\rho_2 \cdots$ is divergent.*

We see in fact that regardless of how great m may be, the set

$$\epsilon_m\rho_m + \epsilon_{m+1}\rho_{m+1} + \cdots + \epsilon_n\rho_n$$

which may contain positive or negative terms, will not approach 0 as a limit.

LEMMA (2) *If in the above series $\alpha<1$, $|\epsilon_m| < A$, the series is convergent*

We see, in fact that the absolute value of the set is less than $\rho_m A 1/(1-x)$, but $\lim \rho_m = 0$.

Now Abel makes a trigonometric development by the use of De Moivres' theorem.

Let

$$\phi(m) = 1 + \frac{m}{1}x + \frac{m(m-1)}{1\cdot 2}x^2 + \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}x^3 + \cdots .$$

Let

$$x = a + bi, \qquad m = k + k'i; \qquad \phi(m) = p + qi;$$

$$\sqrt{a^2+b^2} = \alpha; \qquad x = \alpha[\cos\phi + i\sin\phi]$$

$$\frac{m-\nu+1}{\nu} = \delta\nu[\cos\gamma\nu + i\sin\gamma\nu]$$

$$\binom{m}{\mu} = \alpha^\mu\delta_1\delta_2\cdots\delta_\mu[\cos(\mu\phi+\gamma_1+\gamma_2+\cdots+\gamma_\mu)$$
$$+ i\sin(\mu\phi+\gamma_1+\gamma_2+\cdots+\gamma_\mu)].$$

Abel here treats the real and imaginary parts separately, the question of convergence depends, as above, on whether α is greater than or less than unity. The case where it is equal to one, he treats at length, separately. One has the feeling that the last word has been said.

Yet that again is not the case. We have a variety of proofs that, if any power of a binomial can be expressed as a series of positive integral powers, this is the series. But why should such a series exist *a priori?* Omar sensed this difficulty. De Castillon met it, in the case of rational powers of the binomial, and, if his method were strengthened by showing that the algebraic operations with an infinite series were legitimate, and, that an extension by continuity considerations from the rational to the irrational is permissible, we might find the whole here. In a few cases Newton showed that the series which he developed did what they were supposed to do. Perhaps the easiest thing to do would be to find a general proof independent of the binomial theorem that $\dot{x}^n = nx^{n-1}\dot{x}$; then, with the aid of Taylor's Theorem with remainder, give MacLaurin's proof for the binomial case. What a long distance from Euclid II, 4!

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SOME APPLICATIONS OF EXTENDED ANALYTIC GEOMETRY

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1. Introduction. This paper deals with techniques developed for the case of three axes in extended analytic geometry,* together with some of the applications in number theory which are suggested thereby. While the subject has intrigued the writer on its own merits, it is only natural that readers will be more inclined to give it their attention if they can be shown that it solves problems in other fields.

2. The basic system. On the 3-axes plane the positive halves of the x, y, and z axes radiate from the origin at standard position angles of 0°, 60°, and 120°.† The three coördinates of a point designate directed segments which are parallel to the respective axes and are applied in vector sequence, starting from the origin. It is evident that the order of application is immaterial, and that one set of coördinates designates one and only one point.

We are now prepared to state, by way of preview, some of the reasons why the case of three axes should receive special attention in any study of the proposed system: (1) It is, of course, the simplest case of the n-axes plane when $n>2$. (2) The fact that it constitutes a plane analogue for solid analytic geometry invites comparison of the two subjects at many points, and suggests studies in the new field paralleling those in the old. (3) The application to number theory and particularly to Diophantine equations, is facilitated by a combination of four happy circumstances: (a) The equation of a line or a curve is a single one rather than a pair, as in the solid case; (b) the equation of a straight line through points with integral coördinates may be written with integral coefficients (this fails when $n>3$ unless the axes are spaces unequally, as will be explained later); (c) the points designated by all-integral coördinates lie in a simple pattern at the vertices of equilateral triangles filling the plane; (d) the "one degree of freedom" in choosing the coördinates of a given point is very helpful, as we shall see.

As a matter of fact, the simplicity of results for the 3-axes plane seems less surprising when we note that, from one point of view, its loci are actually the projections on a plane of the surfaces of solid analytic geometry. The plane may be considered to pass through the origin and to be perpendicular to the line segment from (0, 0, 0) to $(-1, 1, -1)$, if we use the right-handed or clockwise XYZ system. The observer's eye is "at infinity" in the direction indicated by the segment. But though this background consideration helps one at times to predict the nature of certain loci, and even gives interesting significance, from the standpoint of projection, to changes in the spacing of the three axes, it is neither essential nor desirable as a habitual way of viewing the system. Certainly the point of view which regards loci on the n-axes plane, when $n>3$, as

* See An Analytic Geometry for N Variables, by R. S. Underwood, this MONTHLY, vol. 52, May, 1945. Henceforth this will be referred to as Paper I.

† For the relevant figures, see Paper I.

"projections of configurations in n-space," is far-fetched as well as, probably, barren and unprofitable.

3. The basic equations. The coördinates x_i of a point on the n-axes plane are related to the rectangular coördinates (X, Y) with references to a superimposed pair of axes by the equations,*

$$X = \sum_{i=1}^{n} x_i \cos \theta_i; \qquad Y = \sum_{i=1}^{n} x_i \sin \theta_i; \qquad \theta_i = (i - 1)\pi/n. \tag{1}$$

In the case of three axes, θ_1, θ_2, and θ_3 are respectively 0°, 60°, and 120°, and the fundamental equations, after simplification, take the form

$$2x + y - z = 2X; \qquad y + z = \frac{2}{\sqrt{3}} Y. \tag{2}$$

Throughout this paper the letters x, y, and z denote the coördinates of a general point on the 3-axes plane, while X and Y are the corresponding rectangular coördinates on the superimposed 2-axes plane.

4. The straight line. Theorem 2 of Paper I may be restated for the case of three axes as follows:

THEOREM 1. *The locus of the equation*

$$Ax + By + Cz = D \tag{3}$$

is a straight line when $B = A + C$; otherwise it consists of all the points on the 3-axes plane.

COROLLARY. *The general equation of a straight line on the 3-axes plane may be written in the form*

$$Ax + By + (B - A)z = C. \tag{4}$$

The proof of the theorem may be obtained by solving (3) simultaneously with the two equations in (2). When the determinant of the coefficients of x, y, and z is not zero, real values for these letters are found to correspond with each real pair X and Y; otherwise the equations are inconsistent except for points (X, Y) on a line.

At this point it may be noted parenthetically that when the analogous method is tried on the equation of first degree in four variables, it is found that if θ_1, θ_2, θ_3, and θ_4 are 0°, 45°, 90°, and 135°, respectively, the general equation of the straight line on the 4-axes plane becomes

$$Ax + By + (\sqrt{2}\, B - A)z + (B - \sqrt{2}\, A)u = C,$$

so that the possibility of integral coefficients is barred. If, however, the four

* Paper I, equations (7), p. 256.

angles are made respectively 0°, arctan (4/3), 90°, and 180° − arctan (4/3), the equation becomes

$$Ax + By + \frac{(5B - 3A)}{4} z + \frac{(5B - 6A)}{5} u = C.$$

This will explain the parenthetical statement following item (b) in the third paragraph of this paper.

5. The equation of a line through two given points. By a convenient extension of the familiar definition of a line's slope which will apply generally when $n > 2$, we may define the slope* m of the line (4) as $-A/B$, including zero and infinite slopes. Geometrically, we may fix the slope of (4), or of any line on the n-axes plane, as "minus the ratio of the y-intercept to the x-intercept," where x and y are the first two variables.

The equation of the line through the points (a, b, c) and (d, e, f) is

$$(5)\qquad \begin{aligned} &(e + f - b - c)x + (a + f - c - d)y + (a + b - d - e)z \\ &\qquad = ae + af + bf - bd - cd - ce, \end{aligned}$$

as may be verified by substitution. It follows that, for this line,

$$(6)\qquad m = \frac{(b + c) - (e + f)}{(a + f) - (d + e)} \cdot$$

Given two points, or the slope and one point on a line, its equation may be written almost immediately. For example, for the line through $(2, -3, 4)$ and $(5, 0, -1)$, $m = -\frac{1}{4}$, whence the left side of the equation is $x + 4y + 3z$ and the right side is 2, as found and checked by using the coördinates of both points.

6. Generalized coördinates. From (2) it follows that if (a, b, c) are a given set of coördinates of a point $P(X, Y)$, then any set of coördinates of P, say x, y, and z, are related to a, b, and c as follows;

$$(7)\qquad 2x + y - z = 2a + b - c; \qquad y + z = b + c.$$

Now if we eliminate y and z successively from (7) we get $y = a + b - x$ and $z = c - a + x$. Therefore, the coördinates

$$(8)\qquad (x,\ a + b - x,\ c - a + x),$$

where x is arbitrary, may be called the generalized coördinates, (abbreviated as G.C.) of the point (a, b, c). The G.C. prove to be extremely useful in the theory, since they allow a choice of coördinates of the given point which satisfy an extra condition. To illustrate, the special coördinates of the point $(-2, 3, 4)$ which satisfy the equation $3x + 4y + 5z = 6$ are $(-7, 8, -1)$, which we find by substituting the G.C., $(x, 1 - x, 6 + x)$, in the equation and solving for x.

* This is consistent with the statement in Paper I (p. 255) that $m_{xy} = \partial y/\partial x$, $m_{yz} = \partial z/\partial y$, etc. Here $m = m_{xy}$.

7. An application to Diophantine equations. Suppose we seek simultaneous integral solutions of the equations

$$5x - 6y - 7z = 4 \tag{9}$$

$$7x + 11y - 3z = 5. \tag{10}$$

Without benefit of illumination from geometry we might try eliminating letters, getting successively $34x+95y=23$, $97x+95y=74$, and $97x+34z=3$. From the first of these equations, $x=(23-95y)/34$ so that we must solve the congruence $95y\equiv 23 \pmod{34}$, or some equally tedious one, to get integral solutions.

Consider next the geometric method. Upon multiplying (10) by k and adding to (9), we get

$$(5 + 7k)x + (-6 + 11k)y + (-7 - 3k)z = 4 + 5k, \tag{11}$$

In solid analytic geometry we would interpret (11) as the equation of a family of planes whose members pass through the line of intersection of the planes (9) and (10). On the 3-axes plane the interpretation is essentially the same; but in this case we can find the equation of the common line simply by fixing k so that

$$-6 + 11k = (5 + 7k) + (-7 - 3k). \tag{12}$$

This yields $k=4/7$, whence (11) becomes

$$63x + 2y - 61z = 48. \tag{13}$$

We may call (13) the equation of the line of intersection of the loci of (9) and (10). Only points on this line have coördinates which satisfy (9) and (10) simultaneously; and furthermore, every point on the line has exactly one such set of coördinates.

The line (13) is "infinitely many layers deep" in the sense that all coördinates of every point on it satisfy its equation. (This statement is not true for a typical locus on the 3-axes plane such as, for example, the plane $x+y+z=1$ and the area $x^2+y^2+z^2=1$, which are respectively one and two layers deep.) It follows that we may assign a value at will to one of the letters, thus using the one degree of freedom, and then "scan" the line with the remaining two coördinates, reaching every point on it. Letting $x=0$ by preference we find, upon solving the simple congruence,

$$61z \equiv -48 \pmod{2}, \tag{14}$$

that $z=2n$ and $y=24+61n$. Thus all of the lattice points on the line (or points whose coördinates are all integers) are designated by the coördinates $(0, 24+61n, 2n)$, where n is an integer. To find the special coördinates of these points which satisfy (9) and (10) simultaneously, we substitute their G.C. $(x, 24+61n-x, 2n+x)$ in (9) [and also in (10) for checking purposes]. This yields $x=37+95n$, and hence, from the G.C., one form of the complete integral solution of

(9) and (10) is

$$x = 37 + 95n, \qquad y = -13 - 34n, \qquad z = 37 + 97n. \tag{15}$$

In comparing the merits of the method used above with the purely algebraic one by which a letter is eliminated initially, it should be emphasized that neither method is *per se* the "more efficient" one. In both cases it is sometimes necessary to solve a second congruence to complete the solution, and each method excels on occasion, when the problem is chosen to that end. However, there are obvious simplifications of the geometry-guided method in many special situations, as when the locus of one of the original equations is itself a straight line, or can be made a line by the interchange of two letters, or again when both loci are lines. In such cases the geometric background casts helpful light on the various situations.

Nevertheless, the full potentialities of the methods associated with the 3-axes plane are not seen as long as we neglect equations of the second or higher degree. Evidently a useful result would be a simple method for obtaining the equation of the curve of intersection of two loci in general. Such a method is found in the next article.

8. Equations of curves. Normally the locus of a single equation is an area, or, from another point of view, a family of curves which collectively cover some determinable area on the 3-axes plane. But sometimes the loci are single curves which are analogous to the cylindrical surfaces of solid analytic geometry "seen on edge." In fact, we may translate the equation

$$f(X, Y) = 0 \tag{16}$$

into the equation in three variables of the designated curve simply by replacing X and Y by the expressions for them in (2). For example, the equation of the circle $X^2 + Y^2 = a^2$ becomes, on the 3-axes plane,

$$x^2 + y^2 + z^2 + xy - xz + yz = a^2. \tag{17}$$

By the *curve of intersection* of the loci of two equations we mean the curve containing all of the points, and only the points, which have coördinates that satisfy both equations simultaneously. The points have also, of course, other coördinates which satisfy the equation of the curve but not of the intersecting loci. In fact, the curve of intersection, like (13), is infinitely many layers deep, and in particular its equation is satisfied by the members of three unique sets $(0, y, z)$, $(x, 0, z)$, and $(x, y, 0)$. Thus when one variable is made zero, the remaining two variables and two axes are sufficient for plotting the curve.

THEOREM 2. *All points on the curve of intersection of the loci of* $f(x, y, z) = 0$ *and* $g(x, y, z) = 0$ *are also on the curve obtained by replacing* x, y, *and* z *in those equations by* x', $x+y-x'$, *and* $z-x+x'$, *respectively, and then eliminating* x'.

In fact, the curve thus obtained usually is the curve of intersection, though

it may contain additional points.* For convenience we shall call it the *plus curve of intersection.*

The proof of the theorem follows. If we replace y and z by $Y/\sqrt{3}+X-x$ and $Y/\sqrt{3}-X+x$ respectively [from (2)] and then eliminate x, we get a relation between X and Y, say $F(X, Y)=0$, which must hold for all coördinates (x, y, z) of points on the curve of intersection. Since $F(X, Y)=0$ itself designates a curve, real or imaginary, this curve must coincide with or include the curve of intersection. To change back to three variables we replace $Y/\sqrt{3}+X$ and $Y/\sqrt{3}-X$ by $x+y$ and $z-x$ respectively. The complete operation is telescoped in the theorem.

For an application, we shall find integral solutions of

$$x^2 + xy + ay^n = z^2 \tag{18}$$

by solving it simultaneously with

$$x + y - z = m. \tag{19}$$

Through the substitutions of Theorem 2 in (19), we get $x'=2x+y-z-m$. Using this value of x' in the expressions for y and z, $(x+y-x'$ and $z-x+x')$, we find that if we replace x, y, and z in (18) by $2x+y-z-m$, $z-x+m$, and $x+y-m$, respectively, we get the equation of the plus curve of intersection of (18) and (19). More simply, we may replace x, y, and z by $y-z-m$, $z+m$, and $y-m$, respectively, getting the same curve, though now, with $x=0$, it is not infinitely many layers deep. This yields

$$y = \frac{z^2 + 2mz - (m+z)^2 + a\,(m+z)^n}{z - m},$$

so that, if $z=m+1$, y is the integer $a(2m+1)^n-m^2$. We then substitute the G.C. of the points $[0, a(2m+1)^n-m^2, m+1]$ in (19), getting $x=a(2m+1)^n-(m+1)^2$. Using the G.C., we then have the following solution of (18), with m arbitrary:

$$x = a(2m + 1)^n - (m + 1)^2;\ y = 2m + 1;\ z = a(2m + 1)^n - (m^2 + m). \tag{20}$$

The accomplishment here is less than it will appear to be at first, since integral solutions for (18) might be obtained by replacing z by $x+u$ and solving for x. However, the fact that a true algebraic result may often be obtained from transformations other than those suggested by the geometric background is to be expected. At times the geometric method leads to an easy solution which in turn suggests an even simpler non-geometric approach. Though this cannot be said to refute the usefulness of the geometry, it does indicate the nature of the game which the writer has found it necessary to play in his search for a conquest with the new tools which he could not (at least without considerable difficulty) duplicate otherwise, even with the aid of "hindsight." We shall close with three theorems which seem to meet this requirement. It is to be understood that all "solutions" are restricted to sets of integers.

* For example, the plus curve of intersection of the loci of $x^2+y+z+1=0$ and $x^2+1=0$ is $y+z=0$ (the z-axis), though $x^2+1=0$ has no locus and hence there is no curve of intersection.

THEOREM 3. *If*

$$f(u, v) = 0, \qquad (u = 2x + y - z, v = y + z) \tag{21}$$

considered as an equation in x, y, *and* z, *is solvable when* $x=0$ (*or* $y=0$, *or* $z=0$), *it is also solvable when either of the other two letters is zero, and furthermore, it has infinitely many solutions; but if the first premise fails,* (21) *has no solutions.*

THEOREM 4. *The equations*

$$x^2 + y^2 = xz, \quad \textit{and} \tag{22}$$

$$z^2 + xy + yz = n \tag{23}$$

have no common solution when $n=2, 5, 6, 8, \cdots$; *that is, when* n *is an integer not expressible in the form* a^2+ab+b^2, *with* a *and* b *arbitrary.*

THEOREM 5. *The equations*

$$x^2 + y^2 + z^2 + xy - xz + yz = n \quad (n \neq 0), \quad \textit{and} \tag{24}$$

$$x^2 + y^2 - z^2 + xy + xz - yz = 0 \tag{25}$$

have no common solution.

Proofs. (Theorem 3.) Equation (21) is that of the curve $f(2X, 2Y/\sqrt{3})=0$. If this curve passes through a lattice point with one set of coördinates (a, b, c), its equation will be satisfied by all coördinates of the point, in accord with previous statements and also as shown by replacing (a, b, c) by the G.C., $(x, a+b-x, c-a+x)$. It will therefore still be satisfied when x or y or z is replaced by zero. But if the curve "misses" all lattice points, its equation cannot have solutions.

(Theorem 4.) The equation

$$(x^2 + y^2 - xz) + (z^2 + xy + yz - n) = 0 \tag{26}$$

will evidently be satisfied by all common solutions of (22) and (23). [Incidentally (26) is the equation of the curve of intersection, though it is not here obtained by use of Theorem 2.] But (26) is the circle $X^2+Y^2=n$, and hence common solutions of (22) and (23) can occur only when n has the form a^2+ab+b^2, since otherwise, by Theorem 3, (26) has no solutions. [Actually (22) and (23) have common solutions, namely $(-4m, 4m, -8m)$, when $n=16m^2$; and do not have common solutions when $n=a^2+ab+b^2$ with $a>0$ and $b>0$.]

Theorem 4 may be generalized in various ways.

(Theorem 5.) Adding the terms of (24) and (25), we get $n=2(x^2+xy+y^2)$. But since, as in the preceding proof, n must have the form a^2+ab+b^2 to yield solutions for (24), and since n as found above is even, it is necessary that $a=2A$ and $b=2B$, whence $n=2(x^2+xy+y^2)=4(A^2+AB+B^2)$. This again requires that x and y, and then A and B, both be even, and so on. The argument by "infinite descent" leads to the conclusion of the theorem.

NOTES ON QUARTIC CURVES

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1. Introduction. In these notes certain properties of the Cartesian graph of the quartic function

$$y = ax^4 + bx^3 + cx^2 + dx + e$$

are considered. Some of these, though well known, are linked to other relationships which have turned up in classwork. Thus certain characteristics are pointed out which may assist in the sketching, and add to the understanding of quartic curves.

For convenience, a translation of axes will be applied to remove the term in x^3. Also, a change in the scale of the y-coördinates will, without loss of generality, allow the quartic to be taken in the form

$$y = f(x) = x^4 + px^2 + qx + s. \tag{1}$$

From the expression in (1) it will be seen that of all the lines which meet the quartic curve it is only those of slope q which, upon solving for the points of intersection, will give quadratic equations in x^2. This means that lines of slope q can be used to fix points on the quartic which can be paired so that their abscissas are numerically equal but opposite in sign. In other words the quartic curve (1) is skew-symmetric with respect to the y-axis along lines of slope q. Thus the y-axis is the locus of the midpoint of all chords of slope q, when these chords are drawn to points properly paired. Furthermore, since the equation of the tangent to the quartic at the point (x_1, y_1) can be written in the form

$$y = (4x_1^3 + 2px_1 + q)x - 3x_1^4 - px_1^2 + s, \tag{2}$$

it is seen that the y-intercept of the tangent is unchanged when $-x_1$ is substituted for x_1. It follows that the two tangents drawn to the quartic from the properly paired endpoints of any chord of slope q will meet on the y-axis. On account of these properties this line may therefore be considered as a pseudo-diameter of the quartic.

2. Further significance of the coefficients. In addition to this significance attached to the coefficient q it is well to note a relationship which exists between two quartic curves whose equations of form (1) differ only in the sign of q. Denote such two quartics by $y=f(x, q_1)$ and $y=f(x, -q_1)$. Then by (1) there exists the identity

$$f(x, q_1) \equiv f(-x, -q_1),$$

which shows that the two quartic curves are symmetrical with respect to the y-axis. Therefore sufficient information in regard to curves of all quartics like (1) can be gathered by limiting q to positive values. Each quartic equation of form (1) for which $q=0$ can be represented by a curve which is symmetrical

with respect to the y-axis.

The significance of the parts played by the constants q and s in equation (1) should now be evident. Variation of the parameter s brings about a translation, while changes in the parameter q affects the skew-symmetry of the quartic with respect to the y-axis. But when the coefficient p is allowed to change there will result such variations in the graphs that it seems best to divide the discussion into two parts according as the quartics have $p \geqq 0$, or $p<0$. The latter group will present more points of interest and will be considered first. Therefore in the following discussion, unless otherwise stated, the coefficient p will represent a negative number.

There are three lines of slope q that may be of use in making a sketch of a quartic curve of this second group. The first is the tangent-secant line

$$y = qx + s \tag{3}$$

which is tangent to the graph of (1) at the point $(0, s)$. It will also cut the quartic at the two points where $x^2 = -p$.

The second is the double-tangent line

$$y = qx + s - \frac{p^2}{4} \tag{4}$$

which is tangent to (1) at the two points where $2x^2 = -p$.

The third line is the secant through the two points of inflection. These points exist only when $p<0$. This line, the inflection secant, may be found by dividing the quartic function in (1) by $1/12f''(x)$ until the remainder $r(x)$ is of first degree or lower. Let the quotient be $q(x)$. It is then true that

$$f(x) = \frac{1}{12}f''(x)\cdot q(x) + r(x). \tag{5}$$

The function $r(x)$ will equal the function $f(x)$ for each of the two values of x for which the second derivative $f''(x)$ vanishes. Therefore, $y=r(x)$ is the equation of the line through the two points of inflection. It turns out that

$$\frac{1}{12}f''(x) = x^2 + \frac{p}{6}, \qquad q(x) = x^2 + \frac{5p}{6},$$

and the equation of the inflection secant is

$$y = r(x) = qx + s - \frac{5p^2}{36}. \tag{6}$$

The two points of inflection P_1, P_2 have $x^2 = -p/6$. It will be seen also that the inflection secant (6) will cut the quartic again in two points Q_1 and Q_2, the abscissas of which are found from the equation $q(x) = x^2 + 5p/6 = 0$. For the two functions $f(x)$ and $r(x)$ are equal for each of the two zeros of the function $q(x)$.

3. The quartic and its inflection secant. At this point it is well to note certain properties or relations between the quartic (1) and its inflection secant (6). Assume that the four points Q_1, P_1, P_2, Q_2 on the inflection secant are taken in this order from left to right. Since the diameter $x=0$ bisects all chords of slope q, it follows that the midpoints of the segments Q_1Q_2 and P_1P_2 coincide, whence

$$Q_1P_1 = P_2Q_2. \tag{7}$$

The quartic curve cuts off two equal segments on the inflection secant.

Since the abscissas of these four points are known, then by means of the x-projections of the line segments it will be seen that

$$P_1Q_2:P_1P_2 = (\sqrt{5}+1):2, \quad \text{and} \quad P_1P_2:P_2Q_2 = 2:(\sqrt{5}-1).$$

It is true therefore that

$$P_1Q_2:P_1P_2 = P_1P_2:P_2Q_2. \tag{8}$$

This shows that the line segment P_1Q_2 is divided by the point P_2 in the "golden section." A similar situation exists for the line segment Q_1P_2 and the point P_1.

From plane geometry it is known that for a circle of radius r the side of the regular inscribed decagon c_{10} is $(r/2)(\sqrt{5}-1)$. It follows from the preceding statement that the three segments Q_1P_1, P_1P_2, and P_2Q_2 are so related that if

$$r = P_1P_2, \quad \text{then} \quad c_{10} = Q_1P_1 = P_2Q_2. \tag{9}$$

A sketch will show that the inflection secant and the quartic curve enclose three distinct areas. A student, E. W. Hofler, has pointed out and proved by integration that two of these areas are equal, and that the largest area of the three is equal to the sum of the other two, (problem E817, this MONTHLY, May, 1948). It is also true for all values of p that the area enclosed by the quartic (1) and any line of slope q is divided by the y-axis into two parts of equal area.

4. Sketching the curve, bend points. The three parallel lines (3), (4) and (6) will prove to be of use in sketching a quartic curve of this group. For notation let T_1 be the point of contact $(0, s)$ of the tangent-secant line. Let S_1 and S_2 (left to right) be the two points in which this line (3) cuts the curve. For the double tangent (4) let T_2 and T_3 be the two points of contact. The abscissas of these points are known. On the inflection secant the four points Q_1, P_1, P_2 and Q_2 have already been used. If three real bend points exist and if q is positive, let B_1 be the bend point, a maximum point. It is located on the curve a little to the right of T_1. The other two bend points B_2 and B_3 are points of minima. They will be near, but to the left of the points T_2 and T_3. This description covers what is usually the part of the quartic curve which is of greatest interest. This part is somewhat like a W with curved turns. Changes in the parameter p cause changes in the size of the W-part. This part lies between the tangent-secant line (3) and the double-tangent line (4). Their distance apart measured in the y-

direction of the y-axis is always $p^2/4$ units. The spread of the W-part is noted from the points S_1 and S_2 on line (3) which may be considered as the upper extremities of the W. Their abscissas are given by $x^2 = -p$. The spread of the lower part of the W is given by the points T_2 and T_3 which lie on the line (4) for which the abscissas are $2x^2 = -p$. These lines and points will locate the W-part and will show how its size varies as the parameter p moves over the range $p < 0$.

Next, consider the conditions for the existence of real points of inflection and three real bend points. The first and second derivatives of (1) are

$$f'(x) = 4x^3 + 2px + q, \quad \text{and} \quad f''(x) = 12x^2 + 2p. \tag{10}$$

From the second derivative it is seen that the quartic will have none or two points of inflection according as $p \geqq 0$, or p is negative.

Turning to the bend points, the condition for the existence of three real bend points is that the first derivative function, or its equivalent the cubic function

$$\frac{1}{4} f'(x) = x^3 + \frac{p}{2} x + \frac{q}{4},$$

will have three distinct zeros. From the theory of cubics it is known that this requires that the following inequality holds

$$\left(\frac{p}{6}\right)^3 + \left(\frac{q}{8}\right)^2 < 0.$$

Expressing q as a function of p, also solving for p in terms of q, will give the two equivalent statements

$$q^2 < -\frac{8}{27} p^3, \quad \text{or} \quad p < -\frac{3}{2} q^{2/3}. \tag{11}$$

It is only when the relations in (11) hold that the quartic curve (1) will have three bend points.

It may be of interest to note that through the three real bend points of a quartic it is possible to pass a parabola of the form $y = ax^2 + bx + c$. Its equation can be found by dividing the quartic function $y = f(x)$ in (1) by the function $1/4 f'(x)$. Let $Q(x)$ be the quotient and $R(x)$ the remainder. It is true then that

$$f(x) = Q(x) \cdot \frac{1}{4} f'(x) + R(x).$$

Assume that the first derivative function $f'(x)$ vanishes for each of the three distinct values $x = r_1, r_2, r_3$. It is seen then that the two functions $f(x)$ and $R(x)$ are equal for each of the three values $x = r_1, r_2, r_3$. Furthermore, the zero of the function $Q(x)$ will also make the two functions $f(x)$ and $R(x)$ equal. Performing the division will give $Q(x) = x$ and

$$y = R(x) = \frac{p}{2}x^2 + \frac{3}{4}qx + s. \tag{12}$$

This is the equation of the parabola through the three bend points. It will also pass through the point $(0, s)$, a fourth point of the quartic. When $p=0$, there is no point of inflection and only one real bend point. In such a case $y=R(x)$ in (12) represents the equation of the secant of the quartic through the only real bend point and the point $(0, s)$.

To complete the part of this paper which concerns the influence of the coefficient p on the graphs of the quartics represented by (1), turn now to the case when $p \geqq 0$. The sketching of the quartic is then relatively simple. Its lower part is somewhat like a U askew according to the function of q. The curve is skew-symmetric to the y-axis along lines of slope q and lies above the tangent line $y=qx+s$. If a part of the curve has been located by points for positive values of x, then since the line $x=0$ is the diameter for chords of slope q such chords will locate corresponding points of the quartic to the left of the y-axis.

5. Certain tangents to the quartic. As a final topic, consider the tangents to the quartic from points on its pseudo-diameter, the line $x=0$. From the equation of the tangent given in (2) it is seen that b, its y-intercept, is

$$b = -3x_1^4 - px_1^2 + s. \tag{13}$$

This is independent of the coefficient q. Therefore, *a one parameter family of quartics as in* (1) *with fixed values for the coefficients p and s, but with q as an arbitrary parameter, has the property that all the tangents to the curves of this family at the points where $x=x_1$ will have the same y-intercept.*

A maximum value of b in (13) greater than $b=s$ will exist only when $p<0$. It is

$$b_{\max} = s + \frac{p^2}{12}\cdot$$

The two tangents that have this y-intercept have their points of contact where $6x_1^2 = -p$. They are the two tangents of inflection.

From every point on the y-axis where $y \leqq s+p^2/12$ one, two, three or four tangents can be drawn to the quartics of (1) for which $p<0$. From the points on the y-axis between $(0, s)$ and $(0, s+p^2/12)$ four tangents can be drawn. From the point $(0, s)$ three tangents can be drawn. One of these is the tangent-secant line given in (3). The other two have their points of contact where $3x_1^2 = -p$. These two points lie on the secant line $y=qx+s-2p^2/9$. This secant will meet the quartic again at the two points where $3x_1^2 = -2p$. It turns out that the tangents at the two latter points are parallel to the two inflection tangents. From all points on the y-axis below the point $(0, s)$ two tangents can be drawn. However, at the point $(0, s-p^2/4)$ only one tangent exists. This is the double tangent given in (4).

The slopes of the two tangents of inflection are given by the expression

$$q \pm \left(\frac{-2p}{3}\right)^{3/2}.$$

If the slope of one inflection tangent is zero, then the slope of the other is $2q$.

MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, University of California
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A GENERALIZED INTEGRAL TEST FOR CONVERGENCE OF SERIES

MORGAN WARD, California Institute of Technology

The following useful generalization of the familiar Maclaurin-Cauchy integral test for convergence of real series deserves to be better known. It is apparently due to G. H. Hardy,* who made a redundant hypothesis on $f(t)$. The integrals may be taken either in the sense of Riemann or in the sense of Lebesgue.

THEOREM. *Let $f(t)$ be a complex-valued function of the real variable in the interval $1 \leqq t < \infty$, such that $f'(t)$ exists and is integrable to $f(t)$ over any finite interval $1 \leqq t \leqq T$. Then if $\int_1^\infty f'(t)dt$ is absolutely convergent, the series $\sum_1^\infty f(n)$ and the integral $\int_1^\infty f(t)dt$ converge and diverge together.*

Proof: By Abel's partial summation formula, we have

$$\sum_{r=1}^{n} a_r b_r = A_n B_n - \sum_{r=1}^{n-1} A_r(b_{r+1} - b_r),$$

where $A_r = a_1 + a_2 + \cdots + a_r$, $(r = 1, 2, \cdots, n)$.

Let $s_n = \sum_{r=1}^n f(r)$. Then on taking $a_r = 1$ and $b_r = f(r)$ in the summation formula, we find that

$$s_n = nf(n) - \sum_{r=1}^{n-1} r(f(r+1) - f(r)).$$

Now if $[t]$ denotes as usual the greatest integer in t, then

* G. H. Hardy: Proc. London Math. Soc. (2), vol. 9, 1910, pp. 126–144.

$$r(f(r+1)-f(r)) = \int_r^{r+1} [t]f'(t)dt.$$

Also

$$nf(n) - 1\cdot f(1) = \int_1^n \frac{d}{dt}(tf(t))dt,$$

or

$$nf(n) = f(1) + \int_1^n f(t)dt + \int_1^n tf'(t)dt.$$

On substituting these expressions into the formula for s_n, simplifying and transposing, we obtain the formula

$$s_n - \int_1^n f(t)dt = f(1) + \int_1^n (t-[t])f'(t)dt.$$

Now $|(t-[t])f'(t)| < |f'(t)|$. Hence the infinite integral $\int_1^\infty (t-[t])f'(t)dt$ is convergent, and

$$\lim_{n\to\infty}\left(s_n - \int_1^n f(t)dt\right) \text{ exists.} \tag{1}$$

Now assume that the integral $\int_1^\infty f(t)dt$ is convergent. Then $\lim_{n\to\infty}\int_1^n f(t)dt$ exists. Hence by (1), $\lim_{n\to\infty} s_n$ exists; that is, the series $\sum_1^\infty f(n)$ is convergent.

The converse result is a little more troublesome. Assume that $\sum_1^\infty f(n)$ converges. Then

$$\lim_{n\to\infty} f(n) = 0, \tag{2}$$

and by (1),

$$\lim_{n\to\infty}\int_1^n f(t)dt \text{ exists.} \tag{3}$$

Now $f(T) = f(1) + \int_1^T f'(t)dt$. But since $\int_1^\infty f'(t)dt$ converges, $\lim_{T\to\infty}\int_1^T f'(t)dt$ exists. Hence $\lim_{T\to\infty} f(T)$ exists, so that by (2),

$$\lim_{t\to\infty} f(t) = 0. \tag{4}$$

Now

$$\left|\int_1^T f(t)dt - \int_1^{[T]} f(t)dt\right| = \left|\int_{[T]}^T f(t)dt\right| \leqq \max_{[T]\leqq t\leqq T} |f(t)|\,(T-[T])$$
$$< \max_{t\geqq [T]} |f(t)|.$$

Hence by (4)

$$\lim_{T\to\infty}\left(\int_1^T f(t)dt - \int_1^{[T]} f(t)dt\right) = 0.$$

But

$$\lim_{T\to\infty}\int_1^{[T]} f(t)dt$$

exists by (3). Hence $\lim_{T\to\infty}\int_1^T f(t)dt$ exists; that is $\int_1^\infty f(t)dt$ is convergent.

As an example, suppose that $f(t)=t^{-1}e^{-i\mu\log t}$, μ real. Then $f'(t)=0(1/t^2)$ and the conditions of the theorem are met. But

$$\int_1^T f(t)dt = \frac{i}{\mu}(e^{-i\mu\log T}-1).$$

Hence $\int_1^\infty f(t)dt$ diverges. Therefore $\sum_1^\infty 1/n^{1+i\mu}$ diverges.

Again, suppose that $f(t)=e^{it^\alpha\theta}/t^\beta$, where α and θ are real, and Rl $\beta>\alpha>0$, $\theta\neq 0$. Then $f'(t)$ is continuous and of order $t^{-1-\mu}$, where $\mu=\text{Rl }\beta-\alpha$, in the range $1\leqq t<\infty$. Hence the conditions of the theorem are met. Now the infinite integral $\int_1^\infty f(t)dt$ is easily seen to converge on making the change of variable $s=t^\alpha$. Hence the infinite series $\sum_1^\infty e^{in^\alpha\theta}/n^\beta$ converges. In particular then, if β is real, we see that the two real series

$$\sum_1^\infty \frac{\cos n^\alpha\theta}{n^\beta} \quad\text{and}\quad \sum_1^\infty\frac{\sin n^\alpha\theta}{n^\beta}$$

both converge if $\beta>\alpha>0$ and $\theta\neq 0$.

The ordinary integral test is included as a special case if we use Lebesgue integrals; for if $f(t)$ is real, continuous and tends to zero steadily, $f'(t)$ exists almost everywhere and $f(t)=\int_1^t f'(s)ds+f(1)$. Since $|f'(t)|=-f'(t)$, the hypotheses of the theorem are evidently satisfied.

GEOMETRY OF THE SQUARE ROOT OF THREE

C. S. Ogilvy, Trinity College, Hartford, Conn.

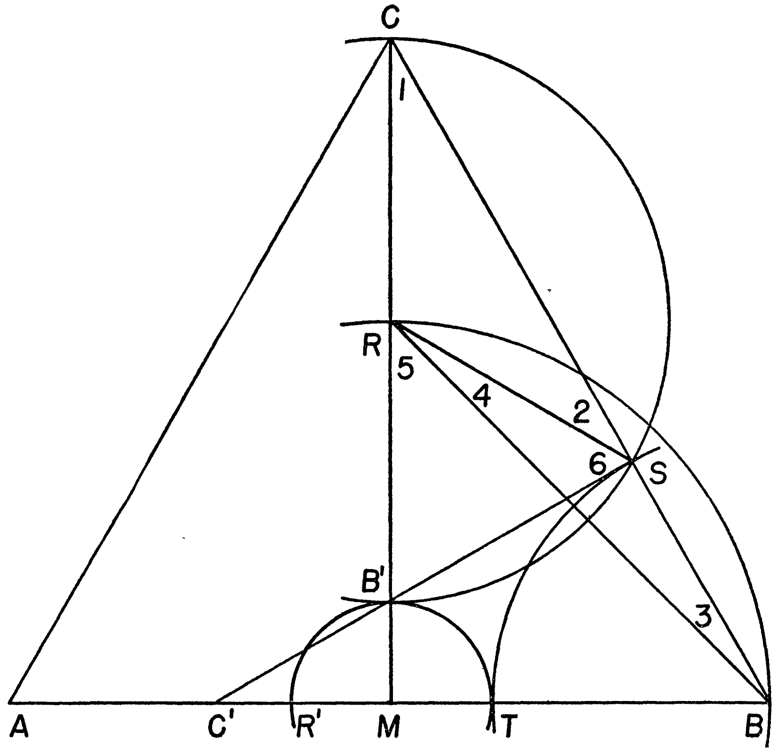
That the diagonal of a square is incommensurable with its side and the quotient is representable by the continued fraction

$$1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cdots}}}$$

is easy to prove geometrically. The corresponding fact that the altitude of an equilateral triangle and half its side are incommensurable and the quotient representable by

$$1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cdots}}}}$$

is not quite so self-evident. A purely geometric proof follows.



Referring to the figure, lay off $MR=MB$. MB cannot be contained twice in MC, because $BC=2MB>MC$. Therefore the first quotient is 1 and the first remainder is $RC=r_1$.

Draw an arc with R as center and RC as radius. $\angle 1=\angle 2=30°$. Therefore $\angle RSB=150°$. But $\angle 3=15°$. Hence $\angle 4=15°$ and $RS=SB$. With B as center and BS as radius, draw an arc cutting BM at T. Then $BT=BS=SR=RB'=RC=r_1$. Now r_1 cannot be contained twice in BM; for if it were, it would be contained 4 times in CB; that is, $4r_1\leqq CB$, an impossibility because $3r_1=(CR+RS+SB)>CB$. On the other hand, r_1 must be contained at least once in BM, since $BM=MR$ and $r_1<MR$. Therefore the second quotient is 1 and the second remainder is $MT=r_2$.

Since $MB'=MT$, an arc may be drawn with center at M, through T and B', cutting MA at R'. Draw SB', cutting AB at C'. Now $\angle 4+\angle 5=60°$ and

$RB'=RS$. Therefore triangle $RB'S$ is equiangular, and $\angle 6=60°$. Since $\angle 2=30°$, triangle BSC' is a 30°–60° right triangle. Hence $BC'=2BS=2BT$, and $C'T=TB=r_1$. Now triangle $MB'C'$ is similar to triangle MBC. $MR'=r_2$ is contained in MC' only once, as above, and hence only twice in TC', with remainder $R'C'$. Therefore the third quotient is 2 and the third remainder is $R'C'$.

The procedure with triangle $MB'C'$ will be exactly that used on triangle MBC. Thus the process is non-terminating, producing for subsequent quotients alternately 1 and 2, which is what is meant by the assertion that the ratio MC/MB can be expressed as

$$1+\frac{1}{1}+\frac{1}{2}+\frac{1}{1}+\frac{1}{2}+\cdots.$$

CONSECUTIVE CUBES WITH DIFFERENCE A SQUARE

VICTOR THÉBAULT, Tennie, Sarthe, France

1. E. P. Starke, in his solution to Problem E 702, this MONTHLY, vol. 53, 1946, p. 464, transforms

$$(n+1)^3-n^3=3n^2+3n+1=r^2 \tag{1}$$

into the Pell equation

$$(2r)^2-3(2n+1)^2=1, \tag{2}$$

and gives the solution

$$n_{k+4}=14n_{k+2}-n_k+6$$

containing all integer values of n.

2. Running, loc. cit., p. 465, notes that the first values for r are

$$1,\quad 13,\quad 181,\quad 2521,\quad 35113,\quad 489061,\quad 6811741,$$

and remarks that these numbers are all of the form $a^2+(a+1)^2$. He asks if the remark applies to all values of r satisfying (1).

The answer is affirmative, for, if we consider the equation

$$3n^2+3n+1=[a^2+(a+1)^2]^2,$$

we find easily that it reduces to

$$4X^2-3Y^2=1,$$

an equation of the same form as (2).

The numbers a satisfy

$$a_k=4a_{k-1}-a_{k-2}+1. \tag{3}$$

The property noted by Running is therefore general.

3. The values of a_k given by (3), the first being

$$a_1 = 0,\ a_2 = 2,\ a_3 = 9,\ a_4 = 35,\ a_5 = 132,\ a_6 = 494,\ a_7 = 1845, \cdots,$$

verify also

$$a_k + a_{k+1} + 2(a_k \cdot a_{k+1} + 1) = (a_k - a_{k-1})^2,$$

and

$$12a_k^2 + 12a_k + 9 = 3[(2a_k + 1)^2 + 2]$$

is a perfect square. For instance, if $n = 5$,

$$12(132)^2 + 12(132) + 9 = 210681 = 459^2.$$

4. Finally, (1) shows that r is the length of the side BC corresponding to the angle $A = 120°$ in a triangle ABC having for other sides the consecutive integers n and $n+1$.

In such a special triangle, *the number r measuring the largest side equals the sum of the squares of two consecutive integers a and $a+1$.*

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College and Institute for Advanced Study

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FUNDAMENTAL IDENTITY OF VECTOR ALGEBRA

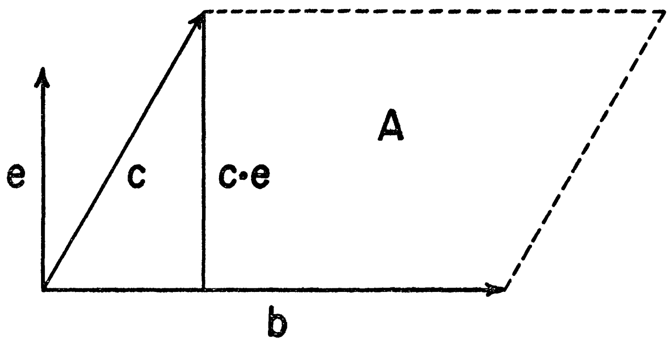
C. J. COE and G. Y. RAINICH, University of Michigan

One of the most valuable features of vector analysis as applied to geometry and physics is the fact that the discussion and results can be free from any reference to a particular coordinate system. This makes it desirable to define the vector operations in geometric form and only develop the corresponding coördinate formulas later as aids in the application. In this way the question of the invariance of these quantities under transformation of coordinates does not arise.

However, simple geometric proofs of the fundamental identity,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \tag{1}$$

would seem to be lacking, since the proofs offered in the texts are extremely various and usually unsatisfactory from the above point of view. The following geometric proof is suggested, as resting directly on the geometric definitions of the operations involved and as clearly showing the geometric significance of the various terms.



Consider first the case in which **a** is a unit vector **e**, coplanar with **b** and **c** and perpendicular to **b** on the same side as **c**. Inspection of the figure, in which the plane of the paper is that of **b**, **c**, **e** shows that $\mathbf{e}\times(\mathbf{b}\times\mathbf{c})$ has the direction and sense of **b** and thus that the equality,

$$\mathbf{e} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c}\cdot\mathbf{e})\mathbf{b} \tag{2}$$

is true in direction and sense. Furthermore, the length of the left member is the area A of the parallelogram formed on **b** and **c**, while the length of the right member is the product of the altitude and base of this parallelogram. Thus equation (1) holds for $\mathbf{a}=\mathbf{e}$, since the second term of its second member vanishes in this case.

Likewise, in the case where **a** is a unit vector **f**, coplanar with **b** and **c**, but perpendicular to **c** on the same side as **b**, we have merely to replace **e** in equation (2) by **f** and interchange **b** and **c** to find,

$$\mathbf{f} \times (\mathbf{b} \times \mathbf{c}) = -(\mathbf{f}\cdot\mathbf{b})\mathbf{c}, \tag{3}$$

and thus equation (1) is satisfied for $\mathbf{a}=\mathbf{f}$, since the first term of its second member vanishes in this case. And finally, if **a** is a unit vector **g**, perpendicular to both **b** and **c**, equation (1) holds since every term is zero.

Except in the trivial case in which **b** and **c** are parallel, any vector **a** is of the form $\mathbf{a}=k\mathbf{e}+l\mathbf{f}+m\mathbf{g}$, where k, l, m are scalars, and it follows that equation (1) holds in every case, since the equation involves **a** linearly and has been seen to hold for $\mathbf{a}=\mathbf{e}$, **f** and **g**.

THE PROBLEM OF A NON-VANISHING GIRDER ROUNDING A CORNER

NORMAN MILLER, Queen's University

In the usual problem of a girder being carried around a corner from one passageway to another, the girder is etherialized into a line segment (fig. 1). The problem gains in interest as well as in reality if we suppose the girder to have positive width. Moreover, the geometry to which it gives rise extends beyond the "practical" problem of the girder.

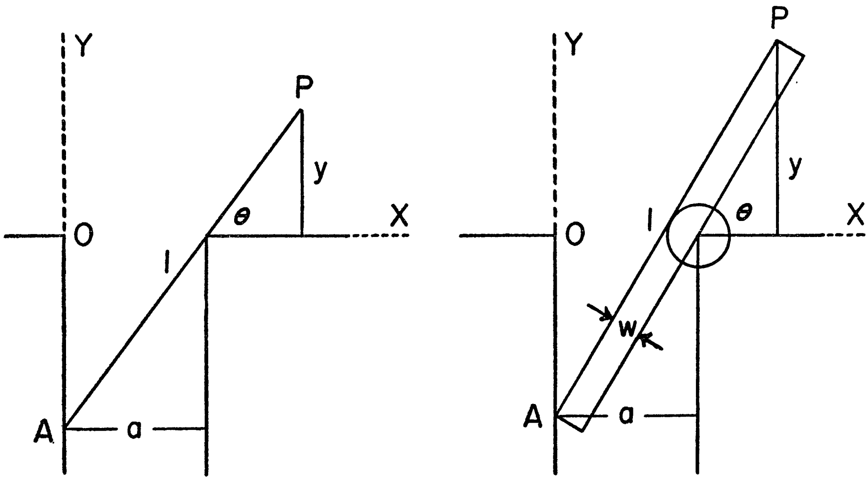


Fig. 1 Fig. 2

Denote by w and l the width and length of the girder, supposed square, and by a the width of the passage from which the girder is being moved. We take $w<a<l$. Then, with the notation of fig. 2, the problem is to find the maximum of y where

$$y = l \sin\theta - (a - w \csc\theta) \tan\theta$$
$$= l \sin\theta - a \tan\theta + w \sec\theta.$$

Since

$$dy/d\theta = l \cos\theta - a \sec^2\theta + w \sec\theta \tan\theta,$$

the maximizing value of θ must satisfy

$$l \cos^3\theta - a + w \sin\theta = 0. \tag{1}$$

Attention to the graphs of the separate terms of this equation shows that it has exactly one root between 0 and $\pi/2$ and one between $-\pi/2$ and 0. If $w=0$, these two roots are equal in absolute value, but otherwise unequal.

The root of (1) which lies between 0 and $\pi/2$ gives the maximum value of y and hence the minimum width of the horizontal passageway which will allow the girder to turn the corner.

The root of (1) between $-\pi/2$ and 0 has also some interest. The further discussion will be concerned not with the problem of a physical girder but with the curve which is the complete locus of the point P. This locus will be defined as follows: A variable straight line touches a fixed circle of radius w and intersects in a variable point A a fixed straight line which is at a distance a from the centre of the circle. The locus in question is that of a point on the moving line which is at a distance l from A, where $w<a<l$. In case $w=0$ the locus is a Conchoid of Nicomedes which (in fig. 1) is symmetrical about OX. In case $w>0$ the locus is a sort of distorted Conchoid, having two points of contact with the fixed circle and containing an unsymmetrical loop. The parametric equations of this locus, with the axes indicated in fig. 2, are

$$x = l \cos \theta, \qquad y = l \sin \theta - a \tan \theta + w \sec \theta,$$

which give the single quartic equation

$$(xy - wl)^2 = (x - a)^2(l^2 - x^2).$$

This equation becomes that of the Conchoid when $w=0$.

When the point A (fig. 2) is on the negative y-axis, the angle θ is positive and when A moves up to the positive y-axis θ becomes negative. Thus all values of θ between $-\pi/2$ and $\pi/2$ apply to points on the curve and a critical point occurs on the loop for one positive value of θ and for one negative value (fig. 3). For $w>0$ the abscissas of these points are different as well as the absolute values of their ordinates.

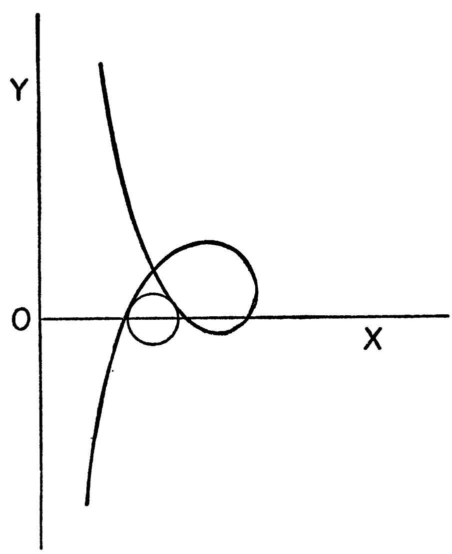


Fig. 3

Example. In the solution of the following numerical problem the maximizing root of (1) is easily identified: A $1\frac{1}{2}$ foot square beam 50 feet long is carried

through a passageway 12 feet wide into a lane at right angles to the passageway. Find the minimum width of the lane which will allow the beam to turn the corner while remaining horizontal.

Here $w=1.5$, $a=12$, $l=50$, and (1) reduces to

$$50\cos^3\theta - 12 + 1.5\sin\theta = 0. \tag{2}$$

This equation has the positive root given by $\cos\theta=3/5$, $\sin\theta=4/5$, for which we verify that $d^2y/d\theta^2<0$. This root gives the maximum value of y, which is found to be 26.5. The required minimum width of the lane is then 26.5 feet. The minimizing root of (2) is the negative value of θ given by $\cos\theta=0.64$, $\sin\theta=-0.79$, each correct to two figures. For this root it is easily verified that $d^2y/d\theta^2>0$. The corresponding point on the curve, the lowest point on the loop, has the approximate coordinates $(32, -23)$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 847. *Corrected. Proposed by Albert Newhouse, University of Houston*

Let a, b, A be the given parts of a triangle in the ambiguous case. Show that the area of the triangle is given by

$$K = \tfrac{1}{2}b\sin A\left[b\cos A \pm (a^2 - b^2\sin^2 A)^{1/2}\right].$$

E 856. *Proposed by J. T. Hurt, Texas Agricultural and Mechanical College*

Let N be an integer of p digits. If the last digit is removed and placed before the remaining $p-1$ digits, a new number of p digits is formed which is $(1/n)$th of the original number. Find the most general such number N.

E 857. *Proposed by M. S. Knebelman, Washington State College*

Evaluate the s by s determinant whose element in the $(i+1)$st row and $(j+1)$st column is $d^{m+j}x^{n+i}/dx^{m+j}$.

E 858. *Proposed by Henry Scheffé, University of California at Los Angeles*

Give an example of an even function with continuous derivatives up to order n, which has neither a maximum nor a minimum at $x=0$.

E 859. *Proposed by C. W. Trigg, Los Angeles City College*

If the faces of a hexahedron are equilateral triangles congruent to the faces of a regular octahedron, then the radii of the inscribed spheres are in the ratio 2:3.

E 860. *Proposed by Leo Moser, University of Manitoba*

Show that if all the faces of a polyhedron have central symmetry then it can be dissected by a finite number of plane cuts and the pieces fitted together to form a solid cube.

SOLUTIONS

Distribution of Suits in a Bridge Hand

E 798 [1948, 502]. Correction to editorial note.

E. M. Berry and Bart Park have pointed out an error in the formula given in the editorial note. The initial factor should be 4, 12, or 24 according as three, two, or none of the n's are equal.

Incircle of Equilateral Triangle

E 820 [1948, 317]. *Proposed by Kaidy Tan, Chip-Bee Institute, Amoy, Fukien, China*

If ABC is an equilateral triangle, and P is any point on the circumference of the inscribed circle, prove synthetically that $(PA)^2+(PB)^2+(PC)^2$ is constant.

I. *Solution by R. P. Stephens, University of Georgia.* If r is the radius of the circle on which P lies and R is the radius of the circumcircle of ABC, then

$$(PA)^2 = r^2 + R^2 - 2r \text{ (projection of } OA \text{ on } OP).$$

We find similar expressions for $(PB)^2$ and $(PC)^2$. Hence the sum of the squares is equal to

$$3r^2 + 3R^2 - 2r \text{ (sum of projections of } OA, OB, OC \text{ on } OP).$$

But OA, OB, OC, considered as vectors, will form a closed figure whose projection on any line will be zero. Therefore

$$(PA)^2 + (PB)^2 + (PC)^2 = 3r^2 + 3R^2 = \text{constant}.$$

This theorem is a special case of a more general theorem which is easily proved in the same way. It is: *If $A_1 \cdots A_n$ is any regular n-gon with center O and radius R, and if P is any point on a concentric circle of radius r: then* $\Sigma(PA_i)^2 = n(r^2+R^2)$.

II. *Solution by Leo Moser, University of Manitoba.* (*Editorial Note.* Although the following solution is analytic, its simplicity and the possibility of numerous

generalizations, cause it to merit consideration.)

Take A: $(1, 0, 0)$, B: $(0, 1, 0)$, C: $(0, 0, 1)$, P: (x, y, z). The inscribed circle is the intersection of a sphere $x^2+y^2+z^2=c_1$ and a plane $x+y+z=c_2$. Hence

$$\begin{aligned}(PA)^2 + (PB)^2 + (PC)^2 \\ &= (x-1)^2 + y^2 + z^2 + x^2 + (y-1)^2 + z^2 + x^2 + y^2 + (z-1)^2 \\ &= 3c_1 - 2c_2 + 3 = \text{constant}.\end{aligned}$$

III. *Solution by William Gustin, Indiana University.* We shall establish the considerably more general theorem:

Let $A_1, \cdots, A_n$ be n points in a euclidean space and let $k_1, \cdots, k_n$ be n real numbers whose sum is unity. The locus of a point P for which

$$\sum k_i(PA_i)^2 = k, \text{ a constant,}$$

is a sphere whose center is the centroid G of the points A_i with weights k_i, and whose radius r is given by

$$r^2 = k - \sum k_i(GA_i)^2.$$

We shall employ vector methods, regarding points in the space as vectors. Let the function f be defined at every point P in the space as follows:

$$f(P) = \sum k_i(PA_i)^2 = \sum k_i(P\cdot P - 2P\cdot A_i + A_i\cdot A_i).$$

Since $\Sigma k_i=1$, the point $G=\Sigma k_iA_i$ is the centroid of the points A_i with weights k_i, and we have

$$f(P) = P\cdot P - 2P\cdot G + \sum k_i(A_i\cdot A_i),$$

whence

$$f(P) - f(G) = (PG)^2. \tag{1}$$

This proves the theorem.

Also solved by Murray Barbour, Paul Brock, W. E. Byrne, William Douglas, Ragnar Dybvik, J. M. Feld, Frank Harary, B. A. Hausmann, Sam Kravitz, B. R. Leeds, D. W. Matlack, Norman Miller, Leo Moser (a second solution), C. S. Ogilvy, Margaret Olmsted, A. P. Rhodes, C. C. Richtmeyer, Joseph Rosenbaum, A. Sisk, Kirk Stewart, Sieh Su and the proposer. C. W. Trigg (four solutions).

Editorial Note. Gustin's generalization (stated for the plane case only) can be found as Cor. 3, p. 99, in M'Clelland's *The Geometry of the Circle*, Macmillan, 1891. Many elegant theorems can be derived from it. Thus we have:

(*A*) *$\Sigma k_i(PA_i)^2$ is a minimum when P coincides with the centroid G of the points A_i with weights k_i.*

(*B*) *If $A_1, \cdots, A_n$ are vertices of a regular polygon (polyhedron) of circumradius R, and if P is any point on a concentric circle (sphere) of radius r, then*

$$\sum (PA_i)^2 = n(R^2 + r^2).$$

This is obtained from Gustin's relation (1) by taking $k_1 = \cdots = k_n = 1/n$. and (for the plane case) is the generalization stated above by Stephens. This generalization was also obtained, in different manners, by Rosenbaum and Trigg. Trigg also established the generalization for three-space.

(C) *If P is any point on the circumcircle (circumsphere) of a regular polygon (polyhedron) $A_1 \cdots A_n$ of circumradius R, then*

$$\sum (PA_i)^2 = 2nR.$$

This follows immediately from (B) by taking $r = R$. This result, for the polygon, appears as Cor. 2, p. 99, in M'Clelland.

(D) *If a_1, a_2, a_2 denote the sides of a triangle $A_1A_2A_3$ of incenter I, inradius r, and area K, and if P is any point on the incircle, then*

$$\sum a_i(PA_i)^2 = 2rK + \sum a_i(IA_i)^2.$$

In Gustin's relation (1) take $k_i = a_i/s$, where $s = a_1 + a_2 + a_3$. Then G coincides with I and we have

$$\sum a_i(PA_i)^2 = sr^2 + \sum a_i(IA_i)^2.$$

But $sr^2 = 2rK$.

This generalization of the given problem appears as ex. 7, p. 102, in M'Clelland.

(E) Rosenbaum pointed out that it can be proved that *if P is a point on a circle concentric with a regular polygon $A_1A_2 \cdots A_n$, then*

$$\sum (PA_i)^{2s} = constant$$

for $s = 1, 2, \cdots, n-1$. For a proof of this see problem 3774 [1938, 190]. A special case is given as ex. 6, p. 101, in M'Clelland, where we find a proof of the fact that

$$\sum (PA_i)^4 = 6nR^4$$

if P is on the circumcircle of the regular polygon.

The Unscrupulous Athletes

E 821 [1948, 364]. *Proposed by B. H. Brown, Dartmouth College*

A sports promoter hired nine not too scrupulous athletes, and formed three cross-country teams A, B, and C, of three men each, which he took on tour for a series of dual and triple meets. If a team could win, it always would; but a losing team could be bribed to run more slowly. Except for this failing, the athletes always ran as automata, and always finished in the same order, with no ties. (In a cross-country meet the winner receives 1 point, the next man 2 points, *etc.* The team score is the sum of the points received by its members, and low score wins.)

In dual meets, team A beat team B, B beat C, and C beat A, much to the annoyance of the promoter who unjustly accused the men of dishonesty.

In an honest triple meet, team A won, whereupon team C went to team B with the following dishonest but logical offer: "No conceivable dishonesty on the part of our C team can enable your B team to beat or even tie A; but if you will slow down, our C team can beat A, and we will divide the profits with you."

Determine completely the composition of each team in terms of the relative excellence of its members.

Solution by Fritz Herzog, Michigan State College. We first introduce the following notations: Let X and Y be any two of the three teams. Then $[XY_i]$ is to denote the number of those members of team X that are faster than the ith member (in order of speed) of team Y, and $[X_iY]$ is to denote the number of those members of team Y that are slower than the ith member of team X, for $i=1, 2, 3$.

Obviously

$$(1)\qquad \begin{aligned} 0 \leqq [XY_1] \leqq [XY_2] \leqq [XY_3] \leqq 3, \\ 3 \geqq [X_1Y] \geqq [X_2Y] \geqq [X_3Y] \geqq 0. \end{aligned}$$

Also, $[XY]$ is to denote the number of those among the nine pairs, consisting of one member of team X and one member of team Y, for which the former is faster than the latter. We have

$$(2)\qquad \sum [XY_i] = \sum [X_iY] = [XY],$$

where the summation is always taken over i from 1 to 3, and

$$(3)\qquad [XY] + [YX] = 9.$$

Let $S_2(X, Y)$ denote the score of team Y minus the score of team X in the dual meet between these two teams, and let $S_3(X, Y)$ denote the same difference in the triple meet.

Now it is easily seen that

$$(4)\qquad S_2(X, Y) = 2[XY] - 9.$$

Furthermore, if Z is the third team, $S_3(X, Y)=S_2(X, Y)+[ZY]-[ZX]$, and so, by (4) and (3),

$$(5)\qquad S_3(X, Y) = 2[XY] - [YZ] - [ZX] = -2[YX] + [ZY] + [XZ].$$

According to the conditions of the problem, the five quantities $S_2(A, B)$, $S_2(B, C)$, $S_2(C, A)$, $S_3(A, B)$, and $S_3(A, C)$ are to be positive. Hence, by (4),

$$(6)\qquad [AB] \geqq 5; \qquad [BC] \geqq 5; \qquad [CA] \geqq 5;$$

and, by (5), $2[AB]-[BC]-[CA]\geqq 1$ and $-2[CA]+[BC]+[AB]\geqq 1$, whence by addition

$$[AB] - [CA] \geqq 1; \tag{7}$$

from (7) and (6),

$$[AB] \geqq 6. \tag{8}$$

Let k_i denote $[AB_i]$, for $i=1, 2, 3$. Then the k_ith member of team A is faster than the ith member of team B, and we conclude that

$$[CA_{k_i}] + [B_iC] \leqq 3, \qquad i = 1, 2, 3, \tag{9}$$

where $[CA_0]$ is to mean zero. Addition of the three inequalities (9) in conjunction with (2) and (6) yields

$$\sum_i [CA_{k_i}] \leqq 4. \tag{10}$$

We shall now investigate the various possibilities for the $k_i=[AB_i]$. In the first place, we show that the three inequalities $k_i \geqq i$ cannot hold simultaneously. For, by (2), (1), and (10), we would then have $[CA]=\Sigma[CA_i]\leqq\Sigma_i[CA_{k_i}]\leqq 4$, which contradicts (6). Now, by (2), (8), and (1), we must have $\Sigma k_i=[AB]\geqq 6$ and $0\leqq k_1\leqq k_2\leqq k_3\leqq 3$. This leaves for k_1, k_2, k_3 only two possibilities, namely, 2, 2, 2 and 0, 3, 3.

If $k_1=k_2=k_3=2$, then (10) gives $3[CA_2]\leqq 4$, hence $[CA_2]\leqq 1$. Therefore, by (2), (6), and (1), we must have $[CA_3]=3$. Since $k_i=[AB_i]=2$ for $i=1, 2, 3$, the result of the dual meet between teams A and B must be $AABBBA$; and since $[CA_3]=3$ all three members of team C are faster than the slowest member of A. Consequently, team C could slow down in such a way that the triple meet would result in the order $AABBBCCCA$, which would mean a tie (12 points each) for A and B. This contradicts the conditions of the problem.

Thus the only possibility left for k_1, k_2, k_3 is 0, 3. 3. Again (10) gives $2[CA_3]\leqq 4$, hence $[CA_3]\leqq 2$. Since $[AB]=6$, we have, by (7), $[CA]=5$. These facts, together with (1) and (2), yield $[CA_1]$, $[CA_2]$, $[CA_3]=1, 2, 2$. If these values of the $[CA_i]$ are substituted in (9) one obtains $[B_1C]\leqq 3$, $[B_2C]\leqq 1$, $[B_3C]\leqq 1$, so that, by (6), $[B_1C]$, $[B_2C]$, $[B_3C]=3, 1, 1$. The nine values of $k_i=[AB_i]$, $[CA_i]$, and $[B_iC]$, obtained above, are seen to be realized only if the (honest) triple meet ends in the order

$$B\ C\ A\ C\ A\ A\ B\ B\ C. \tag{11}$$

We still have to convince ourselves that (11) is actually a solution of the problem. In the dual meets A beats B, B beats C, and C beats A by scores of 9 to 12, 10 to 11, 10 to 11, respectively. In the (honest) triple meet the scores of A, B, C, are 14, 16, 15, respectively. Team B can arrange in numerous ways to let C beat A in the triple meet (for instance, by having the three members of B finish in the last three places so that the scores of C and A would be those of the dual meet between them). However, no matter how the first two members of team C are slowing down they will always decrease the score of A by at least

the same amount as that of B; thus C is unable to cause B to beat or even tie A.

Also solved by Murray Barbour, B. B. Dressler, Roger Lessard, Leo Moser, S. T. Parker, C. W. Trigg, and the proposer.

Editorial Note. In connection with this problem see the proposer's paper "The Scoring of Athletic Contests," *Scientific Monthly*, vol. LXII, pp. 233–237. Following the proposer, let the situation of team A beating team B, B beating C, and C beating A be called the *ring paradox*, and let the situation in which a losing team may control the relative ranking of the others be called the *control paradox*. In most of the solutions to the given problem the solvers first obtained the fifteen instances of the ring paradox that exist for this problem. Of these only three are such that A wins in an honest triple meet. By elimination, only one of these, in turn, is an instance of the control paradox where C may beat A.

Four Distinct Integers

E 823 [1948, 365]. *Proposed by Max LeLeiko, Rutgers University*

Find four distinct non-zero integers a, b, c, d such that

$$\begin{aligned} &a^2 + b^2 + c^2 + d^2 \\ &\quad = (1/5)[(a + b)^2 + (a + c)^2 + (a + d)^2 + (b + c)^2 + (b + d)^2 + (c + d)^2] \\ &\quad = (1/7)[(a + b + c)^2 + (a + b + d)^2 + (a + c + d)^2 + (b + c + d)^2] \\ &\quad = (1/3)(a + b + c + d)^2. \end{aligned}$$

Solution by Margaret Olmsted, Augustana College, Illinois. The three relations are equivalent to $\sum a^2 = \sum ab$. If we set $c = a - b$, $b = a + b$ in this we get $3a = 4b$. Therefore a set of solutions is obtained by taking $a = 4x$, $b = 3x$, $c = x$, $d = 7x$. Since the equations are symmetric in a, b, c, d, any permutation of a solution is also a solution.

Not all solutions are thus obtained, however, as no x will yield the solution 1, 1, 1, 3.

Also solved by Murray Barbour, L. J. Burton, Daniel Finkel, Roger Lessard, Leo Moser, W. V. Parker, and the proposer.

Editorial Note. Parker found a set of solutions given parametrically by $a = r^2$, $b = s^2$, $c = (r \pm s)^2$, $d = r^2 \pm rs + s^2$. This set, like that obtained above, does not contain all solutions of the equations.

This problem is equivalent to problem 4269 [1947, 480], and a general solution may yet be furnished when the solutions to the latter problem are published.

Tetrahedra Circumscribing a Paraboloid of Revolution

E 830 [1948, 427]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The six planes bisecting the adjacent dihedral angles around the base of a

tetrahedron, taken four by four, form fifteen tetrahedra circumscribed about a common paraboloid of revolution.

Solution by the Proposer. It is easy to show that the orthogonal projections of the vertex A of the tetrahedron $ABCD$ on the six planes bisecting the interior and exterior dihedral angles along the edges BC, CD, DB all lie on the plane through the midpoints of the edges AB, AC, AD. Therefore these six planes are all tangent to the paraboloid of revolution having its focus at A and its vertex at the midpoint of the altitude through A. This proves the theorem.

The analogous theorem for the plane may be similarly established.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4336. *Proposed by Orrin Frink, Pennsylvania State College*

Find the arc of fixed length $l>2a$, lying below the x-axis and joining the points $(-a, 0)$ and $(a, 0)$, which includes between itself and the x-axis an area of lowest possible center of gravity. This will be the form actually assumed by a weightless flexible arc supported at its ends if it is holding water.

4337. *Proposed unsigned*

If the numbers R_{nk} are defined by

$$\frac{1-z^2}{\sin \pi z}\prod_{k=2}^{n}\sin(\pi z/k) \equiv \sum_{k=0}^{\infty} R_{nk}z^k,$$

prove that $\underline{\lim}\, R_{nk}^{-1/k}$ is equal to the first prime exceeding n.

4338. *Proposed by R. Bouvaist, Vincelles, Saône-et-Loire, France*

For any given triangle ABC inscribed in a circle (O), there are three points α, β, γ on (O) such that the segments determined by the sides of angle BAC on the tangents to (O) at α, β, γ have α, β, γ, respectively, for their midpoints. Show that the orthocenter of triangle $\alpha\beta\gamma$ is the midpoint of BC.

4339. *Proposed by Paul Erdös, Syracuse University*

Prove that if $n=2^{\mu}(2k+1)$, $k>1$, then 2^n-1 has a composite factor congruent to 1 modulo n.

4340. *Proposed by N. S. Mendelsohn, University of Manitoba*

Let $f(n)$ be the number of distinct equivalence relations connecting n elements. Show that

$$f(n)=\sum_{k=1}^{n}\sum_{j=0}^{k}(-1)^n\frac{(k-j)^n}{j!(k-j)!}=\frac{n^n}{n!}\left(\frac{1}{0!}\right)+\frac{(n-1)^n}{(n-1)!}\left(\frac{1}{0!}-\frac{1}{1!}\right)+\cdots$$
$$+\frac{(n-r)^n}{(n-r)!}\left(\frac{1}{0!}-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots+(-1)^r\frac{1}{r!}\right)+\cdots,$$

and find an asymptotic formula for $f(n)$ as $n\to\infty$.

Note. For equivalence relations and their connection with partitions see, Birkoff and MacLane, *Survey of Modern Algebra*, pp. 159 ff.

SOLUTIONS

A Sphere Related to the Tetrahedron

4206 [1946, 341]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Consider spheres with centers at the vertices of a tetrahedron $ABCD$ and radii equal respectively to k times the sum of the squares of the three opposite edges. Show that the sum of the squares of the distances from the four vertices to the center of the sphere orthogonal to the four spheres is equal to

$$[2(4k+1)]^2R^2-2k(2k+1)\Sigma,$$

where R is the radius of the circumsphere and Σ means the sum of the squares of the six edges. Consider particular cases.

*Solution by the Proposer.** 1. If we designate by a, a', b, b', c, c' the lengths of the edges BC, DA, CA, DB, AB, DC of the tetrahedron $T\equiv ABCD$, then the squares of the radii of the specified spheres (A, m), (B, n), (C, p), (D, q) are defined by

$$m^2=k(a^2+b'^2+c'^2),\qquad n^2=k(a'^2+b^2+c'^2),$$
$$p^2=k(a'^2+b'^2+c^2),\qquad q^2=k(a^2+b^2+c^2).$$

* Translated by W. E. Byrne, Virginia Military Institute.

Let ω and r be the center and radius of the sphere (ω) orthogonal to the spheres (A, m), (B, n), (C, p), (D, q). We have

$$\overline{A\omega}^2 - r^2 = k(a^2 + b'^2 + c'^2), \qquad \overline{B\omega}^2 - r^2 = k(a'^2 + b^2 + c'^2),$$
$$\overline{C\omega}^2 - r^2 = k(a'^2 + b'^2 + c^2), \qquad \overline{D\omega}^2 - r^2 = k(a^2 + b^2 + c^2),$$

from which it follows that

$$\overline{A\omega}^2 - \overline{B\omega}^2 = k[(a^2 - a'^2) - (b^2 - b'^2)], \cdots .$$

But, if G is the centroid of T, we have

$$\overline{AG}^2 = [3(a'^2 + b^2 + c^2) - (a^2 + b'^2 + c'^2)]/16,$$
$$\overline{BG}^2 = [3(a^2 + b'^2 + c^2) - (a'^2 + b^2 + c'^2)]/16,$$

whence

$$\overline{AG}^2 - \overline{BG}^2 = -[(a^2 - a'^2) - (b^2 - b'^2)]/4, \cdots .$$

Hence the quantities $\overline{A\omega}^2 - \overline{B\omega}^2$, $\overline{B\omega}^2 - \overline{C\omega}^2$, $\overline{C\omega}^2 - \overline{D\omega}^2$, $\overline{D\omega}^2 - \overline{A\omega}^2$ are proportional to the quantities $\overline{AG}^2 - \overline{BG}^2$, $\overline{BG}^2 - \overline{CG}^2$, $\overline{CG}^2 - \overline{DG}^2$, $\overline{DG}^2 - \overline{AG}^2$. We obtain thus* the

THEOREM. *The locus of the center ω of the sphere (ω), as k varies, is the line OG which joins the circumcenter and the centroid of T.*

Furthermore,†

$$\overrightarrow{O\omega} = -4k\overrightarrow{OG}.$$

2. Application of Stewart's theorem to the triangle AOG and the point ω gives

$$G\omega \cdot \overline{AO}^2 + O\omega \cdot \overline{AG}^2 = OG \cdot \overline{A\omega}^2 + O\omega \cdot \omega G \cdot OG,$$

or, by the last relation above,

$$\overline{A\omega}^2 = (4k + 1)R^2 - 4k \cdot \overline{AG}^2 + 4k(4k + 1)\overline{OG}^2.$$

If we observe that $\overline{OG}^2 = R^2 - \Sigma/16$, and if we use the previously stated formula for $\overline{AG}^2$, this reduces to

$$\overline{A\omega}^2 = (4k + 1)^2R^2 - k(a'^2 + b^2 + c^2) - k^2\Sigma.$$

Likewise

$$\overline{B\omega}^2 = (4k + 1)^2R^2 - k(a^2 + b'^2 + c^2) - k^2\Sigma,$$
$$\overline{C\omega}^2 = (4k + 1)^2R^2 - k(a^2 + b^2 + c'^2) - k^2\Sigma,$$
$$\overline{D\omega}^2 = (4k + 1)^2R^2 - k(a'^2 + b'^2 + c'^2) - k^2\Sigma.$$

* V. Thébault, Mathesis, 1932, pp. 223–228.

† V Thébault, Mathesis, loc. cit.

Addition of the above equations gives the desired result,

$$\overline{A\omega}^2 + \overline{B\omega}^2 + \overline{C\omega}^2 + \overline{D\omega}^2 = [2(4k+1)]^2R^2 - 2k(2k+1)\Sigma.$$

3. Particular cases. (i) If $k=-\frac{1}{4}$, ω coincides with G, and we have the known result

$$\overline{GA}^2 + \overline{GB}^2 + \overline{GC}^2 + \overline{GD}^2 = \Sigma/4.$$

(ii) If $k=-\frac{1}{2}$, ω coincides with the Monge point M of T, whence‡ we have the

THEOREM. *The sum of the squares of the distances from the Monge point of T to the vertices of T is equal to the square of the diameter of the circumsphere of T.*

(iii) If $k=\frac{1}{2}$, ω coincides with the center M_1 of the Longchamps sphere of T. The expression for

$$r^2 = \overline{A\omega}^2 - k(a^2 + b'^2 + c'^2) = (4k+1)^2R^2 - k(k+1)\Sigma,$$

reduces to§

$$r^2 = 9R^2 - 3\Sigma/4.$$

The Deltoid

4245 [1947, 232]. *Proposed by J. H. Butchart, Arizona State College*

The envelopes of two families of lines, PQ, PQ', making angles of $\pm 30°$ respectively with the tangents to a deltoid at their points of contact P are two deltoids, larger than the given one in the ratio $3^{1/2}:1$. Show that $PQ=PQ'$, where Q, Q' are the points of contact of PQ, PQ' with the respective envelopes, and that the angles between the cusp tangents of the envelopes and the included cusp tangent of the given deltoid are $\pm 10°$.

Solution by R. Goormaghtigh, Bruges, Belgium. We prove a more general result from which the proposed problem follows as a special case. Let P be a variable point on any plane curve Γ, C the center of curvature of Γ at P, ρ the radius of curvature MC, ϕ the angle formed by the tangent of Γ at P and a fixed direction Δ. Consider a straight line PQ making a constant angle α with the tangent to Γ at P, Q being the contact point of that straight line with its envelope (Q).

For any straight line l, invariably attached to the tangent and the normal to Γ at P, the normal to the envelope of l at the contact point passes through C. Hence Q is the projection of C on PQ. This shows that, for any curve Γ, the points Q and Q' corresponding to any two angles α and $-\alpha$ are such that $PQ=PQ'$. Further, if (C) is the evolute of Γ and C_1 is the center of curvature of (C) at C, the angle formed by QC and the tangent PC to (C) being also α, then the point of contact of QC with its envelope, i.e. the center of curvature of (Q), is the projection of C_1 on QC. Hence, the radius of curvature of (Q) at Q is

‡ V. Thébault, Gazeta Matematica (Bucarest), 1933, p. 86.

§ V. Thébault, Mathesis, loc. cit.

$$R = \rho \cos \alpha + \rho_1 \sin \alpha,$$

ρ_1 being the radius of curvature CC_1 of (C).

Let us now consider the case when Γ is any cycloidal curve (i.e. cycloid, epicycloid or hypo-cycloid*), whence

$$\rho = a \sin n\phi,$$

a being a constant. Then we have

$$\rho_1 = d\rho/d\phi = an \cos n\phi$$

and

$$R = a(\sin n\phi \cos \alpha + n \cos n\phi \sin \alpha);$$

or, if ϕ' is the angle $\phi+\alpha$ formed by the tangent to (Q) and Δ, and if β is the angle such that $\tan \beta = n \tan \alpha$, we find

$$R = a \frac{\cos \alpha}{\cos \beta} \sin n(\phi' - \alpha + \beta/n).$$

Therefore, *if* Γ *is a cycloidal curve of index* n, *the straight line passing through a variable point* P *of* Γ *and making with the tangent at that point a constant angle* α *envelopes a cycloidal curve of the same index*; *if* β is the angle such that $\tan \beta = n \tan \alpha$, the ratio r of similitude of the second cycloidal curve to the first is $\cos\alpha/\cos \beta$, and the orientation of the second differs from that of the first by an angle $\alpha-\beta/n$.

When, as in the present problem, Γ is a deltoid, $n=3$; if $\alpha=\pm 30°$, then $\beta=\pm 60°$, $r=3^{1/2}$, $\alpha-\beta/n=\pm 10°$. In the general case, when α is chosen so that $\tan \alpha = n^{-1/2}$, then $r=n^{1/2}$.

Arithmetic Progressions

4257 [1947, 346]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In any arithmetic progression show that the difference (sum) of the product of n consecutive terms and the product of n other consecutive terms is always divisible, provided n is even (odd), by the sum of the greatest and least of the terms. Prove also the corollary: If a and b are positive integers and if $a+b+1=p$ is prime, then

$$a!b! \pm 1 \equiv 0 \pmod{a+b+1},$$

the sign being $+$ or $-$ according as a and b are even or odd.

Solution by P. A. Clement, Student, University of California at Los Angeles. For any two integers r and s we have $r\equiv -s \bmod (r+s)$, $r+d\equiv -(s-d)$, $\cdots$, $r+(n-1)d\equiv -[s-(n-1)d] \bmod (r+s)$. Then, by multiplication

* See Yates, A Handbook on Curves and Their Properties, 1947, pp. 81–85, 126.—Ed.

$$r(r+d)\cdots[r+(n-1)d] \equiv (-1)^n s(s-d)\cdots[s-(n-1)d] \bmod (r+s).$$

This is equivalent to the proposed problem. One notes that the two sets of consecutive terms need not be taken from the same progression (so long as the two progressions have the same common difference) and that the statement "by the sum of the greatest of either set and the least of the other" could replace the corresponding statement in the proposal.

Now in the above result put $d=1$, $s=a+b$, $r=1$, $n=x$. Then we have

$$a! \equiv (-1)^a(b+a)(b+a-1)\cdots(b+1), \bmod (b+a+1).$$

Multiplication by $b!$ gives

$$a!b! \equiv (-1)^a(b+a)! \equiv (-1)^{a+1} \bmod (b+a+1),$$

where the last step follows by Wilson's theorem since $a+b+1$ is prime. If $b+a>1$, then $b+a$ is even, and a and b are both even or both odd, and the congruence is thus equivalent to the proposed corollary. In the trivial case $a+b=1$, the sign is immaterial.

The corollary is equivalent to a problem in Uspensky and Heaslet, *Elementary Number Theory*, p. 157, no. 3.

Also solved by Murray Barbour, N. J. Fine, William Gustin, Roger Lessard. C. R. Phelps, and the Proposer.

Infinite Series and Product

4259 [1947, 418]. *Proposed by Richard Bellman, Princeton University*

If

$$\sum_{k=1}^{\infty} \frac{n_k x^{n_k}}{1+x^{n_k}} = x\prod_{k=1}^{\infty}(1+x^{n_k}), \qquad |x|<1,$$

show that except perhaps for order $n_k=2^k$.

Solution by Shih-fang Li, Yenching University, Peiping, China. From the hypothesis, $n_k \geqq 0$ for all k. Since

$$f_k(x) \equiv x^{n_k}$$

are all analytic for $|x|<1$, we know that the product

$$F(x) \equiv \prod_{k=1}^{\infty}(1+x^{n_k})$$

converges everywhere in the unit circle and is itself analytic there.

By the given equality, since $F(x)\neq 0$ for $|x|<1$, we have*

* For theorems employed in this solution see K. Knopp, *Theory and Application of Infinite Series*, pp. 437, 438.

$$\frac{F'(x)}{F(x)} = \sum_{k=1}^{\infty} \frac{f_k'(x)}{1 + f_k(x)} = F(x). \tag{1}$$

Solving and using the condition $F(0)=1$, we obtain

$$F(x) = (1 - x)^{-1} = \sum_{k=0}^{\infty} x^k. \tag{2}$$

On the other hand, we may expand $F(x)$ in a power series. The corresponding finite product is

$$\begin{aligned} P_s(x) \equiv \prod_{k=1}^{s} (1 + x^{n_k}) = 1 + x^{n_1} + x^{n_2} + x^{n_1+n_2} + x^{n_3} \\ + x^{n_1+n_3} + x^{n_2+n_3} + x^{n_1+n_2+n_3} + \cdots + x^{n_1+n_2+n_3+\cdots+n_1}. \end{aligned} \tag{3}$$

Comparing (3) with (2), we have, except for order, $n_1=1$, $n_2=2$, $n_3=4$, $n_4=8$, $\cdots$, and by induction $n_k=2^{k-1}$ (instead of 2^k as given in the proposal).

Also solved by P. T. Bateman, M. S. Klamkin, and the Proposer.

Upper Density of a Sequence

4268 [1947, 479]. *Proposed by Paul Erdös, Syracuse University*

Let $a_1<a_2<\cdots$ be an infinite sequence of integers of upper density greater than $1/k$. (Denote by $f(n)$ the number of a's up to n, then the upper density is defined as $\overline{\lim} f(n)/n$ as $n\to\infty$.) Then for suitable t the equation

$$a_t = a_{i_1} + a_{i_2} + \cdots + a_{i_r}, \qquad 1 < r < k$$

is solvable. In fact, there are infinitely many t with this property.

Solution by the Proposer. Consider the k sequences

$$a_1 + a_j,\ a_1 + a_2 + a_j,\ \cdots,\ a_1 + a_2 + \cdots + a_k + a_j; \quad j = k + 1,\ k + 2, \cdots.$$

Each of them has upper density greater than $1/k$ and is a translation of the sequence $a_{k+1}, a_{k+2}, \cdots$. Thus the integers represented by them cannot all be different. Hence we must have

$$a_1 + a_2 + \cdots + a_{l_1} + a_{j_1} = a_1 + a_2 + \cdots + a_{l_2} + a_{j_2}.$$

Assume (without loss of generality) $l_2>l_1$. Then

$$a_{j_1} + a_{l_1+1} + a_{l_1+2} + \cdots + a_{l_2} + a_{j_2}$$

as required. It is easy to see that a_{j_1} can be found in infinitely many ways.

The integers $\equiv 1 \pmod{k}$ show that the theorem is false if the density is $1/k$.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

College Algebra. By H. A. Simmons. New York, The Macmillan Company, 1948. 10+619 pages. $4.00.

This book covers the topics usually given in college algebra. It seems to be an attempt to make a textbook that the college freshman can read himself. In the preface the author states, "In experience, we have found that the danger of over-explaining in *College Algebra* is mild in comparison with the opposite danger. Consequently, instead of attempting merely to give rigorous proofs with a minimum of words, we have tried to give rigorous proofs and at the same time add the extra word, phrase, illustration, or example that serves to 'put the idea across' immediately to the student."

This is a very laudable aim. However the reviewer feels the result is unnecessarily wordy. For example, the chapter on exponents and radicals contains 36 pages, while most standard texts cover the same material in less than half that many. Negative exponents are defined twice; on page 152 and again on page 161.

The text seems to emphasize rules for doing particular things rather than general principles, for example, the three sets of exercises for story problems leading to linear equations in one unknown.

A feature of the book is the large number of well graded exercises. There are so many that the answers (given for odd numbered problems only) cover 34 pages. The function idea is stressed throughout the book and in fractions it is always emphasized that the variables cannot assume values to make the denominator zero.

J. A. WARD

Mathematics of Finance. By J. B. Linker and M. A. Hill, Jr. New York, Henry Holt and Company, 1948. 8+171+83 pages. $2.90.

This book fits into the established pattern of text-books on the Mathematics of Finance. One of the authors' stated purposes is to make students "rely more upon their own efforts than to depend upon the instructor to explain each new step." Insofar as this formidable task can be accomplished through careful, explicit presentation, together with strategic use of line diagrams and illustrative examples, this text should be successful.

The explanation of the amortization schedule and its construction is perhaps better than average. The treatment of the method of equation of value, on the other hand, should probably be more explicitly set forth.

The subject of annuities is developed by stages. The reader is led through

the cases devolving upon how the size of the payment period compares with the size of the conversion period, arriving finally at the so-called general annuity. Further classification is made under the headings ordinary annuity, annuity due, and deferred annuity.

The topic of approximate investors' rate by the average net income method (usually presented in the chapter on bonds) has been omitted.

The problems are abundant and carefully stated, even if somewhat lacking in variety.

There is a 31 page introductory treatment of probability and life insurance.

In a 12 page appendix, the author gives an adequate summary of the elementary algebraic tools used in the text.

G. F. ROSE

Intermediate Algebra for Colleges. By W. L. Hart. Boston, D. C. Heath and Co., 1948. 7+316 pages. $2.50.

This is the most recent addition to the author's series of textbooks on algebra intended for first year college students. It is the most elementary one of the series and is intended to be suitable for students who have forgotten practically all of the techniques of algebra learned in high school. The topics treated are those normally covered in three semesters of high school algebra and the book is designed either to prepare students for the study of college algebra or to serve as preparation for courses in other fields which require a knowledge of elementary algebra as a prerequisite.

In common with practically all books of this kind the emphasis is on problem solving and the development of techniques, and scant attention is given to the number system of algebra or to a logical foundation for algebra. Irrational numbers, for example, are not even mentioned in the first half of the book. However, the author gives an excellent presentation of elementary algebraic techniques and the book will no doubt prove to be useful in many colleges which feel the need for courses of this type.

H. P. EVANS

Higher Algebra. By W. L. Ferrar. New York, Oxford University Press, 1948. 6+315 pages. $5.00.

This book is a sequel to the author's *Higher Algebra for Schools.* Like its predecessor, it is addressed to a superior group of British students and is not adapted to the traditional algebra courses given in American colleges.

The book consists of ten chapters dealing with the following subjects: finite series, infinite series and approximations, complex numbers, difference equations and generating functions, theory of equations, partial fractions, inequalities, and continued fractions.

The topics are few in number but are treated in great detail, and some are developed at greater length than in comparable textbooks. Among these may be

mentioned the exposition of linear functions of a complex variable, usually reserved for works on function-theory; the existence theorems for partial fractions; and the inequalities associated with the names of Maclaurin, Tchebychef, Weierstrass, Hölder, and Minkowski.

The book contains an abundance of well-chosen illustrative examples and classified problems, and some excellent advice on how to attack difficult problems. It is well adapted to individual study and is recommended to ambitious students desiring to cultivate algebraic techniques and to those preparing for competitive examinations in mathematics.

LOUIS WEISNER

Algebra for College Students. By J. R. Britton and L. C. Snively. New York, Rinehart and Co., 1947. 11+529 pages. $3.00.

Somewhat more than half of this volume has been published separately under the title *Intermediate Algebra* and has already been reviewed (*cf.* this MONTHLY, vol. 55, p. 376 (1948)). The present review will confine attention to the remainder of the text.

Typical topics include approximate numbers, logarithmic and exponential equations, investment problems, determinants, probability. Complex numbers are at first associated with vectors, afterwards treated in polar form for the student with trigonometric training. The chapter on inequalities is unusually full—eighteen pages long. The theory of equations includes, along with the elementary topics, a derivation of Descartes' rule of signs and considerable attention to numerical methods such as Horner's and the method of successive approximations, but excludes the algebraic solution of cubic and quartic equations. In regard to mathematical induction, a proof utilizes three steps, of which the second is showing the inheritance property. Review lists which occasionally finished chapters early in the book are noticeably absent in the latter half.

In the preface we read: "efficiency in manipulation alone is not the aim of the book. The major stress is on the important underlying ideas." Nevertheless most of the exercises for the student seem to be routine applications of methods developed in the text. Little opportunity is provided the student to learn to reason in algebra.

As noted in the *Intermediate Algebra* review, the writing is smooth and flowing. In conformity with this style, new material is frequently introduced from an intuitive approach, with perhaps a more rigorous viewpoint being adopted later. This mixture occasionally produces disturbing logical complications.

R. A. GOOD

NEW BOOKS RECEIVED

Introduction to the Algebraic Geometry of a Plane. By J. W. Archbold. London, Edward Arnold, 1948. 8+300 pages. $6.00.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS, 1947–48

Kappa Mu Epsilon, Mount St. Scholastica College

The year's theme for *Kansas Gamma* Chapter of *Kappa Mu Epsilon* was *The true value of science in the liberal arts program.* The following discussions were heard:

True value of science in the liberal arts program, by Sister Helen Sullivan, O.S.B.

Student obligation to evaluate the role of science, by Frances Knightley

Is full academic life possible without a correct understanding of the hierarchy of the branches of knowledge?, by Victoria Fritton

Mathematics as a logical system of thought with a unique position in the hierarchy of knowledge, by Jean Moran

Non-Euclidean geometries,—foundations and development, by Sister Marietta Leuken, O.S.B.

The contribution of the particular sciences to learning, by Elaine Carson and Carrie Nelle Bremmer

Constructions with ruler and compass—three classical problems, by Gloria Jaskowiak.

Kansas *Gamma* was host to the four Kansas Chapters of Kappa Mu Epsilon at Mt. St. Scholastica College. About forty active members of Kappa Mu Epsilon attended this meeting to discuss local fraternity problems.

Officers for 1948–49 are: President, Gloria Jaskowiak; Vice-President, Gertrude Harrison; Secretary, Noreen Hurter; Treasurer, Mary Alice Weir; Publicity Chairman, Frances Donlon; Chapter Musician, Mary Jane Martin; Corresponding Secretary and Faculty Sponsor, Sister Helen Sullivan, O.S.B.

Kappa Mu Epsilon, Bowling Green State University

The following papers were presented to the *Ohio Alpha* Chapter of *Kappa Mu Epsilon*:

The history of aims of the national and local organizations of Kappa Mu Epsilon, by Prof. F. C. Ogg

A phase of the relativity theory, by Prof. D. W. Bowman of the Physics Department

Mathematics in economics and social science, by Prof. L. F. Manhart

Mathematics in industry, by Prof. D. M. Krabill

Mathematics in business, by Prof. H. R. Mathias

Mathematics in chemistry, Lewis Miller of the Chemistry Department

Teaching of mathematics, by Prof. Martha Gesling of the Department of Education

Beauty in mathematics, by Prof. F. C. Ogg.

The officers for next year are: President, Dallas Henry; Vice-President, Gordon Domick; Secretary, Donna Stroh; Treasurer, Arthur Miller; Advisor, F. C. Ogg; Corresponding Secretary, H. R. Mathias.

Pi Mu Epsilon, University of Oklahoma

The *Oklahoma Alpha* Chapter of *Pi Mu Epsilon* heard the following papers during 1947–48:

The development of mathematics, by Dr. C. E. Springer

Hyperbolic functions, by Dr. Arthur Bernhart

Greek astronomy, by Mr. Dewey McKnelly

The life of Gauss, by Mr. Truman Wester

Mathematical oddities and paradoxes, by Mr. L. D. Gregory

The essence of relativity, by Mr. R. B. Deal.

At the annual *Pi Mu Epsilon* banquet, the principal speaker was Dr. Lawrence H. Snyder, Dean of the Graduate College, who spoke on *Heredity and modern life.*

Kappa Mu Epsilon, Central Michigan College of Education

Several social gatherings and regular meetings were held by the *Michigan Beta* Chapter of *Kappa Mu Epsilon* during 1947–48. Among the papers presented were:

Star solids, by Miss Gertrude Pratt

Mathematical paradoxes, by James Laux.

The officers for 1948–49 are: President, William Kumbier; Vice-President, Jarold Brown; Secretary, Frances Woodbury; Treasurer, George Germain; Corresponding-Secretary, Mr. D. A. Sudborough.

Pi Mu Epsilon, Oregon State College

The *Oregon Beta* Chapter of *Pi Mu Epsilon* held five evening meetings during the academic year 1947–48. Programs were as follows:

Semantics of relativity, by F. H. Young

Five regular solids, by J. L. Ericksen

Sequential analysis, by J. F. Kahn

The solution of calculus problems by means of soap films, by E. E. Adams, and H. F. White

Finite summations, by R. D. Stalley.

There were forty-four members initiated during the year.

The officers for 1947–48 were: President, N. B. Smith; Vice-president, L. R. Stark; Secretary, Miss Marjorie Sims; Treasurer, Prof. G. A. Williams.

The officers elected for 1948–49 are: President, LaVerne Rickard; Vice-President, P. E. Harper; Secretary, F. Illig; Treasurer, Prof. G. A. Williams.

Pi Mu Epsilon, Ohio State University

The *Pi Mu Epsilon* Chapter at Ohio State University heard the following talks during 1947–48:

Perron's solution of the problem of Kakeya, by Dr. J. W. T. Youngs of Indiana University

Infinity, by Dr. R. L. Swain

Sequences of integers, by Dr. W. Scott

Electronic digital computers, by Dr. H. D. Huskey (Bureau of Standards, Washington, D. C.).

The following officers were elected for the year 1947–48: Faculty adviser, Prof. E. J. Mickle; Director, W. M. Myers; Vice-Director, J. E. Adney; Secretary-treasurer, R. A. Dean.

Graduate Mathematics Club, Ohio State University

The following talks were presented to the *Graduate Mathematics Club* during 1947–48:

On the extension of transformations, by Dr. E. J. Mickle

Length and area, by Dr. L. C. Young (University of Capetown, South Africa)

The structure of abstract algebras, by Prof. G. Birkhoff (Harvard University)

Introduction to distance in an abstract space, by Dr. M. Frechet (Paris, France)

On integral extensions of a commutative ring, by Dr. Harold Charland

Generalization of theorems, by Dr. H. Blumberg

Geometry, retrospect and prospect, by Dr. M. Hall

Analytic methods in non-linear mechanics, by Dr. C. E. Sealander

Topology of measure spaces, by Dr. O. W. Rechard (University of Wisconsin)

Representation problem for Frechet surfaces, by Dr. J. W. T. Youngs (Indiana University)

The present state of electronic digital computers, by Dr. H. D. Huskey (Bureau of Standards, Washington, D. C.).

Kappa Mu Epsilon, Montclair State Teachers College

The *New Jersey Beta* Chapter of *Kappa Mu Epsilon* at Montclair State Teachers College heard the following talks given at monthly meetings:

Einstein's theory of relativity

A discussion of the mathematics convention at Atlantic City

Artillery mathematics

The trends of education in England, by Dr. D. R. Davis

Non-Euclidean geometry

Mathematical tricks, by Dr. I. Brune.

Special events of the year included a joint Christmas party with Sigma Phi Mu, the mathematics club, and a banquet given in honor of Dr. Howard Fehr, Professor of Mathematics, who is leaving to join the Mathematics Department of Columbia University Teachers College.

Officers for the year of 1948–49 are: President, Robert Lundquist; Vice-President, Erwin Winguth; Secretary, May Christensen; Treasurer, Gloria Senapole; Corresponding Secretary, George Kays.

Mathematics Club, Montclair State Teachers College

Papers presented to the *Mathematics Club, Sigma Phi Mu,* during the year were:

Semantics, by Dr. I. Brune
Construction of polyhedra, by Wilma Freeze
Topology, by Clifford Swisher
The abacounter, by Dr. N. Lazar of Columbia
Mathematics in life insurance, by G. C. Campbell
Atomic energy, by Dr. Alyea of Princeton University.

The social program consisted of a fencing match, the annual Christmas Party with *Kappa Mu Epsilon,* and the annual Picnic.

Officers for the year of 1948–49 are: President, June Boswell; Vice-President, Joan Alexander; Secretary, Evelyn Pass; Treasurer, Herbert Gebner.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

THE JOURNAL OF SOUTHEASTERN RESEARCH

A new publication, *The Journal of Southeastern Research,* will provide Southeastern engineers and scientists their own medium for recording research results. The Technical Section of the new journal will present articles of interest to research experts only; its News Section will present factual reports for business executives who wish to keep informed about scientific progress in the region. The first issue was published in January, 1949.

Dr. W. G. Pollard, Director, Oak Ridge Institute for Nuclear Studies, is one of the fourteen Southeastern scientists who has been named to the Advisory Board of the Journal.

PERSONAL ITEMS

Professor Tibor Rado of Ohio State University is the first appointee to the new rank of University Research Professor. To accept the new appointment, he resigned from his position as chairman of the Department of Mathematics.

Mr. Rudolph Feige has been appointed to an instructorship at the University of Cincinnati.

Professor W. W. Gandy of Northwestern State College, Louisiana, has been appointed to an instructorship at Agricultural and Mechanical College of Texas.

Mr. W. T. Guy, Jr. of the University of Texas has received an appointment as graduate assistant at California Institute of Technology.

Mr. Joseph Hilsenrath of the Naval Ordnance Laboratory has accepted a

position as scientific educational advisor with the National Bureau of Standards.

Professor Walter Lyche of Wartburg College has been appointed to an instructorship at Augustana College.

Professor W. E. Milne, head of Department of Mathematics, Oregon State College, is on leave of absence and is spending the year at the Institute of Numerical Analysis.

Miss Margaret Owchar, formerly teaching assistant at the University of Minnesota, has been appointed to an instructorship at Rockford College.

Associate Professor E. D. Rainville of the University of Michigan is on sabbatical leave and is located at the California Institute of Technology.

Professor P. V. Reichelderfer of Ohio State University has been named Acting Chairman of the Department of Mathematics.

Mr. Abraham Rosenfeld has been appointed Training Officer, Physics and Mathematics, at the Ordnance School, Aberdeen Proving Ground, Maryland.

Dr. Andrew Sobczyk of Watson Laboratories has been appointed to an assistant professorship at Boston University.

Assistant Professor D. B. Sumner of Louisiana State University and Agricultural and Mechanical College has been appointed to a lectureship (temporary) at the University of Toronto.

Assistant Professor S. L. Thompson of Alabama Polytechnic Institute has been promoted to an associate professorship.

Dr. Annita Tuller of Hunter College has been promoted to an assistant professorship.

Dr. Herschel Weil of Brown University has accepted a position as a mathematician with the General Electric Company, Schenectady, New York.

Professor Emeritus W. S. Hall of Lafayette College died December 17, 1948 at the age of eighty-seven years. He was a charter member of the Association.

Dr. Frank Irwin, associate professor emeritus of the University of California and a charter member of the Association, died on December 25, 1948 at the age of eighty years.

Dr. W. D. MacMillan, professor emeritus of astronomy and mathematics, University of Chicago, died November 14, 1948 at the age of seventy-seven years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE THIRTY-SECOND ANNUAL MEETING OF THE ASSOCIATION

The thirty-second annual meeting of the Mathematical Association of America was held at Ohio State University, Columbus, Ohio, on Friday, December 31, 1948, in conjunction with the annual meetings of the American Mathematical

Society, the Association for Symbolic Logic, and the National Council of Teachers of Mathematics. About five hundred and seventy-eight persons attended the meetings, including the following three hundred and sixty-three members of the Association:

V. W. ADKISSON, University of Arkansas
J. E. ADNEY, JR., Ohio State University
R. P. AGNEW, Cornell University
A. A. ALBERT, University of Chicago
H. H. ALDEN, Ohio State University
C. B. ALLENDOERFER, Haverford College
E. W. ANDERSON, Iowa State College
R. D. ANDERSON, University of Pennsylvania
R. V. ANDREE, University of Wisconsin
MAX ASTRACHAN, Antioch College
FRANK AYRES, JR., Dickinson College
W. L. AYRES, Purdue University
GLADYS F. BADGER, Roosevelt High School, Chicago
A. H. BAILEY, Georgia Institute of Technology
N. H. BALL, U. S. Naval Academy
F. R. BAMFORTH, Ohio State University
T. A. BANCROFT, Alabama Polytechnic Institute
GRACE M. BAREIS, Ohio State University
I. A. BARNETT, University of Cincinnati
JUNA L. BEAL, Butler University
H. M. BEATTY, Ohio State University
J. H. BELL, Michigan State College
W. J. BELLMER, University of Dayton
A. A. BENNETT, Brown University
THEODORE BENNETT, Marietta College
W. D. BERG, Kenyon College
Brother ALFRED BERNARD, Manhattan College
BARBARA B. BETTS, D. C. Heath and Co.
WILLIAM BETZ, University of Rochester
R. H. BING, University of Wisconsin
A. H. BLACK, Southern Illinois University
C. J. BLACKALL, University of Toledo
HENRY BLUMBERG, Ohio State University
L. M. BLUMENTHAL, University of Missouri
R. P. BOAS, JR., Math. Reviews
J. G. BOWKER, Middlebury College
M. G. BOYCE, Vanderbilt University
G. F. BRADFIELD, De Paul University
HELEN J. BRADLEY, University of Tennessee
A. T. BRAUER, University of North Carolina
RICHARD BRAUER, University of Michigan
H. E. BRAY, Rice Institute
H. W. BRINKMANN, Swarthmore College
J. R. BRITTON, University of Michigan
FOSTER BROOKS, Research and Development Board, U. S. Army
K. E. BROWN, University of Tennessee
R. S. BURINGTON, Bureau of Ordnance, Navy Department
I. W. BURR, Purdue University
HERBERT BUSEMANN, University of Southern California
L. E. BUSH, College of St. Thomas
JEWELL H. BUSHEY, Hunter College
W. H. BUSSEY, University of Minnesota
S. S. CAIRNS, University of Illinois
R. H. CAMERON, University of Minnesota
C. C. CAMP, University of Nebraska
V. B. CARIS, Ohio State University
A. B. CARSON, Army Air Forces Institute of Technology
J. W. CELL, North Carolina State College
R. V. CHURCHILL, University of Michigan
B. B. CLARK, University of Michigan
J. A. CLARKSON, Tufts College
MARY D. CLEMENT, University of Miami
ESTHER A. COMPTON, Cumberland College
J. A. COOLEY, University of Tennessee
W. F. CORNELL, Bowling Green State University
N. A. COURT, University of Oklahoma
R. R. COVEYOU, Oak Ridge National Laboratories
W. H. H. COWLES, Pratt Institute
JANE S. CRONIN, Princeton University
H. E. CRULL, Butler University
J. C. CURRIE, Alabama Polytechnic Institute
H. B. CURRY, Pennsylvania State College
WAYNE DANCER, University of Toledo
VIOLET B. DAVIS, University of Toledo
Rev. L. A. V. DECLEENE, St. Norbert's College
C. H. DENBOW, U. S. Naval Postgraduate School
A. H. DIAMOND, Oklahoma A. & M.
R. P. DILWORTH, California Institute of Technology
H. L. DORWART, Washington & Jefferson College
J. E. DOTTERER, Manchester College
Rev. W. C. DOYLE, Rockhurst College
ARNOLD DRESDEN, Swarthmore College
W. L. DUREN, Tulane University

E. L. EAGLE, University of Arkansas
E. D. EAVES, University of Tennessee
P. D. EDWARDS, Ball State Teachers College
SAMUEL EILENBERG, Columbia University
H. P. EVANS, University of Wisconsin
G. M. EWING, University of Missouri
A. B. FARNELL, Princeton University
F. D. FAULKNER, University of Michigan
WILLIAM FELLER, Cornell University
H. E. FETTIS, Air Material Command
F. A. FICKEN, University of Tennessee
N. J. FINE, University of Pennsylvania
C. D. FIRESTONE, Rutgers University
M. P. FOBES, College of Wooster
CLARENCE FORD, Male High School, Louisville
L. R. FORD, Illinois Institute of Technology
TOMLINSON FORT, University of Georgia
J. S. FRAME, Michigan State College
ORRIN FRINK, JR., Pennsylvania State College
W. A. GAGER, University of Florida
B. E. GATEWOOD, Army Air Forces of Technology
H. M. GEHMAN, University of Buffalo
Rev. F. J. GERST, Loyola University
B. P. GILL, College of the City of New York
J. W. GIVENS, University of Tennessee
A. M. GLEASON, Harvard University
B. C. GLOVER, Otterbein College
E. L. GODFREY, Defiance College
MICHAEL GOLDBERG, Bureau of Ordnance, Navy Department
MICHAEL GOLOMB, Purdue University
A. W. GOODMAN, Rutgers University
RUTH E. GOODMAN, Duquesne University
S. T. GORMSEN, University of Florida
M. J. GOTTLIEB, Newark College of Engineering
S. H. GOULD, Purdue University
F. G. GRAFF, Oberlin College
L. M. GRAVES, University of Chicago
J. B. GREELEY, Utica College of Syracuse University
J. W. GREEN, University of California
L. J. GREEN, Case Institute of Technology
LAURA Z. GREENE, Washburn University
V. G. GROVE, Michigan State College
W. S. GUSTIN, Indiana University
V. H. HAAG, Hershey Junior College
B. F. HADNOT, University of Georgia
FRANKLIN HAIMO, Washington University
MARSHALL HALL, JR., Ohio State University
P. R. HALMOS, University of Chicago
FRANK HARARY, University of Michigan
B. T. HARRIS, The Macmillan Co.
W. L. HART, University of Minnesota
E. E. HASKINS, Fenn College
J. O. HASSLER, University of Oklahoma
G. E. HAY, University of Michigan
C. T. HAZARD, Purdue University
C. H. HEINKE, Capital University
R. G. HELSEL, Ohio State University
FRITZ HERZOG, Michigan State College
E. H. C. HILDEBRANDT, Northwestern University
T. H. HILDEBRANDT, University of Michigan
EINAR HILLE, Yale University
J. J. L. HINRICHSEN, Iowa State College
CLARICE HOBENSACK, Western Hills High School, Cincinnati
H. K. HOLT, Union College
D. L. HOLL, Iowa State College
CARL HOLTOM, Army Air Forces Institute of Technology
E. MARIE HOVE, Hofstra College
Rev. J. A. HRATZ, St. Ambrose College
R. C. HUFFER, Beloit College
RALPH HULL, Purdue University
P. M. HUMMEL, University of Alabama
W. R. HUTCHERSON, Northwestern State College of Louisiana
L. C. HUTCHINSON, Brooklyn Polytechnic Institute
M. A. HYMAN, Naval Ordnance Laboratories
S. J. JASPER, Kent State University
R. L. JEFFERY, Queen's University
E. D. JENKINS, Kent State University
WALTER JENNINGS, U. S. Naval Postgraduate School
A. W. JONES, Rensselaer Polytechnic Institute
B. W. JONES, University of Colorado
MARGARET E. JONES, Ohio State University
P. S. JONES, University of Michigan
MARK KAC, Cornell University
H. S. KALTENBORN, Memphis State College
SIDNEY KAPLAN, Naval Ordnance Laboratories
IRVING KAPLANSKY, Institute for Advanced Study
H. T. KARNES, Louisiana State University
CHOSABURO KATO, Denison University
M. W. KELLER, Purdue University
J. L. KELLEY, University of California
K. D. KELLY, Fenn College
A. J. KEMPNER, University of Colorado
J. R. F. KENT, Triple Cities College
S. C. KLEENE, University of Wisconsin

J. R. KLINE, University of Pennsylvania
P. A. KNEDLER, State Teachers College, Kutztown, Pa.
L. C. KNIGHT, JR., Muskingum College
L. A. KNOWLER, University of Iowa
D. M. KRABILL, Bowling Green State University
MAX KRAMER, University of Illinois
H. W. KUHN, Ohio State University
O. E. LANCASTER, Bureau of Ordnance, Navy Department
A. E. LAMPEN, Hope College
R. E. LANGER, University of Wisconsin
GILLIE A. LAREW, Randolph-Macon Woman's College
E. H. LARGUIER, Spring Hill College
H. D. LARSEN, Albion College
C. G. LATIMER, Emory University
V. V. LATSHAW, Lehigh University
W. I. LAYTON, Alabama Polytechnic Institute
J. S. LEECH, University of Chicago
JOSEPH LEHNER, Hydrocarbon Research, Inc.
A. J. LEWIS, University of Denver
F. A. LEWIS, University of Alabama
MARY BETH LIEBERKNECHT, Iowa State College
B. J. LOCKHART, U. S. Naval Postgraduate School
CHARLES LOEWNER, Syracuse University
L. L. LOWENSTEIN, Kent State University
W. C. LOWRY, Kent State University
C. C. MACDUFFEE, University of Wisconsin
SAUNDERS MACLANE, University of Chicago
INGO MADDAUS, JR., Union College
C. G. MAPLE, North Texas State Teachers College
MORRIS MARDEN, University of Wisconsin
R. H. MARQUIS, Ohio University
W. T. MARTIN, Massachusetts Institute of Technology
MARGARET E. MARTINSON, Washburn University
J. R. MAYOR, University of Wisconsin
N. H. MCCOY, Smith College
S. W. MCCUSKEY, Case Institute of Technology
W. C. MCDANIEL, Southern Illinois University
P. E. MEADOWS, Carroll College
A. E. MEDER, JR., Rutgers University
H. E. MENKE, Heidelberg College
G. M. MERRIMAN, University of Cincinnati
E. J. MICKLE, Ohio State University
H. J. MILES, University of Illinois
D. D. MILLER, University of Tennessee
F. H. MILLER, Cooper Union
L. H. MILLER, Ohio State University
W. E. MILNE, Oregon State College
H. J. MISER, Williams College
B. E. MITCHELL, Millsaps College
C. N. MOORE, University of Cincinnati
F. R. MORRIS, Fresno State College
R. D. MORRISON, Oklahoma A & M
D. C. MORROW, Wayne University
MARSTON MORSE, Institute for Advanced Study
W. B. MOYE, Georgia Teachers College
C. E. MULLAN, Duquesne Light Co., Pittsburgh
J. R. MUSSELMAN, Western Reserve University
W. J. NEMEREVER, University of Michigan
GRETA NEUBAUER, University of Wyoming
C. V. NEWSOM, University of State of New York
E. P. NORTHROP, University of Chicago
F. S. NOWLAN, University of Illinois
C. O. OAKLEY, Haverford College
G. G. O'BRIEN, Washington Missionary College
RUFUS OLDENBURGER, De Paul University
L. F. OLLMANN, Hofstra College
EMMA J. OLSON, Kent State University
MORRIS OSTROFSKY, Duquesne University
E. R. OTT, RUTGERS UNIVERSITY
F. W. OWENS, Pennsylvania State College
HELEN B. OWENS, Pennsylvania State College
GORDON PALL, Illinois Institute of Technology
W. V. PARKER, University of Georgia
H. C. PARRISH, Ohio State University
PHILIP PEAK, Indiana University
SALLIE E. PENCE, University of Kentucky
P. M. PEPPER, University of Notre Dame
MARY PETTUS, Lander College
O. L. PHILLIPS, Mississippi Southern College
A. E PITCHER, Lehigh University
J. C. POLLEY, Wabash College
G. B. PRICE, University of Kansas
D. W. PUGSLEY, Berea College
TIBOR RADO, Ohio State University
LEILA R. RAINES, Cornell University
E. D. RAINVILLE, University of Michigan
J. F. RANDOLPH, University of Rochester
S. E. RASOR, Ohio State University
M. O. READE, University of Michigan
L. M. REAGAN, University of Wichita
O. W. RECHARD, Ohio State University
J. K. RECKZEH, State Teachers College, Jersey City
MINA S. REES, Office of Naval Research

P. V. REICHELDERFER, Ohio State University
ERIC REISSNER, Massachusetts Institute of Technology
C. N. REYNOLDS, West Virginia University
R. F. RINEHART, Research and Development Board, U. S. Army
FRED ROBERTSON, Iowa State College
L V. ROBINSON, University of South Carolina
V. N. ROBINSON, U. S. Naval Academy
W. J. ROBINSON, Centre College
L. D. RODABAUGH Southern Illinois University
P. C. ROSENBLOOM, Syracuse University
ARTHUR ROSENTHAL, Purdue University
M. F. ROSSKOPF, Syracuse University
S. A. ROWLAND, Ohio Wesleyan University
RAPHAEL SALEM, Massachusetts Institute of Technology
CHARLES SALTZER, Case Institute of Technology
R. G. SANGER, Kansas State College
A. C. SCHAEFFER, Purdue University
ROBERT SCHATTEN, University of Kansas
S. A. SCHELKUNOFF, Bell Telephone Laboratories
EDITH R. SCHNECKENBURGER, University of Buffalo
K. C. SCHRAUT, University of Dayton
E. W. SCHREIBER, Western Illinois State Teachers College
VERYL G. SCHULT, Wilson Teachers College
H. M. SCHWARTZ, Brookhaven National Laboratories
C. E. SEALANDER, Ohio State University
C. L. SEEBECK, University of Alabama
WLADMIR SEIDEL, National Bureau of Standards
M. E. SHANKS, Purdue University
H. C. SHAUB, Washington and Jefferson College
C. N. SHUSTER, State Teachers College, Trenton
L. L. SILVERMAN, Dartmouth College
Sister MARY PAULA, Marygrove College
F. C. SMITH, College of St. Thomas
R. E. SMITH, College of William and Mary
W. S. SNYDER, University of Tennessee
ANDREW SOBCZYK, Boston University
T. H. SOUTHARD, Wayne University
C. E. SPRINGER, University of Oklahoma
G. W. STARCHER, Ohio State University
E. P. STARKE, Rutgers University
H. E. STELSON, Michigan State College
R. C. STEPHENS, Knox College
GUY STEVENSON, University of Louisiana
B. M. STEWART, Michigan State College
RUTH W. STOKES, Syracuse University
M. H. STONE, University of Chicago
R. B. STONE, Purdue University
E. B. STOUFFER, University of Kansas
E. G. SWAFFORD, U. S. Naval Academy
OTTO SZASZ, University of Cincinnati
J. S. TAYLOR, University of Pittsburgh
MILDRED E. TAYLOR, Mary Baldwin College
W. C. TAYLOR, University of Cincinnati
R. T. TEAR, Rensselaer Polytechnic Institute
H. P. THIELMAN, Iowa State College
L. O. THOMPSON, University of Detroit
R. M. THRALL, University of Michigan
G. L. TILLER, Utica College of Syracuse University
H. S. TONEY, Wilson College
LEONARD TORNHEIM, University of Michigan
MARIAN M. TORREY, Goucher College
A. W. TUCKER, Princeton University
J. L. ULLMAN, Stanford University
GILBERT ULMER, University of Kansas
E. P. VANCE, Oberlin College
HENRY VAN ENGEN, Iowa State Teachers College
H. E. VAUGHAN, University of Illinois
R. W. WAGNER, Oberlin College
G. L. WALKER, Purdue University
R. J. WALKER, Cornell University
S. E. WALKLEY, University of Illinois
J. L. WALSH, Harvard University
JEAN B. WALTON, University of Pennsylvania
W. R. WASOW, Swarthmore College
K. W. WEGNER, Carleton College
E. T. WELMERS, Bell Aircraft Corporation
INA W. WELMERS, University of Buffalo
F. J. WEYL, Office of Naval Research
E. A. WHITMAN, Carnegie Institute of Technology
P. M. WHITMAN, Johns Hopkins University
D. R. WHITNEY, Ohio State University
G. T. WHYBURN, University of Virginia
R. B. WILDERMUTH, Capital University
F. B. WILEY, Denison University
S. S. WILKS, Princeton University
W. L. WILLIAMS, University of South Carolina
C. O. WILLIAMSON, College of Wooster
R. L. WILSON, University of Tennessee
G. F. WOODSON, Wilberforce University
R. C. YATES, U. S. Military Academy
J. W. T. YOUNGS, Indiana University
J. H. ZANT, Oklahoma A & M
M. A. ZORN, Indiana University

The members of the mathematical organizations and their families were housed in Mack Hall and Canfield Hall. Meals were served in the Mack-Canfield dining hall.

The reception and tea held in the Faculty Club on Tuesday afternoon was well attended. On Wednesday evening, the School of Music of Ohio State University presented a program of music in University Hall Chapel. Miss Janice Murray was the vocalist, and Miss Eleanor Anawalt and Miss Olwen Jones of the University faculty formed a two-piano team.

To entertain the ladies present at the meeting, movies were shown on Wednesday afternoon. On Thursday afternoon, the Math Circle, composed of the wives and women staff members of the Mathematics Department, gave a tea for the visiting ladies.

A dinner for members of the mathematical organizations was held on Thursday evening in the Mack-Canfield dining room. Professor R. E. Langer was toastmaster. Dr. Harlan Hatcher, vice-president of Ohio State University, brought to the guests the greetings of the University on its 75th anniversary. Professor Saunders MacLane spoke on his experiences in Europe during the past year. Professor Marston Morse spoke also about his trip to Italy, and in particular about the Mathematical Congress held at Pisa. At the conclusion of the dinner, a resolution of thanks to the authorities of the University and the local members of the committee on arrangements was presented by Professor R. P. Dilworth and was enthusiastically adopted.

The sessions of the American Mathematical Society began on Tuesday, December 28 and continued through Thursday. On Tuesday evening, the twenty-second Josiah Willard Gibbs lecture was delivered by Professor Hermann Weyl of the Institute for Advanced Study, on the subject "Ramifications, old and new, of the eigenvalue problem." Professor Mark Kac of Cornell University spoke on Tuesday afternoon on "Probability methods in some problems of analysis and theory of numbers." On Wednesday afternoon, Professor A. S. Besicovitch of Cambridge University spoke on "Parametric surfaces" and Professor Lamberto Cesari of the University of Bologna spoke on "Area and representation surfaces." Professor Einar Hille of Yale University delivered his retiring presidential address on Wednesday morning on the topic "Lie theory of semi-groups of linear transformations."

Sessions of the Association for Symbolic Logic were held on Thursday and Friday. The National Council of Teachers of Mathematics held its Ninth Christmas Conference on Wednesday and Thursday. A feature of the conference was the simultaneous meeting of seventeen discussion groups on Thursday morning.

The Pi Mu Epsilon Fraternity held a breakfast meeting on Friday morning. The Kappa Mu Epsilon Fraternity held luncheon meetings on Tuesday and Wednesday.

The Mathematical Association of America held its sessions on Friday morn-

ing and afternoon in Room 306 Pomerene Hall. President L. R. Ford presided at the morning session and Vice-President C. B. Allendoerfer at the afternoon session. The Program Committee for this meeting consisted of J. S. Frame, Chairman, E. D. Rainville and Moses Richardson.

FIRST SESSION OF THE ASSOCIATION

"On the nature of applied mathematics," by Dr. R. S. Burington, Bureau of Ordnance, Navy Department.

"Mathematical aspects of the theory of viscous fluids," by Professor M. E. Shanks, Purdue University.

"Mathematical aspects of aero-elasticity," by Dr. E. T. Welmers, Bell Aircraft Corporation.

"Instructional aids in the teaching of Junior College mathematics," by Professor E. H. C. Hildebrandt, Northwestern University.

SECOND SESSION OF THE ASSOCIATION

"Modern operational calculus for undergraduates," by Professor J. R. Britton, University of Colorado and University of Michigan.

"A topic in the theory of differential equations," by Professor H. W. Brinkmann, Swarthmore College.

MEETING OF THE BOARD OF GOVERNORS

The Board met on Thursday afternoon at 2 in Room 305 Pomerene Hall. Twenty-four members of the Board were present. Among the more important items of business transacted are the following.

The Board voted to approve the appointment of the following Nominating Committee for 1949: R. G. Sanger, chairman, W. E. Milne, W. M. Whyburn.

It was voted that one of the three representatives of the Association on the Policy Committee for Mathematics shall be the Secretary-Treasurer ex officio and that the other two representatives shall be elected by the Board hereafter for three year terms. A ballot resulted in the selection of these representatives: H. M. Gehman, ex officio (1949–1952), C. V. Newsom (1949–1951) and L. R. Ford (1949–1950).

The publication of an Index of volumes 1–55 of the MONTHLY was authorized. The Index is to contain a subject index of Mathematical papers and notes, a short author index, bibliographical material on club topics, miscellaneous official articles about the Association.

Upon recommendation of the Committee on Section Meetings, the Board voted that Sectional Governors shall be elected by a mail vote to be conducted by the Secretary-Treasurer of the Association, and that the Committee be authorized to determine the geographical boundaries of each Section and to allocate members to Sections for voting purposes. It was also voted that a maximum of $35 per year may be expended by the Secretary-Treasurer on be-

half of each Section toward the expenses of its meetings, including payment of travel expenses of an invited speaker.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

President L. R. Ford presided at the annual business meeting, which was held on Friday at 2:00 P.M. in Room 306 Pomerene Hall.

The tellers, L. E. Bush and P. D. Edwards, announced the election of R. E. Langer, University of Wisconsin, as President for the two-year term 1949–1950, and of C. B. Allendoerfer, Haverford College, and R. J. Walker, Cornell University, as Governors for the three-year term 1949–1951.

Announcement was made of the election by the Board of Governors of N. H. McCoy, Smith College, as Second Vice-President for the two-year term 1949–1950.

MEETING OF SECTION SECRETARIES

A meeting of Secretaries of the Sections of the Association was held on Thursday morning in Room 305 Pomerene Hall. Eighteen of the twenty-five Sections were represented. Professor W. V. Parker, Chairman of the Committee on Section Meetings, presided at the meeting.

The following topics were discussed: membership, election of sectional governors, programs of section meetings, finances, coordination of Association activities with those of elementary and secondary-school groups.

Professor Schneckenburger described some of the sources of new members by analyzing a group of recently elected members of the Association. A general discussion of programs led to many helpful suggestions from the representatives of the various sections. The discussion of methods of election of sectional governors and of finances led to the actions taken by the Board of Governors on these matters which are given above.

Professor E. H. C. Hildebrandt, President of the National Council of Teachers of Mathematics, urged the sections to provide a wider range of activities for teachers in the elementary and secondary schools. This was followed by a general discussion.

H. M. GEHMAN, *Secretary-Treasurer*

THE APRIL MEETING OF THE METROPOLITAN NEW YORK SECTION

The seventh annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at Washington Irving High School, New York, N. Y., on Saturday, April 24, 1948. Brother Bernard Alfred, Collegiate Vice-Chairman of the Section, presided at the morning session; Professor W. H. H. Cowles, Chairman of the Section, presided at the business meeting; and Mr. George G. Ross, High School Vice-Chairman, presided at the regular afternoon session.

One hundred and eighteen persons were present, including the following

sixty members of the Association: Brother Bernard Alfred, R. G. Archibald, W. D. Baten, W. C. Bornmann, Samuel Borofsky, C. B. Boyer, A. B. Brown, K. E. Brown, Jewell Hughes Bushey, Hobart Bushey, T. F. Cope, W. H. H. Cowles, J. G. Deutsch, J. N. Eastham, J. M. Feld, Daniel Finkel, R. M. Foster, K. G. Fuller, Harriet Griffin, George Grossman, Frank Hawthorne, G. C. Helme, R. E. Henry, E. Marie Hove, L. C. Hutchinson, R. A. Johnson, L. S. Kennison, E. R. Kiely, Edna Kramer-Lasser, B. R. Leeds, C. H. Lehmann, A. A. LePori, M. E. Levenson, Herman Levy, May H. Maria, A. L. Mayerson, Mary McKenna, F. H. Miller, A. J. Mortola, M. A. Nordgaard, P. B. Norman, Eugene Odin, J. K. Reckzeh, G. J. Ross, H. D. Ruderman, John Salerno, Charles Salkind, Aaron Shapiro, James Singer, E. R. Stabler, Mildred M. Sullivan, R. L. Vitale, H. E. Wahlert, Israel Wallach, Alan Wayne, Margaret C. Weeber, M. E. White, John Williamson, Sister Francis Xavier, R. C. Yates.

The Section elected the following officers for next year: Chairman, Professor R. A. Johnson, Brooklyn College; Collegiate Vice-Chairman, Professor T. F. Cope, Queens College; High School Vice-Chairman, Mr. H. D. Ruderman, Manhattan High School of Aviation Trades; Secretary, Professor James Singer, Brooklyn College; Treasurer, Mr. Aaron Shapiro, Midwood High School. Professor F. H. Miller, Cooper Union, was elected as the Sectional Governor.

Six papers were presented at the meeting.

1. *Practical methods of solving linear Diophantine equations*, by Professor A. B. Brown, Queens College.

If a linear Diophantine equation has a coefficient ± 1, the solutions are obvious. If not, a finite number of substitutions of the form $x=1\cdot x'+ay+\cdots+bz$ will either yield a coefficient ± 1, or disclose incompatibility.

For a system of equations, by taking linear combinations of left and right members of equations, an equation with a coefficient ± 1 may be obtained; otherwise the procedure for a single equation must be used. If an unknown has a coefficient ± 1 in an equation, it can be eliminated from the remaining equations, giving us a system with one fewer equations in one fewer unknowns to be solved. Again incompatibility shows up automatically. There are no formulas to be memorized.

2. *Basic values in junior high school mathematics*, by Miss Mary C. Rogers, Roosevelt Junior High School, introduced by the Secretary.

In a world where quantitative situations must be dealt with repeatedly and accurately, mathematical literacy is as important as the ability to read and write. The speaker held that the special functions of the junior high school are: (1) to provide an adequate and natural continuance of the work of the elementary school; (2) to correct all mathematical retardations and shortages existing among any of its pupils; (3) to provide an expanding and deepening experience with the problems of everyday living; (4) to strengthen and extend the foundations for subsequent experiences with mathematics. It was also stated that the mathematics curricula should include: (1) number and computation; (2) measurement and informal geometry; (3) constructions and interpretation of graphs; (4) an introduction to the functional core of algebra through formulas and equations; and (5) a generous application of the various phases of mathematics to the problem of everyday living. The speaker exhibited a number of charts, drawings, and solids made by students of her school.

3. *Generation, properties, and applications of some curves*, by Lt. Colonel R. C.

Yates, United States Military Academy.

Colonel Yates developed the conics as envelopes of creases formed by folding wax paper. He then discussed a simple ruler-compass construction for the center of curvature of all conics. This led to the consideration of evolutes and their application as caustics in the field of optics, with particular attention to the catacaustic of a circle for various point sources of light, and the diacaustic formed by light rays refracted through a plane surface. The idea of instantaneous centers of motion was then considered as a means of determining tangents to familiar curves such as the limaçon, strophoid, conchoid, and cycloid. The paper closed with remarks on the family of roses and their identification as special epitrochoids and hypotrochoids. Several models were used for illustration.

4. *Mathematical machines*, by Professor F. J. Murray, Columbia University, introduced by the Secretary.

Professor Murray stated that the characteristic aspects of current civilization appear in the utilization of scientific techniques which involve mathematics in a very fundamental way. But the range of application of mathematics is limited by our ability to solve complex problems and carry out computations. Thus the development of mathematical machinery is an important part of technical progress. Applied mathematics in general requires a considerable mathematical development beyond the pure theory, and mathematical machinery poses additional problems. It is similar to an increase in dimensionality. The study of mathematical machines is highly desirable for technical progress, its theoretical development has deep intellectual interest, and to the student of mathematics it offers a fascinating contact with the myriad of technical advances which distinguish our modern culture.

5. *The high school-college articulation group reports*, by Dr. Eugenie C. Hausle, James Monroe High School (introduced by the Secretary), and Professor J. H. Bushey, Hunter College.

Dr. Hausle discussed some of the recommendations submitted by the mathematics sub-committee on articulation between high school and college. Specific recommendations pertaining to methods of teaching and content of courses were dealt with. In particular, it was suggested that, beginning with the eleventh year, more and more of the responsibility for learning should be put upon the pupil. It was also recommended that the problems of coordination of mathematical teaching in the several divisions of the educational system could best be handled by a council for continuing the study of articulation between high school and college.

Professor Bushey discussed these matters from the point of view of the colleges. He pointed out the wide variation in the previous mathematical preparation of college freshmen and recommended a system of testing for placement purposes.

JAMES SINGER, *Secretary*

THE MAY MEETING OF THE ILLINOIS SECTION

The twenty-seventh annual meeting of the Illinois Section of the Mathematical Association of America was held at the Illinois Institute of Technology, Chicago, Illinois, on Friday and Saturday, May 14–15, 1948. Professor John J. Corliss, Chairman of the Section, presided at all meetings.

There were 143 in attendance, including the following 69 members of the Association: M. L. Anthony, D. L. Arenson, H. G. Ayre, Ruth M. Ballard, H. R. Brahana, Winifred V. Berglund, S. F. Bibb, G. M. Bloom, F. R. Brown, E. L. Buell, Laura E. Christman, E. G. Comfort, J. J. Corliss, J. P. Esposito,

Sister Mary Esther, M. D. Eulenberg, I. K. Feinstein, L. R. Ford, Evelyn Frank, D. M. Friedlen, A. E. Gault, G. D. Gore, Cassie G. Greer, Madeleine Grenard, E. D. Hellinger, E. H. C. Hildebrandt, Sister Charlotte Holland, Rose Hornacek, E. C. Kiefer, W. C. Krathwohl, Luise Lange, R. E. Langer, Rose Lariviere, O. F. Latham, H. S. Levin, S. Levy, Leo Liolios, A. T. Lonseth, Sister Mariola, Karl Menger, K. W. Miller, G. E. Moore, C. W. Moran, E. J. Moulton, Mary W. Newson, E. P. Northrup, F. S. Nowlan, Grace M. Nolan, T. B. Ondrak, Gordon Pall, D. W. Pounder, Mary K. Rapp, W. T. Reid, Haim Reingold, Marian Rosenbeck, J. M. Sachs, E. W. Schrieber, H. W. Schwartz, Anice Seybold, Helen Sears, W. T. Scott, Beulah Shoesmith, H. A. Simmons, Albert Soglin, M. H. Stone, L. B. Stelling, M. E. Wescott, L. R. Wilsox, C. C. Wilson.

At business meeting held on Saturday morning the following were elected as officers for the coming year: Chairman S. F. Bibb, Illinois Institute of Technology; Vice-Chairman M. G. Moore, Bradley University; Secretary E. C. Kiefer, Millikin University. The meetings for next year are to be held on Friday and Saturday, May 13–14, 1949, at Bradley University in Peoria, Illinois.

On a motion by Professor L. R. Ford and seconded by Professor G. D. Gore, it was voted that the Illinois Section go on record as favoring, in principle, the encouragement of exceptional mathematical study and achievement in the high schools of Chicago, and that a committee be appointed by the Chairman of the Section to explore the possibilities of furthering these ends by contests and awards under the auspices of the Association. Chairman Corliss appointed Professor L. R. Ford as chairman of this committee.

A motion was passed instructing the Secretary of the Section to send a letter to Mrs. C. E. Comstock of Peoria expressing the feeling of the Section at the death of her husband, Professor C. E. Comstock, who was one of the founders the Section and an active member during his life.

The following papers were presented:

1. *Continued fractions related to the Riccati differential equation*, by Professor W. T. Scott, Northwestern University.

The author discussed a little known algorithm, first used by Euler and Lagrange, for obtaining from any Riccati differential equation,

$$N(x) - P(x) + Q(x)y^2 + y' = 0,$$

a continued fraction. A remainder formula was developed, thereby providing a test to determine whether or not such continued fractions actually converge to a solution of the differential equation. Finally, some special examples were considered.

2. *The use of differentials in thermodynamics*, by Professor Karl Menger, Illinois Institute of Technology.

The purpose of the paper was to bridge the gap between Calculus as taught in elementary mathematics courses and Calculus as applied by physicists. Especially in the field of thermodynamics the discrepancies baffle many beginners. The first law stipulates that the quantity of heat transferred to a gas is $pdv+du$ where p is the pressure, dv the change in volume, du the change of

the internal energy. Since this expression is not a complete differential, the original notation for the quantity of heat, dq, has been superseded by such symbols as $\not{d}q$, dq, and δq, which are not defined in Calculus. However, one can explain the nature of the "incomplete differential" in terms of the most elementary concepts of Calculus One merely has to notice that every thermodynamical process determines, and is characterized by five functions of the time, viz., $v(t)$, $p(t)$, $\theta(t)$, $u(t)$, $q(t)$, denoting the volume, the pressure, the temperature, and the internal energy of the gas, respectively, and the quantity of heat transferred to the gas between an initial moment t_0 and t. An equation of state implies the existence of a function Φ of three variables such that $\Phi[v(t), p(t), \theta(t)] = 0$ for every t. Then the first law can be written in the following differential form

$$\frac{dq}{dt} = p\frac{dv}{dt} + \frac{du}{dt} \quad \text{or} \quad q'(t) = p(t)\cdot v'(t) + u'(t)$$

for every t. An elementary integral form of the first law is

$$q(t) - q(t_0) = \int_{t_0}^{t} p(t)dv(t) + u(t) - u(t_0)$$

where the integral is a Stieltjes integral, that is, the limit of the sums

$$\sum_{i=0}^{n-1} p(t_{i+1}^*)[v(t_{i+1}) - v(t_i)]$$

for subdivisions

$$t_0 < t_1 < \cdots < t_{n-1} < t_n = t, \qquad t_i \leqq t_{i+1} \leqq t_{1+i}^*,$$

for which the maximum of the numbers $t_{i+1} - t_i$ approaches zero.

3. *Notes of the history of calculus with implications for our teaching*, by Professor E. D. Hellinger, Northwestern University.

It seems that investigations on calculus have always started with problems of integration rather than with problems of differentiation. This is shown in the classical Greek mathematics by Archimedes' evaluations of areas bounded by curves. His fundamental ideas were presented, especially those of his heuristic "method" and its relations to precise "geometric proofs." Again, at the beginning of modern mathematics one finds integration procedures; here, particularly, the basic ideas of Cavalieri were sketched. The subsequent development is based on the discovery of the close relation which can be established between the determination of the area beneath one curve and the slope of another one. Certain characteristic points in this line, leading to the proper differential and integral calculus, were indicated. It was pointed out that the pattern of historical development could be made useful for the modern teaching of calculus.

4. *Isaac Newton*, by Professor R. E. Langer, University of Wisconsin.

The speaker presented a brief biography of Newton, in which an attempt was made to project his scientific achievements against the background of the personal incidents of his life and the historical and sociological conditions of his time. In short sketches, the settings of his three great discoveries (the composite nature of white light, the calculus, the law of gravitation) relative to the achievements of his scientific predecessors were given. The episodes of the writing of the Principia, the ensuing collapse of his health, and finally of his later life in London as an official of the British mint were described.

5. *Objectives in the teaching of college mathematics*, by Professor F. S. Nowlan, University of Illinois.

Two main aims are considered in the teaching of college mathematics. These are: (1) Training

in precise thinking and a grasp of basic principles; (2) The acquisition of information and a mastery of certain technical skills. It is pointed out that a disproportionate stress has been placed upon the second of these to the almost total neglect of the first.

Various means are suggested whereby the first objective can be attained. In particular, the student should be required to express himself precisely and without ambiguity, in speech, in writing, and in symbolism. He should cease the misuse of the equality symbol, and the word "equals" must be used correctly. Also, the student must be trained to realize the postulational nature of mathematics, and in any study he must not only reason logically, but he must examine and question his initial assumptions.

6. *Mathematics in the Bachelor of Arts curriculum*, by Professor H. R. Brahana, University of Illinois.

For the last fifteen years it has been possible for freshmen to enter the College of Liberal Arts and Sciences of the University of Illinois without credits in high school mathematics. Recent changes in requirements for graduation make it necessary that a student have some mathematics beyond the elementary school level; the new requirement is satisfied by credit in algebra and geometry in high school, one year of each.

Close scrutiny of the small group of students affected by the changes in requirements reveals some unsuspected facts and also yields facts to support some opinions widely held. Briefly the more important facts are: (1) Almost all students entering the University have taken algebra and plane geometry in high school; (2) The rest have tried to do mathematics; (3) Those who have not done mathematics in high school disappear quickly from the University ; (4) Those who have not done mathematics in high school and who have low intelligence quotients disappear even more quickly; (5) Those who have not done high school mathematics are almost all deficient in knowledge of arithmetic.

Speculating about reasons why after fifteen years of not requiring mathematics for entrance we get so few who do not offer it, two reasons appear: (1) The children like to do mathematics; (2) Their teachers have put up a valiant resistance to those who would remove mathematics from the offerings of the high schools. Many high school teachers believe that the colleges have shirked a duty.

7. *Enrichment of mathematics offerings for juniors and seniors*, Professor M. H. Stone of The University of Chicago.

This discussion of the mathematics curriculum for juniors and seniors is directed to the need for incorporating recent mathematics advances in the course offerings, both as a phase of general education and as an aid to students intending to go on to graduate work in mathematics. The development of mathematics has resulted in a changed point of view as well as in strictly technical advances, both of which should be reflected in higher collegiate instruction. It is in algebra and geometry (including topology) that special efforts in designing new introductory courses at the junior and senior level seem to be needed. However, in the traditional courses some reflection of the vigorous growth of mathematics should also be expected.

E. C. KIEFER, *Secretary*

THE MAY MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association of America was held at the College of St. Thomas in St. Paul, Minnesota, on Saturday, May 8, 1948. Two sessions were held in the forenoon, one at luncheon, and one in the afternoon. Professors R. L. Lokensgard, K. W. Wegner, L. S. Laws, and H. L. Turrittin (Chairman of the Section) presided at the respective sessions. Following the afternoon session, tea was served to the members and

guests by the wives of the mathematics staff of the College.

One hundred and nine persons attended the meeting, including the following forty-seven members of the Association: N. R. Amundson, H. M. Anderson, F. J. Arena, J. E. Bearman, Louise Beasley, Sister Ada Marie Boehm, K. H. Bracewell, R. W. Brink, M. D. Brown, L. E. Bush, W. H. Bussey, R. H. Cameron, E. J. Camp, C. S. Carlson, Elizabeth Carlson, Gladys Gibbens, Sister Seraphim Gibbons, H. W. Godderz, Charles Hatfield, Jr., J. S. Hill, T. W. Jackson, D. A. Johnson, G. K. Kalisch, W. S. Loud, C. C. MacDuffee, Kenneth May, W. H. McBride, W. R. McEwen, A. G. Montgomery, Sigurd Mundhjeld, M. J. Norris, F. R. Ohnsorg, J. M. H. Olmsted, G. C. Priester, P. A. Rognlie, L. W. Sheridan, F. C. Smith, Marion V. Smith, A. H. Speltz, A. G. Swanson, F. J. Taylor, Takashi Terami, Marian W. Thornton, Ella Thorp, H. L. Turrittin, K. W. Wegner, and Irene L. Wente.

The nominating committee, consisting of Professor W. H. Bussey, Chairman, Sister Seraphim Gibbons, and Professor Sigurd Mundhjeld, presented the following slate of nominees for the coming year: Chairman, W. P. McEwen, University of Minnesota, Duluth Branch; Secretary, L. E. Bush, College of St. Thomas; Executive Committee, R. L. Lokensgard, Winona State Teachers College, H. M. Anderson, Gustavus Adolphus College, H. L. Turrittin, University of Minnesota. These were duly elected.

In accordance with instructions given at the 1947 annual meeting, the Secretary, acting for the Executive Committee, presented a proposed set of by-laws for the Section. On the motion by Professor Brink, seconded by Professor Bussey, the proposed by-laws were amended. They were then adopted, subject to approval by the Board of Governors, on the motion by Professor Brink, seconded by Professor Gibbens.

By invitation the Executive Committee, Professor C. C. MacDuffee delivered an address at the second morning session. The title of his address was *The Association Wants To Help*. He lent his support to the principle that the high school teacher should occupy an honored and well-paid position in his local community, for his work is fully as important and exacting as that of the physician or lawyer. In return, the teacher should be a genuine scholar, able to inspire in the minds of his students a high enthusiasm for the great achievements of the human mind and an ambition for productive scholarship. A present tendency to substitute in the name of "education for democracy" a cheaper type of education seems a cynical underestimation of the capacities of American youth. The sections of the Association should take a leading part in demanding a substantial curriculum in our public schools.

The following short papers were presented:

1. *The possibility of a simple iterative solution for simultaneous quadratics*, by Professor W. S. Loud, University of Minnesota.

A sufficient condition that the iterative process $x_{n+1}=f_1(x_n, y_n)$, $y_{n+1}=f_2(x_n, y_n)$ converge to a solution of the system of equations $x=f_1(x, y)$, $y=f_2(x, y)$ is that the matrix of the Jacobian of f_1 and f_2 with respect to x and y, evaluated at the solution, have its characteristic values both less than

unity in absolute value, and that the process be initiated sufficiently close to the solution. The concrete realization of this in the case of the system $x=y^2+A$, $y=x^2+B$ is that if the solution is (x_0, y_0), the relation $|4x_0y_0|<1$ holds. It follows as a consequence that not all systems of the latter type can be solved by the above process. The condition in terms of the parameters A and B are as follows: Let

$$g_1(A, B) = 256(A^2B^2 + A^3 + B^3) + 288AB - 27,$$

$$g_2(A, B) = 256(A^2B^2 + A^3B^3) + 160AB + 125.$$

Then, if g_1 and g_2 are both positive, and A and B both lie between $-5/4$ and $-3/4$, there are two real solutions which can be found by the above process. If g_1 and g_2 have opposite signs, there is one real solution which can be found by the process. In all other cases when g_1 and g_2 are both different from zero, no real solutions can be found by the process. If g_1 or g_2 is zero, there is a solution satisfying $|4x_0y_0|=1$.

By a suitable shift of origin and scale, the solution of almost any quartic equation can be reduced to the solution of the system $x=y^2+A$, $y=x^2+B$.

2. *Mathematics among the Maya Indians*, by Mr. Douglas Guy, Macalester College, introduced by Professor E. J. Camp.

The civilization of the Maya Indians of ancient Middle America produced the first known positional number system with a symbol for zero. The system was much like ours except that it used twenty as its base. In counting time, a further modification was made so that the second position above the units position indicated multiplication by 360 instead of 400. The Maya figured elapsed time from a hypothetical day of creation, about August 1, 3113 B.C. Thus, using commas to indicate that the figures separated by them are to be multiplied by 144000, 7200, 360, 20 and 1, the date *8, 14, 3, 1, 12* gave the exact number of days from the starting date and also approximated the number of years in the three highest "digits."

3. *On a certain type of wire puzzle*, by Professor Charles Hatfield, Jr., University of Minnesota.

The kind of wire puzzle dealt with was the "thread" type which consists of a loop of string (or if mechanical construction permits, loop of wire) around one or more parts of a wire configuration, the object of the puzzle being to disentangle the loop from the rest of it. It lends itself, both in construction and solution, to an analysis which utilizes the distinction between simply and multiply connected regions.

4. *An experimental course in mathematics*, Professor K. W. Wegner, Carleton College.

A two semester-hour course has been offered three times at Carleton in which the main objective was the development of clarity of thinking and expression. It was designed primarily for those who did not take any other college mathematics. In order to make it available to everyone, and since no attempt was made to cover a particular body of subject matter, no prerequisite was placed on the course.

Three main topics were included. The first of these was the topic of permutations, combinations, and probability. No formulas were developed, but each problem was solved from first principles. Stress was put on clarity of expression in describing solutions, whether oral or written. Number systems with bases different from ten were then studied. One period was devoted to playing and studying the game of Nim. The third main topic was from the theory of linear point sets. The class was given several definitions such as limit point, closed set, open set, derived set, etc. They were then given sets of various descriptions to test their understanding of the definitions. As a result of this experiment it is felt that such a course can go a long way in attaining the objective of the development of clarity in thinking and in expression.

5. *A study of the preparation of secondary school mathematics teachers in Minnesota*, by Mr. W. N. Nelmark, Mabel, Minnesota, introduced by the Secretary.

The twenty-one teacher training institutions in Minnesota varied in the number of credits required to graduate from a low of 120 semester hours to a high of 136 semester hours. The average was 126.2 semester hours. Practice teaching requirements averaged 4.3 semester hours, from a low of three semester hours to a high of eight semester hours.

One hundred fifty-six questionnaires were sent to college educators, mathematics professors, and high school mathematics instructors, of which over 77% responded. These people were asked to state which of certain topics in mathematics, education, and other subjects they considered of fundamental importance to a secondary school mathematics teacher, which they considered of considerable importance, which have some value, and which have no value.

Quite amazing were the results of the evaluation by high school mathematics teachers of the different mathematics subjects. Only the three subjects, college algebra, trigonometry, and analytic geometry, were considered of fundamental importance. Those of considerable importance were differential calculus, theory of equations, projective geometry and mathematics of investments. Those subjects having some value were integral calculus, spherical trigonometry, solid analytic geometry, history of mathematics, and mathematical statistics.

The general trend in the teacher preparation program in Minnesota seems to be for a sound general education, with broad fields of majors, increased certification requirements, five years of preparation with increased actual teaching experience, and raising the threshold standards of admission to teacher training institutions.

6. *Solution of a problem of Ramanujan*, by Mr. R. E. Graves, University of Minnesota, introduced by Professor R. W. Brink.

Ramanujan stated without proof that

$$\sum_{n=1}^{\infty} \frac{n}{e^{2\pi n} - 1} = \frac{1}{24} - \frac{1}{8\pi}.$$

This result was proved by Kac, in an unpublished paper, by use of the Poisson sum formula. At the time when Kac communicated his result to the author, he posed the problem of establishing Ramanujan's formula by methods which would have been available to the Indian mathematician. This paper establishes the formula by transforming

$$\sum_{n=1}^{\infty} \frac{n}{e^{2\pi n} - 1}$$

into

$$\sum_{r,n=1}^{\infty} n e^{-2\pi r n},$$

thence into

$$\frac{1}{4} \sum_{n=1}^{\infty} \operatorname{csch}^2 n\pi,$$

and then summing this last series by elementary methods based upon the theorem of residues.

7. *A note on an inequality concerning subharmonic functions*, by Professor S. E. Warschawski, University of Minnesota, introduced by Professor Neal R. Amundson.

In the manner described below, the speaker developed an extension of a theorem of R. M. Gabriel. Let $u(x, y)$ be subharmonic in a region R; let C_1 and C_2 be simple closed rectifiable curves in

R, C_1 in the interior of C_2, such that it is possible to "roll" a circle of radius ρ_2 along C_2 on the "outside" of C_2, and a circle of radius ρ_1 along C_1 on the "inside" of C_1. Then $\int_{C_1} uds \leqq A\int_{C_2} uds$, where A is a constant which depends only on ρ_2 and D/ρ_1 where D is the diameter of C_1. If C_2 is convex and C_1 is a circle, then $A=2$.

8. *A note on cylindrical adsorption*, by Mr. C. E. Sanborn and Professor N. R. Amundson, University of Minnesota. (Mr. Sanborn was introduced by Professor Amundson.)

Consider a thin layer of thickness l and of infinite extent of adsorbent. A solution having a concentration c_0 of some solute is introduced on a circular area of radius R. The solute in solution on passing through the adsorbent is adsorbed. Letting the concentration be c, the amount adsorbed per unit weight q, the volume poured in v, the layer thickness l, and the fractional void volume α, then the following equation can be derived.

$$\frac{\partial c}{\partial r} + 2\pi r l\alpha \frac{\partial c}{\partial v} + 2\pi r l \frac{\partial q}{\partial v} = 0.$$

Assuming that the adsorption isotherm has the form $q=Af(c)$, A a constant, then

$$\frac{\partial c}{\partial r} + 2\pi r l[\alpha + Mf'(c)]\frac{\partial c}{\partial v} = 0.$$

The initial conditions are $c(0, r)=0$ and $c(v, R)=c_0$. The solutions of this system were discussed, as well as the companion problem of the development of the chromatogram, i.e. solution with pure solvent.

9. *On the coefficients of certain multinomial expansions*, by Mr. F. J. Arena, North Dakota State College.

Tremblay has shown (*National Mathematics Magazine*, Vol. XI, 1937, p. 255) how to construct a generalized Pascal arithmetical triangle for the coefficients of multinomials of the form $(x+x^2+\cdots+x^n)^m$. He writes unity n times for the first row. Then a term in any row is found by adding n terms of the above row. Begin with the term above the one sought and to it add the $(n-1)$ terms preceding it. With the use of such tables many problems in probability can be easily solved. For example, we can easily find the probability of getting 12 points with 4 dice.

Mr. Arena gives a rule for constructing such tables of the coefficients of the expansion of multinomials of the form $(1+x+x^2+\cdots+x^n)^{-m}$. In the bottom row write 1 followed by several zeros. Then any term is found by adding the n terms preceding the term sought, subtracting from this the term directly below the term sought, and then changing the sign of this result. The following table, when read from the botton upward, gives the first few coefficients of the expansion of $(1+x+x^2+x^3)^{-m}$ for $m=0, 1, 2, 3, 4, 5$.

1	−5	10	−10	10	−26	50	·	·	·
1	−4	6	−4	5	−16	24	·	·	·
1	−3	3	−1	3	−9	9	·	·	·
1	−2	1	0	2	−4	2	·	·	·
1	−1	0	0	1	−1	0	·	·	·
1	0	0	0	0	0	0	·	·	·

Thus

$$(1+x+x^2+x^3)^{-5} = 1 - 5x + 10x^2 - 10x^3 + 10x^4 - 26x^5 + \cdots.$$

10. *On the integral of a continuous function*, by Professor M. J. Norris, College of St. Thomas.

A method of proving that the Riemann integral of a continuous function on a closed interval exists without introducing the notion of uniform continuity was presented. The upper and lower integrals are shown to be additive. Then the difference of the upper and lower integrals, considered as a function of one end-point, is shown to have a derivative identically zero. It then follows that the upper and lower integrals are equal.

11. *Trends in the teaching of algebra*, by Professor K. H. Bracewell, Hamline University.

One hundred thirty-three replies received from a selected list of leading universities and colleges to whom a questionnaire was sent indicate a downward trend in pre-college preparation in algebra. Considerable variation exists among all higher institutions in both the amount of credit and content of courses in college algebra. Very few cover the chapters on theory of investment and infinite series. Permutations and combinations, probability and mathematical induction are in somewhat greater, but still limited, use. Sixty per cent of all schools reporting divide their algebra sections into elementary and advanced classes. No marked difference in practice was distinguished between universities and colleges.

12. *Concrete expression of mathematical ideas: an exhibit*, by Mr. E. J. Berger, Monroe High School, introduced by the Secretary.

The exhibit included about forty articles made by students at Monroe High School. Some of the articles illustrated properties of various algebraic curves, such as the focal properties of the ellipse and parabola. Others were measuring devices such as Jacob's staff, transit, clinometer, and sextant.

L. E. Bush, *Secretary*

THE MAY MEETING OF THE INDIANA SECTION

The spring meeting of the Indiana Section of the Mathematical Association of America was held at Purdue University, Lafayette, Indiana, on May 8, 1948.

Eighty-two persons attended the meeting including the following forty-seven members of the Association: J. L. Beal, Stanley Bolks, C. F. Brumfiel, Lee Byrne, G. E. Carscallen, K. W. Crain, H. E. Crull, Rev. H. F. De Baggis, M. W. De Jonge, R. H. Downing, Sister M. Virgilia Dragowski, F.O.S.F., P. D. Edwards, Ky Fan, E. L. Godfrey, Noel Gottesman, S. H. Gould, G. H. Graves, W. S. Gustin, Smith Higgins, Jr., Carl Holtom, H. K. Hughes, H. F. S. Jonah, P. S. Jones, M. W. Keller, E. L. Klinger, Florence Long, Sister Mary Ferrer McFarland, R.S.M., Karl Menger, G. T. Miller, P. M. Nastucoff, Paul Overman, Philip Peak, J. C. Polley, P. M. Pepper, C. K. Robbins, Arthur Rosenthal, A. E. Ross, G. X. Saltarelli, L. S. Shively, R. B. Stone, Raimond Struble, Anna K. Suter, G. L. Walker, M. S. Webster, A. M. Welchons, K. P. Williams and M. A. Zorn.

At the business meeting it was decided that the fall meetings which for the past several years have been held jointly with the Mathematics Section of the Indiana Academy of Science will be discontinued. The spring meeting of 1949 will be held at the University of Notre Dame.

P. D. Edwards, Ball State Teachers College, was elected Section Governor. Other officers elected at the meeting are: Chairman, H. E. Wolfe, Indiana Uni-

versity; Vice-chairman, A. E. Ross, University of Notre Dame. P. M. Pepper, University of Notre Dame, continues as Secretary. P. D. Edwards was appointed chairman of a committee of three, the other members to be chosen by him, to investigate the possibility of the Indiana Section compiling a report on the curricula of the colleges of Indiana similar to the report of a committee of the Michigan Section.

Professor Karl Menger of the Illinois Institute of Technology gave an interesting hour lecture entitled *Are Variables Necessary in Calculus?* Professor Menger's paper is to be published in the MONTHLY.

Professor P. S. Jones, University of Michigan, on invitation of the Section, gave a report entitled *Report of a Study of the High School Mathematics Prerequisite to Various College Curricula in Michigan College.* He described the activities and findings of the Committee on High School Mathematics of the Michigan Section of the Mathematical Association of America composed of Professors H. W. Alexander, C. C. Richtmeyer, and the speaker. Further information on this subject is presented in the February, 1949, issue of the MONTHLY in the paper by C. C. Richtmeyer.

The following papers were presented:

1. *On the use of a single axis, and of the unit circle in the teaching of trigonometry,* by Professor J. C. Polley, Wabash College.

The first part of the paper was a discussion emphasizing the lack of both utility and theoretical importance in the so-called vertical axis in the development of trigonometric theory, concluding with the opinion that it might better be abandoned in favor of a system in which the coordinates of points are defined relative to a single axis and a point thereon. The rest of the paper was devoted to a discussion on the more extensive use of the unit circle in trigonometry. In illustration the forms for the sine and cosine of the sum and difference of two angles, and those for the sum and difference of the sines and of the cosines of two angles were derived.

2. *On the teaching of determinants,* by Professor A. E. Ross, University of Notre Dame.

It was the purpose of the speaker to derive the usual properties of determinants from a set of assumptions connected as directly as possible with the solution of systems of linear equations. He considered an $n\times n$ matrix $A=(a_{ij})$ and the related systems of equations (I) $a_jx_j=\beta$. He showed that if a function $V(A)=V(\alpha_1\cdots\alpha_n)$ has the properties

$$V(\alpha_1\cdots c\cdot\alpha_k\cdots\alpha_n) = cV(\alpha_1\cdots\alpha_n) \tag{1}$$

$$V\left(\alpha_1,\cdots,\alpha_k+\sum_{j\neq k}c_j\alpha_j,\cdots,\alpha_n\right) = V(\alpha_1\cdots\alpha_n) \tag{2}$$

$$V(I) = V(e_1\cdots e_n) = 1 \tag{3}$$

then

$$\begin{aligned} V(\alpha_1\cdots\alpha_n)x_k &= V(\alpha_1\cdots a_kx_k\cdots\alpha_n) = V(\alpha_1,\cdots,\textstyle\sum a_jx_j\cdots\alpha_n) \\ &= V(\alpha_1\cdots\beta\cdots\alpha_n) = V_k. \end{aligned} \tag{II}$$

Thus if $V(A)\neq 0$, then system (I) has solutions x_j and $x_j=V_k/V$ (Cramer's rule). Following Artin, he proved the "product" formula, and, specializing one of the factors, showed that $V(\alpha_1\cdots\alpha_n)$ is the desired multilinear form with the correct rule of signs for the individual terms. The existence

of V with the properties (1), (2), and (3) is proved by induction. In teaching, one may employ areas of parallelograms and volumes of parallelopipeds (which do have properties (1), (2), (3)), together with (II) to derive an equivalent of Cramer's rule without the use of determinants in the usual elementary sense, and thus pave the way for the general geometrical theory.

3. *Determination of the area of a triangle from its sides*, by Professor E. L. Godfrey, Defiance College, Defiance, Ohio.

The presentation of the theory of simultaneous equations and determinants may well include a method of determining the area of a triangle from the equations of its sides, as well as that commonly given using its vertices.

4. *Exponent laws for integral powers*, by Professor M. A. Zorn, Indiana University.

The exponent laws for integral powers of the same base are derived by means of a modified induction principle.

5. *An introduction to a new theory of elementary complex geometry*, by Mr. E. L. Klinger, Purdue University.

To each point in the kth, three-dimensional complex space are assigned coordinates of the form $(x+ik, u+iv)$, or briefly (z, w), where x, u and v are real variables and k is any real constant. After a discussion of distances, it was shown that the equation of any line, except one lying in planes that are perpendicular to the z-axis, had the form $Ax+Bw+C=0$, where A, B and C are real or complex constants, except when $B=0$. In this case any line through (Z, W) may be represented by the system $z=x+ik$, $w=L(w)+ia$, if L and a are real constants and L an arbitrary one. Certain derived formulas involving the angle γ of intersection of two lines were discussed.

6. *Simplicially interlocking spheres*, by Professor William Gustin, Indiana University.

In an n-dimensional euclidean space let there be given $n+1$ closed spheres S_k such that the simplex T spanning the $n+1$ centers of these spheres is non-degenerate, and such that the simplex spanning any subset of the centers is covered by the spheres with those centers. According to a known theorem, due jointly to Knaster, Kuratowski, and Masurkiewicz, there exists a point common to all the spheres S_k and the simplex T. In this note such a point is found by elementary means.

7. *Short formulations of Boolean algebra, using ring operations*, by Dr. Lee Byrne, Purdue University.

Much interest has attached to recent formulations of Boolean algebras intended to emphasize their character as rings, and thus featuring especially ring operations. Most of these are relatively long, and Dr. Byrne's note was concerned with the question whether a simple formulation of this type might show appreciably more brevity. Leaving closure (and non-emptiness) assumptions tacit, he presented four "transformation" postulates, followed by ten theorems, which suffice to show the system to be a ring, a Boolean ring (i.e. one in which every element is idempotent), and a Boolean algebra (i.e., a Boolean ring with unit). The number of transformation axioms appears to be about two less than in previous versions with a similar approach.

8. *On the eigenvalues of symmetric kernels*, by Professor Ky Fan and Mr. Norman Haaser, University of Notre Dame.

Let the kernel $K(s, t)$ be real symmetric in $a \leqq s$, $t \leqq b$, and such that the classical Hilbert-Schmidt's theory is applicable. (I) Let ξ be a real number and let the eigenvalues λ_i of K be so arranged that $|\lambda_i-\xi| \leqq |\lambda_{i+1}-\xi|$, $(i=1, 2, \cdots)$. Then for any fixed integer $m>0$, $|\lambda_j-\xi|$ is the

greatest value which can be taken by the $G. L. B.$ of the expression $(\|(\xi K-I)^m f\|/\|K^m f\|^{1/m}$ when f is orthogonal to $j-1$ arbitrarily fixed functions. (II) From the case $m=1, j=1$ of (I) one obtains directly the inclusion theorem of D. H. Weinstein (*Proc. Nat. Acad. Sci.*, vol. 20, 1934, pp. 529–532. (III) If the eigenvalues of K are bounded from below and so arranged that $\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_i \leqq \cdots$, then as limiting case $\xi = -\infty$ of (I), for any fixed even integer m, λ_j is the greatest value which can be taken by the $G. L. B.$ of $(K^{m-1}f, f)/(K^m f, f)$ when f is orthogonal to $j-1$ arbitrarily fixed functions. If, in addition, K is positive definite, then it can be shown, as was proved by L. Collatz (*Math. Zeitschr.*, vol. 46, 1940, pp. 692–708) and R. Iglisch (*Math. Ann.*, vol. 118, 1942, pp. 263–275), that the above characterization of λ_j holds for any integer m, even or odd, and $(K^{m-1}f, f)/(K^m f, f)$ is non-increasing with respect to m. (IV) If K, K', K'' are three real symmetric kernels such that

$$K(s, t) = \int_a^b K'(s, r)K''(r, t)dr,$$

and if their respective eigenvalues λ_i, λ_i', λ_i'' are so arranged that

$$|\lambda_i| \leqq |\lambda_{i+1}|, \qquad |\lambda_i'| \leqq |\lambda_{i+1}'|, \qquad |\lambda_i''| \leqq |\lambda_{i+1}''|,$$

then $|\lambda_{i+j+1}| \geqq |\lambda_{i+1}'| \cdot |\lambda_{j+1}''|$ holds for all $i, j \geqq 0$. This inequality implies that, for the composite kernel K, the series $\sum |\lambda_n|^{-1}$ converges. This is a particular case of a thorem due to Lalesco-Gheorghini (cf. Hille-Tamarkin, *Acta Math.*, vol. 57, 1931, p. 31).

P. M. PEPPER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Joint Meeting with American Society for Engineering Education, Troy, New York, June 20–21, 1949.

Thirty-first Summer Meeting, Boulder, Colorado, August 29–30, 1949.

Thirty-third Annual Meeting, New York City, December 30, 1949.

ALLEGHENY MOUNTAIN, West Virginia University, Morgantown, May 7, 1949.

ILLINOIS, Bradley University, Peoria, May 13–14, 1949

INDIANA, University of Notre Dame, May 7, 1949

IOWA, Drake University, Des Moines, April 15–16, 1949

KANSAS, Kansas State College, Manhattan, April 2, 1949

KENTUCKY, Centre College, Danville, May 14, 1949

LOUISIANA-MISSISSIPPI, University of Mississippi, Oxford, April 8–9, 1949

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, University of Virginia, Charlottesville, Spring, 1949

METROPOLITAN NEW YORK, Brooklyn College, April 9, 1949

MICHIGAN, Wayne University, Detroit, April 2, 1949

MINNESOTA, Gustavus Adolphus College, St. Peter, May 7, 1949

MISSOURI, University of Missouri, Columbia, April 9, 1949

NEBRASKA, Lincoln, May, 1949

NORTHERN CALIFORNIA

OHIO, Ohio State University, Columbus, April 2, 1949

OKLAHOMA

PACIFIC NORTHWEST, Oregon State College, Corvallis, March 25–26, 1949

PHILADELPHIA, Haverford College, November 26, 1949

ROCKY MOUNTAIN, Colorado School of Mines, Golden, April 22–23, 1949

SOUTHEASTERN, University of Alabama, University, March 18–19, 1949

SOUTHERN CALIFORNIA

SOUTHWESTERN

TEXAS, Denton, April 8–9, 1949

UPPER NEW YORK STATE, University of Buffalo, April 30, 1949

WISCONSIN, Lawrence College, Appleton, May 14, 1949

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ON THE NATURE OF APPLIED MATHEMATICS*

R. S. BURINGTON, Bureau of Ordnance, Navy Department**

1. Introduction. The mathematician is almost always irresistibly attracted by unquestionable logical deduction, where the assumptions are clearly and explicitly assigned and the conclusion follows with decision and certainty [1]. To a very great extent he can choose his mathematical models and materials and can more or less decree or dictate the characteristics and manner in which his materials behave, and he can do it in such ways that the sometimes muddy streams of nature cannot interfere [2]. But the problems of nature are considerably different, and so are the problems of the applied mathematician. In such problems we are given, say, a part of the premises, perhaps a portion of the conclusions; also portions of one or both of the premises and conclusions may be given, not as precisely determined data, but as data with varying degrees of probability and plausibility. As in pure mathematics, rigorous logical reasoning is needed to form a well-mapped structure joining the assumptions and conclusions, but it is the foundation anchorage upon which the structure rests which is so uncertain, and hence apt to be precarious. It is about the nature of these problems of applied mathematics, and in particular those of anchorage, that I shall speak today.

It is inevitable that many of the points of view I shall discuss are similar to those held by others. Some of these views have been developed by the great masters over the years and may be found in the writings of many. I have found certain works of Russell, Langer and Carnap particularly pertinent. Various matters which I will raise will doubtlessly strike a familiar note to many of you as being a part of your own experience.

I shall not attempt actually to define pure mathematics, nor shall I try to define physics or engineering. I shall not quarrel over whether one is the other or not, or as to what are the boundaries between them. Actually, in applied mathematics, and in the world at large, there are no such hard and fast compartmentations. When I speak of applied mathematics, I include all of those fields commonly known as mathematical physics, physical mathematics, engineering mathematics, mathematical engineering, mathematical economics, biomathematics, mathematical statistics, and the like. Where one branch of pure mathematics is applied to another branch of pure mathematics, I shall not consider it applied mathematics in the sense of this paper, even though in some sense all pure mathematics is applied mathematics.

2. On mathematical applications to natural, economic and other types of phenomena. A study of almost any field of natural, physical, economic, bio-

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** The views expressed in this paper are those of the author and are not necessarily those of the Navy Department.

logical, or industrial science discloses the fact that the actual reactions of the objects or situations under observation to various influences are almost always to a considerable extent complicated. Often the reactions are due to the influence of factors which may not be those of principal concern in the study; yet even though they be irrelevant or merely incidental, they may present certain difficulties and distractions and so becloud the real issues involved. It almost always happens that when a full account is taken of all these factors, the resulting formulation of the actual problem at hand appears well hidden. Consequently, the pertinent factors and/or characteristics in the field of application are very likely to be obscured by distracting and confusing features. The result is that the problem for which a solution is desired is quite apt to be set in such a fashion as to be beyond the realm of possible mathematical solution, if not beyond any type of solution.

Such difficulties as mentioned above make it necessary and desirable to view any natural, physical, economic, . . . situation F in the light of what it might be interpreted to be, or actually might be, if the objects or situations could be stripped of the lesser distracting elements, leaving only the essential and fundamental features, skeleton and flesh. Just how to determine what is essential and fundamental is very often distressingly difficult, and commonly requires long study and keen insight; many attempts may be required. Sometimes it is helpful to view the field F from the overall point of view rather than the microscopic. The importance of reconnaissance before becoming immersed in details should be realized. This may help in eliminating or circumventing various pitfalls which may hinder progress. In other cases this overall point of view may not be particularly useful. In considering a troublesome field it is often productive of progress to begin with the simplest cases rather than undertake more complicated situations involving the less familiar and obvious. The product of all these considerations is a certain simplification and idealization or model M of the original situation and formulation of the problem [3]. Sometimes more than one such idealization is desirable. It is such idealized and simplified models M which are made the subject of analysis, and this may entail extensive mathematical analyses.

When the better-established concepts and related intuitive notions prove not to be successful as guides in the study of a field F, or preconceptions prove too confining, it may be necessary to use a high type of free mathematical construction, using premises which may appear to have little connection with previous experience as a guide. However abstract the premises so devised, whatever deductions from theory that may result must always have a readily identifiable relation with experience. To carry through successfully such an unconventional treatment requires an unusually fine insight [4].

The mathematical analysis of an idealized model M may involve the selection of a set of observable quantities $x_1, x_2, \cdots$ which are assumed to be appropriate for the study of the problem at hand. The relations R involving $x_1, x_2, \cdots$ thus obtained may be algebraic and/or they may involve the functions of

analysis, thus leading to differential equations or the like. It is at this point that the services of the mathematician, *as such*, are needed. It is his business to reveal all that is implied by the mathematical structure R presented. But this may, or may not, be possible, for sometimes existing mathematical techniques will permit the realization of either only a part or perhaps none of the goal. In order to make any progress at all, the mathematician may then be driven to make further qualifying assumptions and a corresponding model M' which may or may not be entirely justified. Such compromising assumptions may either seriously restrict the field of investigation or may lead to results which are incompatible with the original idealized model M and its prototype. In many such cases entirely new fields for pure mathematical investigation become vividly apparent [5].

Thus the applied fields may suggest problems in pure mathematics which need solution, or they may serve to suggest further pure mathematical research (perhaps considerably removed from any possible or immediate applicability to the original applied field), which may require or involve the efforts of many pure mathematicians. Such pure mathematical research should not be confused with applied mathematics.

If one knowingly digresses from a firm foundation for a field F of study, he should not blind himself into thinking he has solved the problem. He may select any premise he wishes, but what he chooses may not apply to his problem, or perhaps at the moment to any other problem. In such cases, he should realize that he is dealing with a purely mathematical situation as he proceeds with it. Perhaps the results obtained will be useful some other time or place, but he should not claim that he has solved the immediate or original problem in F.

It is of considerable importance to realize that the idealized model M is inevitably somewhat different from the actual prototype situation F. As a consequence any mathematical results derived from the model M and interpreted in terms of the situation F may be questioned quite properly as to their validity, applicability, and relevance to the original setting F. In other words, the foundation anchorage upon which the idealized model M rests must be thoroughly investigated in the light of its bearing on the relevance and applicability to the original and primary setting F of the mathematical results derived from the model. If the model M differs too greatly from the original situation F, any related practical considerations or projects may be held back or brought to naught. This is particularly and often vividly appreciated by those who in the course of their professional work apply mathematics to the problems of industry and the business world. No matter how logical and beautiful an analysis is made in connection with an industrial project or consideration, if it fails to meet the realistic tests of relevancy, adequacy, and applicability, sooner or later the work will be recognized by those engaged in the original larger problems F as being inadequate, and perhaps shelved as "being worthless because it is too impractical, academic, · · · ," "not giving us the answer, and it is answers we want," and so on [6].

Sometimes, even though the model M is adequate for the particular situations to be considered involving the prototype F, attempts are made on the basis of deductions made in the system M to say more about the system F than is justified by the connection initially established between F and M. In so doing meanings are attached to the mathematical terms and relations, connotations which are not necessarily valid in F, and which result in fantastic and extravagant conclusions. This must be recognized and guarded against. The literature abounds in examples of this sort of thing, particularly certain types of semi-popular and popular scientific publications.

It is such considerations as these which place bounds on the domains in which simplifying models and idealizing abstractions can be successfully made. To estimate successfully the extent to which the idealization or abstraction can be made safely requires a discriminating and sharp sense of values and a truly adequate, penetrating wealth and depth of understanding. These qualities are indispensable.

In any particular situation whether the model M falls within allowable bounds must, as a general rule, be determined by experiment after the analysis based on the model has been completed. The crucial test as to the true applicability of the theory as deduced mathematically or otherwise stands or falls according as its results agree or disagree at suitable points with the data of observations, that is, with experiments and measurements. The structure must take adequate account of the "facts." However, in studying the degree of agreement or disagreement ample weight must be given to the probable reliability of the test data, and in many cases to the reliability of the premises upon which the model and subsequent analyses were made. In both the premises and in the test data precisely determined information and data are seldom available, but rather are of varying degrees of probability. Whatever the final decision as to the applicability or non-applicability of the mathematical theory, the decision does not necessarily cast any reflection upon the theory's soundness as a well-knit logical structure joining the premises and conclusions, except in so far as the theory involves the use of numerical results or premises which may rightfully be questioned, or involves steps which entail actual errors in calculation or logical reasoning. When the latter types of errors are missing, the inferences to be drawn from the final judgment are likely to bear only upon the legitimacy of the abstractions and simplifications which were made in the construction of the model and its basis, and this in turn rests upon the discernment and intuition used in determining the appropriateness, and sufficiency of the primary idealizing approximations [7].

A crucial test as to the success of a particular model structure, or theory, lies not only in the degree to which the results reached by analyses agree with experiments and measurements, but in the power of the theory to *predict* observations in the prototype field previously unknown, and in the ability of the theory to include *all* already known pertinent facts in the realm it is designed to describe. Often a number of theories or models may be advanced which appear to fit,

within a certain range, the prototype field equally well. This may be a good thing. Try them out. You will find that your structure or theory is an evolutionary thing. Among these theories T there may appear to be one which is the most worthwhile pursuing as the "best bet" for future investigation. Such a theory should be suggestive of new lines of investigation. Furthermore, if in addition to being suggestive and stimulating to new developments, this theory meets the test of prediction, it is very likely to prove to be a truly successful theory, and probably the most suitable to adopt of the several original apparently equally good theories T.

It sometimes happens that a mathematical structure of a theory for an applied field, with all of its simplifying assumptions and related uncertainties, is considered in the light of its predictions as a basis for making important decisions involving considerable financial, personnel, or material risk [**8**, **9**]. There may be many other factors quite apart from the particular structure used which should be considered before final decisions are made. In such cases the experienced qualitative or quantitative judgment of others should be sought, a judgment in which the relative importance of various factors, the reliability of data, and other pertinent items which may lie far beyond the scope of the available mathematical structure, should be given appropriate weights. In making any final decisions, a careful synthesis of both approaches and methods is highly desirable and essential in order to reach the best possible decision. This is no reflection on the work of the applied mathematician as long as he has done all in his power to cover the situation adequately within available time, factual, and other limitations.

A study of almost any field of application of mathematics—be it physical, economic, biological, or other phenomena—reveals some or all of the characteristics and difficulties mentioned above, and more too. Thus the success of almost any broad program in applied mathematics depends upon superior insight and perception, and a high level of background knowledge, experience, and technical excellence on the part of the investigators.

3. **Types of activities in applied mathematics.** There are many ways in which the activities pursued in applied mathematics might be classified. For convenience I will consider the following general types. In listing these categories it is not intended that they should be considered as mutually exclusive or exhaustive. In the following discussion, by a field of activity F is meant any field of activity whatsoever, be it economic, physical, biological, and so on.

I. *Problems involving the formulation of a general mathematical structure and theory for a particular situation or field F being made the subject of study.* The history of mathematical physics abounds with examples in this category. The formulation by Newton of his theory of gravitation, by Maxwell of his theory of electromagnetism, by Einstein of his theory of relativity, and the development of various theories of quantum and wave mechanics by Planck, Heisenberg, Schroedinger, Dirac, von Neumann, and others, are examples of the sort of

developments I have in mind. Certain current theoretical developments in nuclear physics fall into this category. Various recent attempts to formulate mathematical theories of economics furnish other examples. Additional developments may be listed in many other fields.

The formulation of a mathematical structure or theory for a particular field F need not be unique. The numerous theories encountered in cosmology, cosmic rays, and so on furnish excellent examples of this.

In the course of my own professional endeavors, I have encountered and formulated many problems where the general mathematical structure and theory have had to be developed for a field about which relatively little has been known quantitatively. In particular, my experiences with the development of adequate theories of weapons systems furnish excellent examples. The need for such theories arises long before the systems being studied have even reached the drawing board stage [8]. As design and development proceed the theories must be revised as they pass through an evolutionary stage finally culminating in experimental tests and perhaps operational verification. Similar developments of theory are required in operational problems.

It has been stated by many scientists that the art of properly stating the question, that is, the art of formulating a problem in a field F, is often more important than its solution. This is very important indeed. In fact, the obtaining of the solution is often merely a series of reformulations of the given problem, leading eventually to a formulation which itself is the solution to the problem.

II. *Problems involving the formulation of a simplified model or abstraction for the purpose of solving a particular problem or class of problems encountered in a field F.* Such problems may be comparatively simple, but very often may prove difficult, and at times may require a considerable degree of originality and insight. Such formulations may be carried out by analogy or suggestion, or direct deduction, derived from a well-established mathematical structure previously developed for the field F, or for a field which may be considered analogous. Many current problems in radio communication, wave-guide theory, and so on fall into this category. If no precedent exists upon which to base the analysis, and a structure must be devised, the problem is considered as belonging to category I.

III. *The solution of mathematical problems which have their origin in mathematical and physical structures which are well established (or may be so considered at the moment) for the field F.* Many of the problems of conventional circuit theory, strength of materials, structures, heat conduction, dynamics, and the like, fall into this category. While the mathematical structures or formulations needed in such cases may be well established, it still happens often that a great deal of pure mathematical skill is needed in order to reach the desired conclusions.

There are many cases where the physical and mathematical formulations for the field F are fairly well established, but the purely mathematical theories

and techniques needed do not exist, and therefore should be developed. Such problems are a challenge to both the pure and the applied mathematician. The current research efforts in the fields of non-linear analysis form excellent examples, the impetus stemming from problems of applied mathematics (*e.g.*, celestial mechanics, electronics, dynamics), problems in which the usual principles of superposition do not hold, problems in which the equations encountered are not linear, and which when linearized in various ways lead to solutions which are incompatible with the underlying physical problems. The last few years have seen many instances of this sort. Thus, the development of non-linear dynamics by Poincaré, Van der Pol, Minorsky [**10**], Cartwright, Lefschetz, and others has been given impetus by a flood of physical problems in which the traditional linear methods fail completely to furnish the required mathematical techniques necessary for the proper development and understanding of certain physical theories, and the discovery of methods for producing reliable predictions in these fields.

The well-known problems of the motion of airships, surface ships, the motion of a pendulum, and so on give rise to a host of problems in which linear mathematical methods often completely fail to predict the true behavior of the systems under study. Thus the oscillations in such cases may have variable periods. Existing linear theories are completely inadequate to cope with such situations. But the developments of the mathematical theory of non-linear mechanics are contributing much to the proper understanding of such phenomena.

Similarly the fields of electronics, heavy rotating mechanical and electrical machinery, vibrations in airborne structures, strange behaviors in servomechanisms and control systems, electrical networks and filters with variable circuit parameters, and others appear to require the development of more adequate mathematical theories and techniques. These fields present considerable challenge and inspiration to research mathematicians, as well as to applied mathematicians and engineers.

IV. *The numerical solution of problems arising in any of the above categories.* Many of the mathematical problems encountered in applied fields have been solved theoretically. However, in many cases, without careful mathematical organization and planning, the computational difficulties and the labor involved in obtaining numerical solutions are prohibitive; and even with the best of planning the magnitude of the undertaking may be such as to require the use of computing engines. Many problems in the theory of gas dynamics, meteorology, ballistics, and aerodynamics fall into this category. The general field of boundary-value problems of mathematical physics abounds with such problems, such as those involving integral equations, partial differential equations, and the like, both linear and nonlinear. Much of the incentive for current developments in computing machines, computing techniques, theory, and the like, stems from such problems.

Several large-scale calculating engines have been built to obtain the numer-

ical solutions to a large variety of difficult and important problems. These machines, and some others now in the development stage, should make it possible to deal with problems which have heretofore been too complicated to handle. They have inspired the development of new methods of analysis which should be applicable to the problems now confronting us.

A word about the use of such machines may be in order. Before a particular problem is given to one of these engines for solution, it should be ascertained first whether it is important enough to warrant the labor and high expense involved in operating these engines. A mathematician charged with the problems which appear to need the use of such machines should bear this in mind. Unless the problem fully warrants the expenditure, the machine should not be used; in case it is used, the mathematician should exert every effort to set the problem in the most efficient way, and make every effort not to load the problem beyond the anticipated needs.

The development of such machines as the Harvard Mk 1, and the Mk 2 now at the U. S. Naval Proving Ground, Dahlgren, Virginia, the Eniac, the IBM Selective Sequence Calculator, the Bell Computers, and others have served to exert great influence on research in numerical methods. The problems suggested by practical difficulties encountered (*e.g.*, error estimation in computational process, rounding off of errors, coding, programming, and so on) in the use of such machines have led to a number of stimulating and thought-provoking mathematical studies, such as illustrated by the recent works of J. von Neumann [**11**], H. H. Goldstein, H. Aitken [**12**], and many others. Some of the work so stimulated has involved research of a high character in pure mathematics. The results of such investigations will bear additional fruit as better means are found to carry through the solutions of problems which, but a short time ago, no one could or would have dared to undertake.

V. *The statistical analysis and interpretation of data collected in connection with experiments or facts relating to studies in a field F.* A great deal of applied mathematics encountered in practice falls into this category and much of it can be classed as routine.

Wherever studies in a field F involve the collection, processing, and interpretation of data, the formulation of conclusions, and the presentation of related data as supporting evidence, statistics plays an important role. Serious and difficult problems sometimes arise which require a high order of mathematical statistics. Here mathematical theories are applied to the development of statistical methods (*e.g.*, designs for experiments, surveys) and to the determination of whether and to what extent various statistical methods are appropriate to the problems at hand, that is, to what extent the statistical theories are properly anchored both theoretically and subject-wise.

Sometimes one encounters organizations which collect great volumes of information but do little or nothing with it in the way of serious analysis. This should not be considered as applied mathematics or mathematical statistics. The mere collection of facts is not enough for the solution of many problems.

Facts may be plentiful. Any attempt to obtain "all of the data" on a problem is likely to result in such an unwieldy mass of mostly irrelevant information that any solution desired may be needlessly delayed and hindered rather than furthered. To be efficient and useful at an early stage, the gathering of data must be selective, and the best methods for their collection must be used. The available information is first studied and analyzed. Several working hypotheses may be formulated; from each of these are deduced the consequences which should follow if that hypothesis were correct; additional data are then sought and the old data re-examined for evidence which would either bear out or refute the hypothesis under scrutiny.

Contemporary problems in engineering, physics [**13**], economics, and the social and biological sciences abound in situations where the mathematical theories of probability, stochastic processes, ergodic theory, and so on play leading and important roles both in the framing of the mathematical structures and in the techniques involved. Much of the current research work in mathematical statistics stems from pressing problems encountered in such fields. Notable examples of this may be seen in the recent works of Wilks [**14**], Tukey [**15**], Scheffé, and others on order statistics; the sequential tests for statistical hypotheses of Wald [**16**]; the probability theories of Cramer [**17**], Doob, Feller, Kac, Von Mises; the various statistical techniques developed by Neyman, Hotelling, and others.

There appears to be the same distinction between pure mathematical statistics and applied statistics, as between pure and applied mathematics in general. Statistics can be very useful in almost any field F—physical, social, biological, or otherwise. If such work is to have the best possible anchorage in the field of study at hand, the statistical techniques used must be adequate and appropriate, but in addition the theories and principles peculiar to the field F behind the events leading to the statistical data must be taken fully into account. For the best of results a suitable compromise between these two highly complementary disciplines must be found and used intelligently.

4. **On the nature of pure mathematics and its role in applied mathematics.** As discussed earlier, the grounds for the acceptance of a theory in any field F of physical, economic, . . . science consist in the agreement at strategic points of predictions derived from the theory with the data of experiment. This is the basis for the acceptance of the results and predictions obtained by the application of mathematics to any such field F. But such acceptance necessitates also the acceptance of the deductions involved which hinge on the operations of pure mathematical techniques. This raises the question as to what are the grounds upon which the acceptance of pure mathematics is justified.

The development of a pure mathematical theory may be considered to proceed from a set P of definitions and postulates, the selection of which is a matter of choice, which are not subject to proof within the theory. Set P is stated in terms of certain primitive concepts which are left undefined within the theory.

The primitive terms may be interpreted in various ways, but, of course, the interpretation used should be consistent with P. The entire mathematical theory is completely determined when the primitives and postulates have been stipulated, and the principles of logic selected; that is, once the principles of logic which are to be accepted and used in the theory have been chosen, every term of the theory can be defined in terms of the primitives, and every theorem of the theory can be derived logically from P. This is a point of view commonly taken in considering mathematics as an axiomatic deductive system.

It is known that the content (concepts and propositions) of classical arithmetic, algebra and analysis can be derived from a rather simple system of postulates (Peano's) involving three primitives, using the rules of ordinary formal logic, the "axiom of choice," and the definitions of the non-primitive mathematical terms. Any interpretation of the primitives consistent with the postulates turns the postulates and all theorems deducible therefrom into consistent statements [**21**]. Actually, many valid interpretations of the primitives can be made. Moreover, when the primitives are interpreted in their usual meanings (*i.e.*, "0," "natural integer," "successor"), the Peano postulates are consistent, and ordinary mathematics (classical arithmetic, . . .) appears as a valid theory of the mathematical concepts used in their ordinarily intended meanings [**18**, **19**, **20**].

In the case of those branches of mathematics which do not stem from arithmetic (*e.g.*, topology, geometry, fields, group theory, and so on), the situation is quite similar [**22**]. Each such branch of mathematics can be treated as a deductive system stemming from a suitable set of postulates P. If a theorem T is derived from P, by suitable principles of logic, then T is "true" provided the postulates (involving unassigned meanings) are accepted as "true" [**23**, **24**].

In applying any such branch of mathematics to either a specific field of pure mathematics or of applied science, it is necessary to allocate to each primitive some specific meaning and then find out whether this converts the postulates P into consistent ("true") statements. When this is the case, all derived theorems T are also consistent ("true") statements. To put the matter in a slightly different way: Pure mathematical propositions are of the form P implies T, where both P and T involve parameters $x, y, \cdots$. The mathematics does not concern itself with the truth of either P or T; what it does do is to assert, "If P is true, then T is true"; and (with the acceptance of the more common forms of logic) "If T is not true, then P is not true." In other words, if P is true for certain values of $x, y, \cdots$, then T may be asserted as true for those values of $x, y, \cdots$.

If one accepts this view of pure mathematics, he may wonder why it is that mathematics has proved itself of such value in the applied sciences. In any problem relating to field F in which pure mathematics is applied, the role of mathematics is to render explicit assumptions or propositions Q in the field F which are implied by and included in (or can be deduced from) the premises P underlying the treatment of the problem, the propositions Q being hidden and

perhaps heretofore unobserved. Actually from this point of view the mathematical process leads to nothing intrinsically new. The theorems of pure mathematics in themselves give us no information whatever on any field F of physical, economic, or other experimental science. While the results obtained are implied by the premises P, and are thus not intrinsically new, they may be new in a psychological sense. The techniques of logic and pure mathematics applied may disclose much more than might be anticipated from a brief examination of the premises P involved. It is thus not surprising that one hears so much about "the power of mathematics to discover new facts," even though the mathematical reasoning merely reveals the relationships hidden in the premises.

The function of mathematics in an applied field may be thought of as analytic and clarifying, but not in itself predictive. Surely the mathematics contributes much to the gaining of insight into the field F; it is a key to understanding. The predictive power of the treatment in which mathematics is used originates in the initial data of observation and from the fundamental laws or premises assumed for the field F in question. As long as full and proper use is made of mathematical techniques, the failure of a prediction in the field F should be attributed to the inadequacy and unsuitability of the premises and data of observation involved, that is, to a weak anchorage of the mathematical structure in the field F.

A word about geometry is in order. Geometry may be considered as either "pure" or "physical." In pure geometry the mathematician may interpret the primitives in a plurality of ways, and will do so, and he need not give any specific meaning to the primitives. On the other hand, the physical scientist or engineer may be willing to accept only one interpretation of the primitives and the geometry.

In physical geometry, and it is this kind of geometry which is so commonly used, the geometry is a kind of theory of the structure of a physical space. Here the primitives are interpreted as referring to certain kinds of physical entities—perhaps defined, as in the case of measurements, in terms of human (or machine) operations; and the theorems and postulates are construed as factual properties of interpreted physics relating to space, motions and configurations of bodies, and the operations involved in defining the terms used. Hence the truth of this type of geometric theory hinges on physical, experimental, and human considerations as well as those of a mathematical character [25]. This is in distinct contrast to pure geometry, where no such relationship to physics is involved. Pure geometry is concerned primarily with consistencies, and as such is regarded as pure mathematics. As pure mathematics, pure geometry asserts nothing concerning physical space or any specific field of application.

When the terms of a pure geometry are given physical definitions, and the postulates in their physical interpretations can be considered as true, then the theorems of the geometry in the light of their physical interpretation can be taken as necessarily true. Such theorems tell us that if $D_1, D_2, \cdots$ are done,

then $S_1, S_2, \cdots$ will happen [26]. A word in this connection seems appropriate. The act of giving the terms of a pure mathematics a physical setting and definition does not make the theorems of the pure mathematics necessarily true statements in a physical field in which the physical terms that are used appear. As discussed throughout the earlier sections of this paper, the proper anchorage of the postulates of the mathematics in the physical field is also very necessary. As long as an adequate amount of confirming evidence appears in the physical field, and until some disconfirming evidence has been discovered, the theory may be accepted.

A word of caution should be voiced at this point. It is my intention here to emphasize the nature of applied mathematics, the nature of its anchorage in the applied field, and the nature of the pure mathematics which may be involved. In carrying out this intent, I have deliberately stressed the postulational and deductive character of pure mathematics. Actually I could not leave this topic of pure mathematics, its nature, and its relation to applied fields without emphasizing the great importance in both pure and applied fields of the forces of intuition, invention, and discovery, and the motives underlying any developments of value in these fields. Surely they are most important and indispensable elements in both pure and applied mathematics.

Not only is there importance to these intuitive forces, but they may enable one to frame more meaning to the mathematics. It may be that mathematics is much more than it is thought to be when viewed from the deductive postulational basis [27].

In the soul of man there is higher ability which may enable him to formulate concepts and postulates and to so order them that they serve to produce and create new results. It is this sort of thing that cannot be measured in the postulational deductive system of mathematics, and it is a very important part of mathematics, pure and applied.

5. **Further examples illustrating the nature of applied mathematics.** Each of the categories $I, \cdots, V$ of activities discussed in Part 3 contains some or all of the general characteristics discussed at length in the earlier parts of this paper. A comprehensive investigation of a fairly large field F may entail some or all of these categories, and perhaps others.

As additional examples of the principles under discussion, I will mention several fields of activities of considerable current interest. In selecting these illustrations, I have not attempted to be exhaustive, but merely illustrative. Many other fields will doubtless occur to you.

Seismology. In seismology earth waves supply the basic data. Such waves play the same role as sound waves do in acoustics, and as electromagnetic waves do in radio communication. Not too much is known about the structure of the earth's crust or the types of waves that earth materials in place will transmit. Quantitative observational evidence as to the types of waves of this sort is meager. Some information has been obtained from seismographs of earth

waves generated by earthquakes, but such sources are, of course, unpredictable and do not furnish a controlled source. Explosive blasts as generators of earth waves have furnished some controlled quantitative information. One of the basic problems of seismology is to establish observationally the types of waves which earth materials when in place will transmit and support. Classical mathematical geodynamics has indicated that waves in the earth must be elastic in type. In 1885 Lord Rayleigh predicted from theoretical mathematical considerations the earth wave now known as the *Rayleigh Wave*, but the existence of such a wave was not shown observationally until 1931. Lamb predicted from theoretical considerations in 1904 a certain type of wave pattern which could be generated by a vertical impulse force, but the necessary experimental conditions were not succesfully realized until July 16, 1945. One type of surface shear wave, now known as a *Q-Wave*, was predicted by Love from theoretical elasticity considerations. The predicted *Q-Wave* was first observed in 1939, along with a heretofore unobserved and unpredicted wave called the coupled or *C-Wave*, in connection with the study of earth waves generated by dynamite blasts. The 1945, New Mexico tests also showed the existence of still another type of earth wave known as the *Hydrodynamic Wave*, which had not been reported or predicted previously [**28**]. In general, the test results of 1945 differ widely and very significantly from predictions [**29**]. The evidence seems to indicate that existing theories are inadequate, due not only to the lack of suitable observational data, but due also to an inadequate anchorage of the theory, and to inadequate mathematical techniques. Thus, while volumes have been written on the theory of elasticity and elastic waves, there still is need for both quantitative experimental information and a suitable mathematical theory, a theory anchored firmly in the known data which will predict all known types of earth waves, and, if possible, one which will say "here are all the types, there are no others." The field of *Seismology* offers a challenge not only to geophysicists and seismologists, but to mathematicians as well.

Mathematical biophysics. In recent years certain phases of biophysics have become increasingly more analytical in character [**30**]. The trend is indicated by the magnitude of research now underway in the field of mathematical biophysics. One of the principal goals in this work is the construction of a mathematical biology which will stand in the same relation to experimental biology as mathematical physics now stands to experimental physics. A survey of current literature in this field shows the extent to which investigators are trying to build a mathematical biophysical structure which will be firmly anchored in the data of experiment, and which will stand the tests of prediction and those other requirements which any such structure must meet.

Some of the developments now in progress on theories of nervous systems (*e.g.*, mathematical neuropsychology) are quite analogous to, and actually are, theories of communication [**31**]. So little is known—and some of the existing experimental data are such as not to hang together very well—that theories in this field, as well as in the general field of mathematical biology, may well be

expected to change rapidly as time goes on and the anchorages are changed, just as has been the case in the history of cosmological theories.

In a field of this sort there is a considerable tendency to make use by analogy of the results of classical mathematical physics. Such attempts may be productive, but the pitfalls which are so often experienced with the use of analogies should be guarded against. Perhaps a very fresh approach in which the attempt is made to build a structure directly from the known biological data might in the long run prove more fruitful of lucrative advances.

A very recent development in this field is the research of Norbert Wiener on *cybernetics* [32]. In this work a new effort is made to find the common elements in the human nervous system and in the functioning of automatic machines. The attempt is to develop a theory which will cover the entire field of communication and control, both in living organisms and in machines.

The field of communications. In communication engineering, the principal interest is in the accurate and rapid transmission of signals. Here is a field which exhibits activities in all of the Categories $I, \cdots, V$. In spite of the amazing advances which have been made in the field of communications, there still remain many major unsolved problems. Today the propagation of waves along lines, wave guides and surfaces plays a major role in communication practice. This means research over the whole frequency spectrum. Here we see merging the older circuit theory for low-frequency phenomena and the field theory of high-frequency phenomena [33].

Much was developed during the war principally from fundamental research which was carried out prior to the war. A considerable portion of this work remains unpublished, or is not available for general use.

Powerful mathematically-trained groups may well formulate a more general structure upon which the entire subject can stand. The discovery of new types of circuit elements, semi-conductors, new methods of detection and rectification (as the Bell Telephone Laboratories' Transistor), new switching techniques, and the like throws this whole field open to new intensive study.

The use of pulse systems of communications, television, and radar point to the need for a new, more direct approach to circuit behavior with particular attention to transient phenomena. There is some indication that a new superposition principle is needed so that the overall characteristics of the newer systems can be determined readily and directly from those of the individual elements.

The fidelity of transmission is intimately related to noise since the received signal is a function of both the original signal and such additional noise signals as are inevitable in transmission. A study of what happens to noise disturbances when they pass through transmission lines and systems has been the subject of much successful investigation. The work of Rice [34] on the mathematical analysis of noise is noteworthy [35].

Considerable work has been done recently on the propagation of electromagnetic waves [36] in bent pipes or wave guides [37]. It appears that a more general mathematical structure for this field is needed.

Much still remains to be done in the theory and design of antennas, reflectors, and so on. This field provides many opportunities for improving mathematical techniques. A more thorough, rigorous and well-founded formulation of the transmitting, receiving and other properties of antennas is being actively pursued by investigators [**38**]. Here is a good example of an activity of the type of Category I, where a soundly anchored structure for the theory is being actively sought. A better mathematical theory for predicting the directivity and selectivity of an antenna system which takes proper account of geometric features, physical sizes, and neighboring matter is needed. This will no doubt involve the formulation of new general concepts and theorems. For example, the design of an antenna along classical lines for installation in aircraft may prove to be wholly unsatisfactory unless unusual care is taken to account for the radiation properties of the airplane itself. Here is a real problem for both the designer and the mathematician; a completely satisfactory solution is not yet known.

Very recently C. A. Shannon [**39**] has published an important new paper on the mathematical theory of communication. This theory makes extensive use of stochastic processes. Probability measure space plays a central role. Here one finds beautiful examples of Markoff processes which have been studied so extensively in the literature of pure probability theory. Here we find some of the concepts, such as entropy, which have been found so useful in statistical mechanics playing a central role as measures of information, choice, and uncertainty. The received signal, noise, and the transmitted signal appear as chance variables.

The recent penetrating work of Norbert Wiener [**40**] on operations on time series has given us a clear formulation of communication theory as a statistical problem.

These new theories of Wiener and Shannon will undoubtedly find wide application, and as with the cybernetics [**32**] of Wiener, will lead to stimulating discussions and progress both in pure and applied fields.

Fluid mechanics. The field of fluid mechanics, the modern hydrodynamics and aerodynamics, abounds in inspiration to the mathematical physicist. The study of the behavior of bodies in fluids, compressible or incompressible, offers much room for the applied mathematician. The fascinating problems of cavitation, the phenomena of the growth and collapse of cavitation bubbles, the behavior of bodies on entering one fluid medium from another, the motion of bodies within the fluids, and the related dynamics have long been of interest, and in fact are coming into the foreground of mathematical hydrodynamics in a powerful way [**41**]. The tie-up of these studies with thermodynamics, as for example in gas dynamics, jets, and so on is a most fascinating study for the applied mathematician. Here the problems are vital, real, and intriguing. In each of these problems one encounters almost all of the Categories $I, \cdots, V$. Whereas the older hydraulics was essentially empirical, and the older classical mathematical hydrodynamics was too much removed in its anchorage from actual experimental data, the newer theory of fluid mechanics is based on firmer physical principles, and its mathematical anchorage in the science is very much

improved. The subject as it is today is closely correlated with the data of experiment which complement and back up the fundamental structure of the theory.

The newer mechanics draws heavily upon the stronger features of both the older hydraulics and hydrodynamics. In the sense of fluid mechanics, both the mathematical and the physical theories are fundamental to meteorology, oceanography, marine engineering, lubrication theory, and ballistics. Hand in hand with this is emerging a modern mathematical theory of hydrodynamics.

The formation and propagation of *shock waves* in a compressible substance have been the subject of intensive study, because of their importance in the development of high-speed aircraft and in the theory of explosions. The works of G. I. Taylor, L. H. Thomas, J. von Neumann, Seeger, Polachek, Friedrichs, Courant, and Weyl are of particular interest [**42**, **43**, **44**, **45**].

For a long time a well-founded *theory of turbulence* has been sought. It is quite possible that the theory of random functions may help to furnish a mathematical framework for an improved theory of turbulence [**46**, **47**, **48**]. The actual fluctuations in the fluid as measured probably include both random and nonrandom elements. Any theory for turbulence must take adequate account of this. As yet methods for experimentally separating these two classes of elements have not been discovered. Hence the formation of a theory remains a difficult problem.

The transient and steady-state *propagation of sound* in fluids, moving or stationary, and from one fluid to another, offer a stimulating challenge. Such analysis may involve the solution of difficult boundary-value problems. It appears there is considerable room for progress here, perhaps in the judicial use of assumptions of a statistical character [**49**].

For centuries man has studied the *winds and the waves of the sea*. The tides are now well understood; here classical hydrodynamics has been found fairly reliable. Much has been learned about the behavior of winds and waves, particularly macroscopically. Some features of such behavior can be predicted with accuracy while others cannot. In this work there are cases where classical hydrodynamics has lent itself well to a successful theory capable of reliable predictions, particularly in deep water. But there is much to learn yet. During the war years, a great deal was learned experimentally and theoretically about the behavior of waves and surf in shallow water, and mathematical methods have been successfully used in some cases [**50**]. Yet in spite of this, little is known of the behavior of breaking waves, and of water movements between breakers and the beach. One must still seriously question theoretical results when an attempt is made to apply them to actual conditions such as on beaches. The excellent and effective work done by the U. S. Hydrographic Office [**51**] and U. S. Beach Erosion Board [**52**] before and during the war years may be cited. Yet there is still need for a good mathematical theory for this type of water movement. Here, there is much room for mathematical activity in the categories discussed earlier, perhaps in Category I in particular.

Elasticity, plasticity, strength of materials. A classical theory of elasticity has long been in existence and has been successfully used by engineers for design purposes, but the limitations of the theory are serious. Where existing theories or mathematical techniques have been lacking, entirely inadequate, or not available, the engineer has gone ahead and built things anyhow. He has often been forced to build when he is unable to calculate accurately the stresses and strains which take place in complex structures. Consequently his structures were sometimes unsafe, and often times over-expensive, too heavy, too bulky, and the like. He has been remarkably successful, though with his progress he has often made serious mistakes. With the advent of aircraft the need for lighter, stronger materials became absolutely imperative, and the designer was forced to operate under severe limitations as to safety factors and the like. Likewise on land or sea, one can no longer afford to drag around unwieldy, unnecessarily heavy objects. In attempting to meet these needs the designer has learned again how inadequate existing mathematical techniques and theories really are. There is great need for a retooling of mathematical techniques and the development of new ones in these branches of elasticity, where a great deal of experimental information is available and where the ordinary theories of elasticity are known to be well anchored.

Existing theories of elasticity are not adequate for many modern practical design problems. The use of laminated materials, of materials whose elastic characteristics are orthotropic [57, 58] has brought new challenges to the theorist as well as to the designer. Here is a great field for research, both in the development of a truly adequate theory and in the development of tools and new techniques for computing in the theory.

Today many materials are being used where not only the elastic properties [59, 60] are of importance, but where plastic properties and the transition from elastic to plastic to strain hardening states must be adequately accounted for. In plasticity, the strains exceed the small values for which the classical theory of elasticity is valid. Here we have phenomena where, with the action of repeated stresses, and accompanying changes in strains, the material undergoes internal changes of such a character that with the restoring of the earlier stress forces, the material does not exhibit the same strains. In other words the entire earlier history of the material must be taken account of in predicting its performance where a kind of elastic hysteresis phenomenon must be recognized, where it must be determined whether changes in strain are permanent or not. In the mathematical theory of plasticity one objective is to determine the history of the state of stress and strain at all parts of a plastic or semiplastic body when the history of the boundary loadings and displacements is specified. Such problems are now the field for wide investigation. Prager and others are actively engaged in building improved theories for the elastic and plastic properties of materials. Here, as in other fields, to be successful one must take specific note of the experimental procedures used in testing the materials and of the purposes for which the theory should be designed.

Theory of structures (bridge design). Man has been building bridges and other structures for untold ages. The theory and practice of bridge design is considered by many to be well established. But a study of the current literature reveals many serious unsolved problems in the design of safe structures. Many questions relating to rigidity, stability, have been brought to the foreground by recent bridge disasters. One notable example of a bridge disaster, directly attributed to the wrong "anchorage" of the mathematical treatment of design, is the collapse of the bridge which whipped itself to death under aerodynamics forces. A study of some of the works of Steinman [**61**] indicates that here is a real field for investigation—for the creation of a new mathematical structure for bridge design, one which is strongly anchored in the physics of the problem, a theory which can be used for design purposes, one which can furnish criteria for aerodynamic stability, and so on.

This is a field where engineers, physicists, aerodynamicists, and mathematicians can weld their techniques and experiences to good advantage.

Evaluation and analysis of systems. Out of the war we have witnessed the emergence of a new field of applied mathematics, various branches of which are known as evaluation and analysis of systems, weapons systems analysis, operations research, and the like. It is capable of very wide industrial and economic, as well as naval and military, applications. All branches of the Services have made wide use of such fields as a very necessary and useful adjunct to the management of large-scale developments and operations [**8, 9, 53**], and as an aid in making important decisions involving large expenditures of money, material, and human effort. It is in these fields that I have devoted a considerable portion of my energies in recent years. Some indication of the nature of this field can be gleaned from a study of contemporary literature. The foundations of these fields are well established in some respects but not in others; much of a theoretical character remains to be done. Activities of the sort described in Category I play leading roles.

Theories of economic behavior. Time will not permit me to go into the recent impetus given to mathematical economics by the work of Morgenstern and von Neumann on the various types of economic behavior and game theories [**54, 55, 56**]. Here we find many of the categories discussed earlier, particularly Category I, emerging in various interesting ways. Such theories are particularly vulnerable because of their anchorage in actual economic phenomena, since the situations that can now be handled may not be realized actually in the world at large, mainly because of the limited knowledge and incomplete and faulty description of the necessary economic facts. However, we can expect the developments of this field to go through the same evolutionary stages that I have discussed earlier in the case of the physical sciences, and to lead eventually to a well-formulated discipline, and probably new mathematical techniques.

Other contemporary fields. Today the startling research and development progress in certain contemporary branches of cosmology, physics, chemistry,

and engineering, particularly in the fields related to nuclear energy, can be expected to affect profoundly the development of future mathematical and physical theories in a way quite similar to the manner in which celestial mechanics has affected the development of both mathematical and physical theories over the centuries. Each of these fields furnishes many pertinent illustrations of the principles which I have discussed at length in the earlier parts of this paper.

6. Summary and conclusions. To summarize briefly what I have said: In pure mathematics one can choose his own models, but in studying the problems of the world of our experience the situation is different, and so are the problems of applied mathematics. In such problems one is given perhaps a portion of the premises and perhaps a part of the conclusions, though often only in a probability sense. A logical mathematical approach is needed to form a well-organized structure to join the premises and conclusions. The process of constructing such a structure may be readily outlined in the language of the theory of equivalence [3] as follows:

(1) The extraction from the field F under study of a nearly isomorphic (equivalent) model M for F.
(2) Reduction of the model M to an equivalent mathematical model M' amenable to treatment.
(3) A solution of this system M'.
(4) The interpretation of the solution found in (3) in terms of the mathematical model M'.
(5) The interpretation of the solution found in (4) in terms of the model M.
(6) The interpretation of the results of (5) in terms of the original field F.

The crucial test of the results lies in the predictive power of the theory and its ability to include all known relevant facts.

The construction of the models M and M' must be done with very careful attention to the foundation anchorage of the structures M and M', for it is upon this that the predictive power of the theory hinges.

I have indicated in some detail the various categories of applied mathematics which are now very much in the foreground of interest, and have discussed in general the nature of applied mathematics.

In actual applications to applied fields, pure mathematics is an invaluable instrument for understanding, acquiring, and expressing knowledge. It serves as a powerful tool which aids in the disclosure of the implications of given theories (or sets of assumptions) in pure and applied fields. The results so revealed may be new to man, and as such may be classed as discoveries, though in reality the mathematics in itself adds nothing new to the actual content and implications embedded in the given theories (or sets of assumptions). However, without the mathematics the results might not be revealed.

The deductions arising from the mathematical study of an applied field are

a reflection of the content of the premises used so that any predictions made for the applied field stem primarily from these premises, the anchorage, rather than from the pure mathematics itself.

Much use is now being made of mathematical methods and talents both in the world at large and in academic settings. Some fields and problems have become so complicated that they cannot be handled by the older simpler methods. Consequently such fields must be re-examined and analyzed with great care. In the practice of applied mathematics the problem of anchorage is of utmost importance, for if anchorage is overlooked, or is incorrect, the results as interpreted in the world of experience may be grossly wrong. Certainly in industry the production of poorly anchored work will not be long tolerated.

For these reasons when mathematics is used in applied science, it should be used with one's eyes open, with conscious and adequate attention to the anchorage in the applied field.

If there is a lesson which you may carry home to your students and colleagues it is this: Mathematically-inclined workers in applied fields, particularly those who lean toward the formalisms and manipulations of mathematics, will do well to watch the anchorage of their theories and manipulations in the underlying field to which they apply their mathematics, lest in their zeal and enthusiasm they forget the tides and winds, and find their ship with anchor dragging, or lost, upon the rocks of futility.

References and Comments

1. In making this statement it is recognized that what in one age is considered as unquestioned, and a certainty, may not be so considered in a later age.

2. At least, Nature may not interfere in too serious a manner. In a psychological sense, Nature may exert considerable influence on the decisions of the mathematician.

3. For a discussion of this principle in the light of the concept of equivalence, see Burington, Richard S., The Role of the Concept of Equivalence in the Study of Physical and Mathematical Systems, J. Washington Academy of Science, Vol. 38, No. 1, Jan. 15, 1948, pp. 1–11.

4. The extent to which mathematical analysis may be used often depends materially on the ability and acquaintance with mathematical methods and techniques of the individuals concerned. The uninitiated may not even recognize the need for, the possibilities, or the advantages which may be gained from the use of such analyses.

5. I do not mean to infer that all pure mathematical research is suggested by something in applied fields. Indeed, the motivations for pure mathematical research are much more extensive. A discussion of the motivation of pure research is one which would require a separate treatment.

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22. There are instances of pure mathematics which cannot be derived from the Peano System. This has been shown by K. Gödel. See Hempel, ref. 18.

23. To go into what is meant by "true" or "consistency" would take us too far afield. I shall leave these terms undefined. Absolute truth is unobtainable in mathematics. There is no absolute test for the consistency of a system.

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A GENERALIZATION OF TAYLOR'S EXPANSION

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1. Introduction. The purpose of this article is to give a generalization of Taylor's expansion. We develop a power series expansion which can usually be made to converge about twice as rapidly as the Taylor expansion. Special cases of our generalized expansion give rise to an almost unlimited number of ways in which a given function may be expanded into a power series so that rapid convergence is usually possible by proper choice of the series used. Finally, it will be shown that the results of several recently published papers are very special cases of our general expansion.

2. The general theorem. The requirements on the function $f(x)$, which is to be expanded, are identical with those required for the Taylor expansion. It will be assumed that $f(x)$ is continuous in an interval which contains a and x. It will also be assumed that all derivatives of $f(x)$ up to and including the one of order $m+n+1$, where m and n are positive or zero integers, are continuous in the same interval. Throughout the article the greek letter θ will denote a value between a and x. Also, we will denote the binomial coefficients by ${}_pC_q$ with the understanding that ${}_pC_q=0$ if q is greater than p.

THEOREM. *Under the conditions just stated*

$$(1)\quad f(x) = f(a) + \sum_{k=1} \frac{(m+n-k)!}{(m+n)!}\left[{}_mC_k f^{(k)}(a) - (-1)^k {}_nC_k f^{(k)}(x)\right](x-a)^k + R,$$

where

$$R = (-1)^n \cdot \frac{m!n!(x-a)^{m+n+1}}{(m+n)!(m+n+1)!} f^{(m+n+1)}(\theta),$$

θ between a and x.

Proof: We define

$$(2)\qquad F(x) = \int_a^x f^{(m+n+1)}(t)g(t)dt,$$

where $g(t)=(x-t)^m(t-a)^n$. By a well-known formula for repeated integrations by parts*

$$(3)\qquad F(x) = \sum_{i=0}^{n+m} (-1)^i f^{(m+n-i)}(t)g^{(i)}(t)\Big]_a^x .$$

It is not difficult to show by induction that

* See for example Cours D'Analyse Mathematique, Goursat, page 190.

$$g^{(i)}(t) = \sum_{k=0}^{i} (-1)^{i-k} i!\,{}_mC_{i-k}\cdot {}_nC_k (x-t)^{m+k-i}(t-a)^{n-k}.$$

Therefore $g^{(i)}(a)=0$ if $i<n$, and for $i \geqq n$, all terms vanish for $t=a$ except the one for which $k=n$. Hence

$$g^{(i)}(a) = (-1)^{i-n} i!\,{}_mC_{i-n}(x-a)^{m+n-i}. \tag{4}$$

Similarly, $g^{(i)}(x)=0$ if $i<m$, and for $i \geqq m$,

$$g^{(i)}(x) = (-1)^m i!\,{}_nC_{i-m}(x-a)^{m+n-i}. \tag{5}$$

Using (4) and (5) in (3) and changing indices by setting $k=m+n-i$, one easily obtains

$$(-1)^n F(x) = \sum_{k=0} (m+n+k)!\left[(-1)^k {}_nC_k f^{(k)}(x) - {}_mC_k f^{(k)}(a)\right](x-a)^k. \tag{6}$$

We now return to (2). Since $g(t)$ is of constant sign throughout the interval of integration, the mean value theorem for integrals† may be applied to give

$$F(x) = f^{(m+n+1)}(\theta)\int_a^x g(t)dt,$$

θ between a and x. To evaluate the integral we set $t=(x-a)u+a$ and obtain

$$F(x) = f^{(m+n+1)}(\theta)(x-a)^{m+n+1}\int_0^1 (1-u)^m u^n du.$$

This last integral is a Beta function and equals $m!n!/(m+n+1)!$. Therefore

$$F(x) = \frac{m!n!(x-a)^{m+n+1}}{(m+n+1)!} f^{(m+n+1)}(\theta). \tag{7}$$

The theorem is now obtained by eliminating $F(x)$ from equations (6) and (7). This completes the proof.

By inspecting the fundamental equation (1) it is clear that by different choices of m and n, a wide variety of expansions is possible. Some of these are of particular interest. If $n=0$, for example, we have the familiar Taylor finite expansion with a remainder. The case where $n=m$ seems unusually interesting and may be written

$$f(x) = f(a) + \sum_{k=1}^{n} \frac{(2n-k)!}{(2n)!}\,{}_nC_k\left[f^{(k)}(a) - (-1)^k f^{(k)}(x)\right](x-a)^k + R, \tag{8}$$

where

† See for example Advanced Calculus, Sokolnikoff, page 113.

$$R = (-1)^n \frac{(n!)^2(x-a)^{2n+1}}{(2n)!(2n+1)!} f^{(2n+1)}(\theta).$$

The coefficients in (8) are easily computed by setting

$$c_k = \frac{(2n-k)!}{(2n)!} {}_nC_k$$

and using the easily established recursion formula

$$c_{k+1} = \frac{n-k}{(2n-k)(k+1)} c_k, \quad \text{and} \quad c_1 = \tfrac{1}{2}.$$

That the expansion (8) usually converges rapidly can be seen by inspecting the remainder term. Using Sterling's approximation for factorials, it is not difficult to show that

$$\left| R \right| \leqq 2(e/4n)^{2n+1} \left| (x-a)^{2n+1} f^{(2n+1)}(\theta) \right|.$$

The above expression indicates that even for moderate n the remainder term in (8) is usually small.

Finally it should be pointed out that the expansion (1) does not lend itself well to all types of functions since it requires the evaluation of the derivatives at both a and x. For certain types of functions, however, this creates no difficulty as will be illustrated in the next section.

3. Applications. Let us consider equation (8) with $n=1$. For this case we have

$$f(x) = f(a) + \tfrac{1}{2}(x-a)[f'(a) + f'(x)] - \frac{(x-a)^3}{12} f'''(\theta).$$

If we choose $f(x) = x^{n+1}$, it follows readily that

$$x^{n+1} = a^{n+1} + \tfrac{1}{2}(x-a)(n+1)(a^n + x^n) - \frac{n(n^2-1)(x-a)^3\theta^{n-2}}{12}.$$

If, as is usually the case, $x - \frac{1}{2}(n+1)(x-a) \neq 0$, this last equation can be written in the form

$$x^n = \frac{2a + (n+1)(x-a)}{2x - (n+1)(x-a)} a^n - \frac{n(n^2-1)(x-a)^3\theta^{n-2}}{6[2x - (n+1)(x-a)]}.$$

A formula equivalent to the above, without the remainder term, was first given by V. A. Bailey (*Prodigious calculations*, Australian Journal of Science, vol. 3, No. 4, 1941, pp. 78–80). Bailey did not prove the formula. J. S. Frame discussed Bailey's formula in this MONTHLY, April, 1945, pp. 212–214. Frame gave a proof and showed that the error was of the order $(x-a)^3$. More recently, H. S. Wall (this MONTHLY, February 1948, pp. 90–94) derived Bailey's formula in connection with the problem of extracting roots with essentially the same results

as obtained by Frame. When the formula is used recursively for the extraction of roots, the rapidity of convergence is amazing. Wall's article discusses this so well that a numerical example will not be given here. Obviously by taking $n=2$ or 3 in (8), an analogous formula can be developed for powers or roots which is highly superior to the Bailey formula.

As a second illustration we choose $f(x)=e^x$. By choosing $n=4$ and $a=0$ in (8), one easily obtains after obvious simplifications

$$e^x = \frac{1+\frac{1}{2}x+3x^2/28+x^3/84+x^4/1680}{1-\frac{1}{2}x+3x^2/28-x^3/84+x^4/1680}+R,$$

where

$$R = \frac{x^9e^\theta}{25{,}401{,}600(1-\frac{1}{2}x+3x^2/28-x^3/84+x^4/1680)}\cdot$$

Setting $x=1$ in the above formula gives $e=2721/1001=2.718281718\cdots$ which is in error less than 1 in the 7th decimal place. For comparison purposes, using the Taylor finite expansion with $n=4$, gives an error of 1 in the second decimal place. In fact it is necessary to take $n=12$ in the Taylor expansion to get accuracy to 8 decimal places. By taking $n=12$ in formula (8), the error is less than 1 in the 30th decimal place.

4. Extension to infinite expansions. Let p and q be non-negative real numbers not both zero and let m and n become infinite in such a way that $n/m\to p/q$. Under these conditions it is easily shown that

$$\frac{(m+n-k)!}{(m+n)!}\,{}_mC_k \to \frac{q_k}{k!(p+q)^k}, \qquad \frac{(m+n-k)!}{(m+n)!}\,{}_nC_k \to \frac{p^k}{k!(p+q)^k},$$

and the expansion becomes, at least formally,

$$f(x) = f(a) + \sum_{k=1}^{\infty}\left[q^kf^{(k)}(a) - (-p)^kf^{(k)}(x)\right]\frac{(x-a)^k}{k!(p+q)^k}\cdot \tag{9}$$

If in (9) $p=0$, we get the familiar Taylor expansion, whereas for $p=q$ we get

$$f(x) = f(a) + \sum_{k=1}^{\infty}\left[f^{(k)}(a) - (-1)^kf^{(k)}(x)\right]\frac{(x-a)^k}{k!2^k}\cdot$$

To prove the validity of (9) let $f(x)$ have the Taylor series expansions

$$f(u) = \sum_{k=0}^{\infty} f^{(k)}(a)\frac{(u-a)^k}{k!}, \qquad f(v) = \sum_{k=0}^{\infty} f^{(k)}(x)\frac{(v-x)^k}{k!},$$

with respective radii of convergence R_a, R_x, each greater than $|x-a|$. We choose u and v so that $u-a=q(x-a)/(p+q)$ and $v-x=-p(x-a)/(p+q)$. Since $q/(p+q)$ and $p/(p+q)$ both lie between 0 and 1, it follows that u and v as de-

fined above both lie in their respective intervals of convergence. Moreover it is easily seen that $u=v$ so that the Taylor expansions may be equated giving

$$\sum_{k=0}^{\infty}\left[\frac{q}{p+q}(x-a)\right]^k \frac{f^{(k)}(a)}{k!} = \sum_{k=0}^{\infty}\left[\frac{p}{p+q}(a-x)\right]^k \frac{f^{(k)}(x)}{k!},$$

and (9) follows by rearranging terms. Hence (9) is valid for all non-negative values of p and q not both zero provided $|x-a|$ is less than both R_a and R_x.

By choosing a finite number of terms of the infinite expansion, the accuracy is usually not as good as that obtained by using a finite expansion of the same order. Moreover it is difficult to find a bound for the error when the infinite expansion is used.

A CLARIFICATION

Dr. Ladopoulos points out that in his article, *Some Theorems on Cyclic Polygons Inscribed in a Circle*, this MONTHLY, vol. 55 (1948), pp. 301–307, the final corollary can be stated more clearly as follows:

COROLLARY. The figure A_1, A_1', A_1'', $\cdots$, in which successive points correspond under the projectivity π on the circle, possesses Brocard points, a Brocard circle, and first and second Lemoine circles.

MATHEMATICAL NOTES

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A CHARACTERIZATION OF CONVEX SETS

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If X is a subset and p a point of a finite-dimensional Euclidean space E, Xp will denote the set of all points of X which are nearest to p; i.e., $q\in Xp$ provided $q\in X$ and $|p-q|=\inf_{x\in X}|p-x|$. We will establish the following result:

THEOREM 1. *Suppose that $S\subset E$ is closed and has an interior point. Then S is convex if and only if it has the property:*

(Π_1): *For each $p\in E-S$ and $q\in Sp$, there is an unbounded set of points u for which $Su\ni q$.*

This result is related to a well-known theorem which asserts that a non-empty subset C of E is closed and convex if and only if Cp contains exactly one point for each $p\in E$.

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If x and y are distinct points of E, then $\overline{xy}$ will denote the line segment joining them and $\overrightarrow{xy}$ will denote the half-line which contains $\overline{xy}$ and terminates at x. If $q \in X \subset E$, $X^{-1}q$ will denote the set of all points $p \in E$ for which $Xp \ni q$.

LEMMA. *Suppose $X \subset E$, $p \in E$, and $q \in Xp$. Then $\overline{qp} \subset X^{-1}q$. If in addition X is convex, then $\overrightarrow{qp} \subset X^{-1}q$.*

Proof. For each $x \in X$ and $u \in \overline{qp}$ we have $|u-x| \geqq |p-x| - |p-u| \geqq |p-q| - |p-u| = |u-q|$. Hence $\overline{qp} \subset X^{-1}q$. Now suppose in addition that X is convex and $p \in \overline{qu}$. We may assume without loss of generality that q is the origin, and then $u = kp$ for some $k \geqq 1$. Now if $x \in X$ we have $k^{-1}x = y \in X$ and $|u-x| = k|p-y| \geqq k|p-q| = |u-q|$. Hence $\overrightarrow{qp} \subset X^{-1}q$.

Proof of Theorem 1. Let Q denote the set of all points $q \in S$ for which $S^{-1}q$ contains a point other than q itself. Theorem 1 asserts that S is convex if and only if $S^{-1}q$ is unbounded for each $q \in Q$. Now if S is convex it follows from the lemma that it has property (Π_1). Suppose conversely that (Π_1) holds and for each $q \in Q$, let $R(q)$ be the set of all points $p \neq q$ for which $\overrightarrow{qp} \subset S^{-1}q$. Then $R(q)$ is non-empty. For if $q \in Q$, (Π_1) says there is a sequence of points x_α with $|x_\alpha| \to \infty$ and $x_i \in S^{-1}q$ for each i. For each i let y_i be on $\overrightarrow{qx_i}$ such that $|q-y_i| = 1$. Then for some sequence n_α of integers and some $y \neq q$ we have $y_{n_\alpha} \to y$. But then $\overrightarrow{qy} \subset S^{-1}q$, since from the lemma it follows that $\overline{qx_i} \subset S^{-1}q$ for each i.

Now for each $q \in Q$ and $p \in R(q)$, let H_{pq} be the hyperplane which passes through q perpendicular to $\overline{pq}$, and let K_{pq} be the closed half-space which is bounded by H_{pq} and does not contain p.(*) Then $K_{pq} \supset S$. For suppose S contains a point x on the same side of H_{pq} as p. Then $\angle pqx < \pi/2$, so for a point y sufficiently far out on $\overrightarrow{qp}$ we have $\angle yxq > \angle yqx$, whence $|y-q| > |y-x|$, contrary to the fact that $y \in S^{-1}q$.(*) Now define $K = \bigcap_{q \in Q, p \in R(q)} K_{pq}$. Then K is convex and contains S.

Now suppose $z \notin S$ and let W be an open set contained in S. Let C be the convex hull of $\{z\} \cup W$. Then there is a point $v \in \text{Int } C \cap S \cap \overline{E-S}$. For some $\epsilon > 0$, C contains the sphere of radius ϵ and center v. Now let x be a point of $E-S$ such that $|x-v| < \epsilon/2$. Since bounded closed subsets of S are compact, Sx is non-empty, and since $v \in S$, $Sx \subset C$. Now let $q \in Sx$ and $p \in R(q) \cap C$. If $z \in K_{pq}$ we have (since $S \subset K_{pq}$) $C \subset K_{pq}$ and hence $p \in K_{pq}$, which is impossible. Hence $z \notin K_{pq}$, so $z \notin K$. Thus $K \subset S$ and the proof is complete.

By quite a similar proof the following characterization can be established:

THEOREM 2. *Suppose that $S \subset E$ is closed. Then S is convex if and only if it has the property*:

(Π_2): *for each $p \in E$ and $q \in S_p$, $\overrightarrow{qp} \subset S^{-1}q$.*

Although proofs have been given only for finite-dimensional Euclidean spaces, it is not hard to see that these characterizations are valid in any Minkowskian space whose unit sphere has a unique supporting hyperplane at each boundary point. (This uniqueness validates an argument analogous to that between the (*)'s above.) And Theorem 2 can in fact be established for any normed

linear space whose unit sphere is weakly compact and also has the property just mentioned. (Weak compactness assures that if S is closed, then Sp is non-empty for each p.)

NOTE ON THE STRONG LAW OF LARGE NUMBERS

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In elementary books on mathematical statistics one frequently finds a proof of the Weak Law of Large Numbers for the Bernoulli distribution based upon the Tchebycheff inequality. However it is well known that the Tchebycheff inequality is not sharp enough for the "strong" law. In this note we shall prove an elementary inequality which is of the Tchebycheff type but which is sufficient for the proof of the Strong Law of Large Numbers in the Bernoulli case.

Let p be the probability of the occurrence of an event E in a single trial and let m be the number of occurrences of E in n independent trials. Then the Tchebycheff inequality asserts that

$$P_r\left(\left|\frac{m}{n}-p\right|>\epsilon\right)\leqq\frac{pq}{n\epsilon^2}\qquad\text{where } \epsilon>0 \text{ and } q=1-p.$$

We shall prove the following theorem.

THEOREM. *Let ϵ_1 be any positive number and let ϵ be the smallest of ϵ_1, p, and q. Then*

$$P_r\left(\left|\frac{m}{n}-p\right|>\epsilon_1\right)\leqq 2(1-\epsilon)^{n\epsilon}.$$

Proof. Since the probability of exactly m occurrences of E in n independent trials is given by $\binom{n}{m}p^mq^{n-m}$ we have

$$P_r\left(\left|\frac{m}{n}-p\right|>\epsilon_1\right)=\sum_{m>(p+\epsilon_1)n}\binom{n}{m}p^mq^{n-m}+\sum_{m<(p-\epsilon_1)n}\binom{n}{m}p^mq^{n-m}.$$

Now since $\epsilon\leqq\epsilon_1$

$$\sum_{m>(p+\epsilon_1)n}\binom{n}{m}p^mq^{n-m}\leqq\sum_{m>(p+\epsilon)n}\binom{n}{m}p^mq^{n-m}=\sum_{m=[(p+\epsilon)n+1]}^{n}\binom{n}{m}p^mq^{n-m},$$

where, as usual, $[x]$ represents the greatest integer in x. Replacing m by $m+1$, we obtain

$$\sum_{m>(p+\epsilon_1)n}\binom{n}{m}p^mq^{n-m}\leqq\sum_{[(p+\epsilon)n]}^{n-1}\frac{n!}{(m+1)!(n-m-1)!}p^{m+1}q^{n-m-1}$$

$$\leqq\frac{np}{[(p+\epsilon)n]+1}\sum_{[(p+\epsilon)n]}^{n-1}\frac{(n-1)!}{m!(n-1-m)!}p^mq^{n-1-m}.$$

Again replacing m by $m+1$ in the summation, we have

$$\sum_{m>(p+\epsilon_1)n}\binom{n}{m}p^m q^{n-m} \leqq \frac{np}{[(p+\epsilon)n]+1}\cdot\frac{(n-1)p}{[(p+\epsilon)n]} \cdot \sum_{[(p+\epsilon)n]-1}^{n-2} \frac{(n-2)!}{m!(n-2-m)!}p^m q^{n-2-m}.$$

If this process is repeated k times, where $k \leqq [(p+\epsilon)n]$, we get

$$\sum_{m>(p+\epsilon_1)n}\binom{n}{m}p^m q^{n-m} \leqq \frac{np}{(p+\epsilon)n}\cdot\frac{(n-1)p}{(p+\epsilon)n-1}\cdots\frac{(n-k)p}{(p+\epsilon)n-k},$$

where the inequality $[x]>x-1$ has been used and the sum has been replaced by its bound 1. Now let $k=[n\epsilon]$. Then since $\epsilon \leqq q$ it is easily verified that $(n-l)p/(p+\epsilon)n-l$ increases with l and hence

$$\sum_{m>(p+\epsilon_1)n}\binom{n}{m}p^m q^{n-m} \leqq \left(\frac{(n-k)p}{(p+\epsilon)n-k}\right)^{k+1} \leqq \left(\frac{(n-n\epsilon)p}{(p+\epsilon)n-n\epsilon}\right)^{n\epsilon} = (1-\epsilon)^{n\epsilon}.$$

But we also have

$$\sum_{m>(p-\epsilon_1)n}\binom{n}{m}p^m q^{n-m} = \sum_{n-m<(q+\epsilon_1)n}\binom{n}{n-m}p^{n-(n-m)}q^{n-m},$$

and hence the same inequality holds for this sum. Thus

$$P_r\left(\left|\frac{m}{n}-p\right|>\epsilon_1\right) \leqq 2(1-\epsilon)^{n\epsilon},$$

which is the conclusion of the theorem.

Since, for $x>a>0$, we have $(1-a/x)^x<e^{-a}$, we get the following corollary.*

COROLLARY. *If ϵ is less than the smaller of p and q, then $P_r(|m/n-p|>\epsilon) < 2e^{-n\epsilon^2}$.*

From the Corollary it follows that if $0<\epsilon \leqq \min(p, q)$, the probability that $|m/n-p|>\epsilon$ for at least one n greater than n_0 is less than $2\sum_{n_0}^{\infty} e^{-n\epsilon^2}$. Hence the probability that m/n differs from p by more than ϵ for infinitely many n is less than $2\sum_{n_0}^{\infty} e^{-n\epsilon^2}$ for every n_0. But since $\sum e^{-n\epsilon^2}$ converges, $\sum_{n_0}^{\infty} e^{-n\epsilon^2}$ approaches 0 as $n_0 \to \infty$. Hence we get the Strong Law of Large Numbers for the Bernoulli distribution; namely

$$P_r\left(\frac{m}{n}\to p\right) = 1.$$

* The inequality of the Corollary is essentially that given by Uspensky, Introduction to Mathematical Probability, p. 206.

A NOTE ON LINEAR EQUATIONS

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Suppose that from a system of three linear equations in three unknowns x, y, z, we can deduce that $x=1$, $y=2$, $z=3$. Then of course the equations cannot have any solution other than $x=1$, $y=2$, $z=3$. But does it follow that this is actually a solution of the given system of equations? Obviously not, if all methods of deduction are allowed, since the given equations may well be inconsistent, and a false premise implies any conclusion. On the other hand, if the equations $x=1$, $y=2$, $z=3$ are obtained *by forming linear combinations of the given equations*, then the answer is *yes*. Since this is the usual method of solving linear equations, the result has some practical as well as theoretical interest; but I have not seen it mentioned in treatments of the theory of linear equations.

THEOREM. *Suppose that we are given the system of linear equations*

$$\sum_{l=1}^{n} a_{kl}x_l = b_k \qquad (k = 1, 2, \cdots, n). \tag{1}$$

If it is possible to obtain the equations

$$x_1 = c_1, \quad x_2 = c_2, \quad \cdots, \quad x_n = c_n \tag{2}$$

by forming linear combinations of the given equations, then (2) *is a solution of* (1), *and is of course the only solution.*

Proof. By hypothesis, each x_j is a linear combination of the left sides of equations (1). Thus, for any j, the equations

$$\sum_{k=1}^{n} \lambda_{jk}a_{kl} = \delta_{jl} \qquad (l = 1, 2, \cdots, n) \tag{3}$$

have a solution $\lambda_{j1}, \lambda_{j2}, \cdots, \lambda_{jn}$. (As usual, $\delta_{jl}=1$ if $j=l$ and $\delta_{jl}=0$ if $j\neq l$.) Now (3) shows that the matrix (λ_{jk}) is inverse to the matrix (a_{kl}), and hence that (a_{kl}) is non-singular. Consequently, equations (1) have a solution, which can only be (2).

Remark. It is essential that the number of equations be equal to the number of unknowns. Indeed, if we consider a similar statement about m equations in n unknowns, we see that it is false of $m>n$. Consider for example the system $x=1$, $y=2$, $z=3$, $z=4$ of four linear equations in three unknowns. We can obtain $x=1$, $y=2$, $z=3$ in the prescribed way, but this is not a solution of the given system. On the other hand, if $m<n$ the result is trivial, since it is impossible to satisfy the hypothesis.

THE WRONSKIAN FOR LINEAR DIFFERENTIAL EQUATIONS

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1. Introduction. We consider a linear differential equation with constant coefficients

$$(A_0D^n + A_1D^{n-1} + \cdots + A_n)y = 0, \qquad A_0 \neq 0,$$

and the Wronskian of a set of linearly independent solutions. We shall take $e^{\alpha x}$ as a solution corresponding to a simple root and $e^{\beta x}$, $xe^{\beta x}$, $x^2e^{\beta x}$, $\cdots x^{s-1}e^{\beta x}$ as the set of solutions corresponding to a root β of multiplicity s of the auxiliary equation. It is a classical fact that these solutions are linearly independent and consequently their Wronskian does not vanish. The converse of this proposition is also true for the type of functions considered.* By Abel's equation the Wronskian $W(x)$ is given by $W(0)e^{-A_1x/A_0}$ so that our problem reduces to the computation of $W(0)$. It is an intriguing problem to obtain $W(0)$ for roots of various multiplicites; the only thing I have been able to find in the literature is $W(0)$ for the case of simple roots. In that case it is given by Vandermonde's determinant and is of course a special case of our result.

2. Lemma. We shall use the following identity.

LEMMA:

$$\Delta \equiv \begin{vmatrix} x^n & Dx^n & D^2x^n & \cdots D^{s-1}x^n \\ x^{n+1} & Dx^{n+1} & D^2x^{n+1} & \cdots D^{s-1}x^{n+1} \\ \cdot & \cdot & \cdot & \cdot \\ x^{n+s-1} & Dx^{n+s-1} & D^2x^{n+s-1} & \cdots D^{s-1}x^{n+s-1} \end{vmatrix} = x^{ns}\cdot \prod_{k=0}^{s-1}(k!).$$

This is probably a well known result and its calculation is simple enough; after writing the various derivatives explicitly, one may factor x^{n-i} out of the $(i+1)$st column and then x^j out of the $(j+1)$st row so that the total power of x in Δ is $\sum_{i=0}^{s-1}(n-i)+\sum_{j=0}^{s-1}j=\sum_{i=0}^{s-1}n=ns$. The remaining numerical determinant, which we denote by $\Delta(n, s)$, may be computed as follows: subtract the $(s-1)$st row from the sth; then the $(s-2)$nd from the $(s-1)$st, etc., and finally the second row from the first. This leaves a determinant of $(s-1)$ rows in which the ith column has a common factor i. Thus $\Delta(n, s)=(s-1)!\Delta(n, s-1)$; consequently $\Delta(n, a)=(s-1)!(s-2)! \cdots 2!1!$ and the lemma is proved.

3. The Wronskian. Returning to our differential equation, we assume that the auxiliary equation has the roots $\alpha_1, \alpha_2, \cdots, \alpha_k$ of multiplicities $r_1, r_2, \cdots, r_k$. We choose the set of linearly independent solutions $x^{\lambda_i}e^{\alpha_i x}$, $\lambda_i=0, 1, 2, \cdots,$

* Cf. R. P. Agnew, Differential Equations, p. 327.

r_i-1; $i=1, 2, \cdots, k$ and it is the Wronskian of these functions for $x=0$ that we are trying to compute. Since

$$D^p(x^m e^{\alpha x}) = \sum_{\lambda=0}^{p}\binom{p}{\lambda}\frac{d^\lambda x^m}{dx^\lambda}\cdot \alpha^{p-\lambda}e^{\alpha x},$$

we have

$$D^p(x^m e^{\alpha x})\Big]_0 = \binom{p}{m}\cdot m!\alpha^{p-m} = p(p-1)\cdots(p-m+1)\alpha^{p-m} = D^m\alpha^p,$$

where

$$D^m\alpha^p \equiv \frac{d^m\alpha^p}{d\alpha^m};$$

then $W(0)$ has the form

$$W(0)=\left|\begin{array}{cccc|cccc|ccc|ccc}
1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 & 0\cdots & \cdots & 1 & 0\cdots & \cdots \\
\alpha_1 & D\alpha_1 & \cdots & D^{r_1-1}\alpha_1 & \alpha_2 & D\alpha_2 & \cdots & D^{r_2-1}\alpha_2 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\alpha_1^2 & D\alpha_1^2 & \cdots & D^{r_1-1}\alpha_1^2 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\alpha_1^{r_1-1} & D\alpha_1^{r_1-1} & \cdots & D^{r_1-1}\alpha_1^{r_1-1} & \alpha_2^{r_1-1} & \cdots & \cdots & D^{r_2-1}\alpha_2^{r_1-1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\alpha_1^{r_1} & D\alpha_1^{r_1} & \cdots & \cdots & \alpha_2^{r_1} & \cdots & \cdots & D^{r_2-1}\alpha_2^{r_1} & \alpha_3^{r_1} & \cdots & D^{r_s-1}\alpha_3^{r_1} & \cdots & \cdots & \cdots \\
\alpha_1^{r_1+1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\alpha_1^{r_1+r_2-1} & \cdots & \cdots & \cdots & \alpha_2^{r_1+r-1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\alpha_1^{r_1+r_2} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \alpha_3^{r_1+r_2} & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\alpha_1^{r_1+r_2+r3-1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \alpha_3^{r_1+r_2+r_3-1} & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}\right|$$

$W(0)$ is obviously a polynomial in $\alpha_1, \alpha_2, \cdots, \alpha_k$ and it vanishes if and only if $\alpha_i=\alpha_j$ for some $i\neq j$. Hence

$$(A) \qquad W(0) = N(r_1, r_2, \cdots, r_k)\cdot\prod_{j<i}(\alpha_j-\alpha_i)^{\lambda_{ij}},$$

the values of the exponents λ_{ij} depending only on $r_1, r_2, \cdots, r_k$ and not on the values of α_i. In order to compute λ_{12}, we let $\alpha_1=0$; then all elements in the first r_1 columns of $W(0)$ will become zero except those in the main diagonal of the first r_1-rowed principal minor so that

$$W(0)=\left|\begin{array}{ccc|ccc|ccc}
\alpha_2^{r_1} & \cdots & D^{r_2-1}\alpha_2^{r_1} & \alpha_3^{r_1} & \cdots & \cdots & \cdots & \cdots & \cdots \\
\alpha_2^{r_1+1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\alpha_2^{r_1+r_2-1} & \cdots & D^{r_2-1}\alpha_2^{r_1+r_2-1} & \alpha_3^{r+r_2-1} & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\alpha_2^{r_1+r_2} & \cdots & \cdots & \alpha_3^{r_1+r_2} & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\alpha_2^{r_1+r_2+r_3-1} & \cdots & \cdots & \alpha_3^{r_1+r_2+r_3-1} & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \alpha_1^{r_1+r+r} & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}\right|\cdot\prod_{j=0}^{r_1-1}(j!).$$

By our lemma the lowest power of α_2 is obtained from the r_2-rowed first principal minor in the above determinant and this power is r_1r_2. Writing the product in (A) with $\alpha_1=0$, we get

$$\alpha_2^{\lambda_{12}}(\alpha_3-\alpha_2)^{\lambda_{23}}(\alpha_4-\alpha_2)^{\lambda_{24}}\cdots(\alpha_k-\alpha_2)^{\lambda_{2k}}\prod_{j>i>2}(\alpha_j-\alpha_i)^{\lambda_{ij}}$$

and evidently the lowest power of α_2 in this expression is λ_{12}. Hence $\lambda_{12}=r_1r_2$ and quite similarly it follows that $\lambda_{ij}=r_ir_j$, $(i\neq j)$. To find the value of $N(r_1, r_2, \cdots, r_k)$ we note that it is the coefficient of

$$\alpha_1^{0}\cdot\alpha_2^{r_1r_2}\cdot\alpha_3^{(r_1+r_2)r_3}\cdot\alpha_4^{(r_1+r_2+r_3)r^4}\cdots\alpha_k^{(r_1+r_2+\cdots+r_{k-1})r_k}$$

in the expansion of the product in (A), while in the determinant the coefficient of this term is the product of the coefficients of the minor blocks in the main diagonal. Hence by our lemma again

$$N(r_1, r_2\cdots r_k)=1!2!\cdots(r_1-1)!1!2!\cdots(r_2-1)!\cdots1!\cdots(r_k-1)!.$$

Putting these results together we obtain

$$W(x)=\prod_{i=1}^{k}[1!2!\cdots(r_i-1)!]\cdot\prod_{j>i}(\alpha_j-\alpha_i)^{r_ir_j}e^{-A_1x/A_0}.$$

DIOPHANTINE EQUATIONS CONNECTED WITH THE CUBIC FERMAT EQUATION

J. D. Swift, University of California, Los Angeles

1. Introduction. The Diophantine equation, $a^3-z^3=3ab$, is derived from the Fermat cubic equation, $x^3+y^3=z^3$, by the substitutions $x+y=a$, $xy=b$. The new equation may be solved by quite elementary means, and its solutions may be employed with reference to the original to produce an infinite set of Diophantine equations of known solutions. Thus, by combining the results of two related equations, we actually solve infinitely many.

Secondly, all solutions of the derived equation correspond to solutions of

the Fermat cubic in integers of quadratic fields, and when combined with results due to Burnside, may be shown to be essentially the general solution in such fields.

2. Solution when $(b, z)=1$, $(a, b)=1$. (These equivalent conditions correspond to x, y, z coprime in pairs for the Fermat equation.)

a) From $(a-z)(a^2+az+z^2)=3ab$, we have $3\mid(a-z)$ or $3\mid(a^2+az+z^2)$; but these are equivalent, so both are true.

b) Let p be prime, $p\neq 3$, $p\mid a$; and let $3k+j$ be the highest power of p which divides a, $j=0, 1, 2$. Then $p^{3k+j}\mid z^3$; and if $j\neq 0$, then $p^{3(k+1)}\mid z^3$ and therefore ab; thus $p\mid b$, a contradiction. Similarly, let $3i+j$ be the highest power of 3 dividing a and, proceeding as before, either $j=2$ or $i=j=0$. Thus $a=n^3$ or $9m^3$, where $(n, 3)=1$. (This convention with regard to n will be continued throughout.)

c) In the first case, $a=n^3$, we have $z=n^3-3mn$, m arbitrary, since $n\mid z$, and $3\mid(a-z)$; $b=3m(n^4-3mn^2+3m^2)$; and $(3m, n)=1$ is a necessary and sufficient condition that the *GCD* conditions be satisfied.

d) If $a=9m^3$, then $z=9m^3-3mn$, $b=n(27m^4-9m^2n+n^2)$; and again $(3m, n)=1$.

THEOREM: *The solutions of* $a^3-z^3=3ab$, $(b, z)=1$, *are given by the sets*:

$$(1)\quad a = n^3,\quad z = n^3 - 3mn,\quad b = 3m(n^4 - 3mn^2 + 3m^2),$$

$$(2)\quad a = 9m^3,\quad z = 9m^3 - 3mn,\quad b = n(27m^4 - 9m^2n + n^2),\quad (3m, n) = 1,$$

m and n otherwise arbitrary integers.

3. The general solution. Relaxation of the condition $(3m, n)=1$ in (1) and (2) above will not provide the general solution. E.g., $a=25$, $z=10$, $b=195$ cannot be thus obtained. The "primitive" solutions here will permit a square-free *GCD* of b and z which exceeds one. However, if $p\mid(b, z)$, and $p^2\mid b$, then $(a/p)^3-(z/p)^3=3(a/p)(b/p^2)$, where all numbers are clearly integers. Let $r=(b, z)$; then $r^3/3ab$, and $r^2\mid 3a$ if the solution is to be primitive. Let q be square-free, $(q, 3)=1$, but otherwise arbitrary; then q^2 and $3q^2$ represent the possible new factors for a. Combining the results of §2 with the preceding, we obtain the new sets:

$$(1)\quad a = q^2n^3,\quad z = q^2n^3 - 3mnq,\quad b = 3mq(q^2n^4 - 3qn^2m + 3m^2),$$

$$(2)\quad a = 3q^2n^3,\quad z = 3q^2n^3 - 3mnq,\quad b = 3mq(3q^2n^4 - 3qn^2m + m^2),$$

$$(3)\quad a = 9q^2m^3,\quad z = 9q^2m^3 - 3mnq,\quad b = nq(27q^2m^4 - 9qm^2n + n^2),$$

with the new conditions for sets (1) and (2) that $(m, q)=1$, and for set (3), $(n, q)=1$.

THEOREM: *The three sets above give all primitive solutions of* $a^3-z^3=3ab$. *All solutions in integers result from the relaxation of the GCD restrictions imposed or, alternatively, from the substitutions* $a\to sa$, $z\to sz$, $b\to s^2b$, *where s is an arbitrary integer.*

4. Derived equations. To any solution obtained above, an integral solution of $x^3+y^3=z^3$ will correspond if and only if a^2-4b is a perfect square, or, equivalently, $x^2-ax+b=0$ has integral solutions. The only such solutions occur when one of x, y, z is zero; that is, if $z=0$ or $b=0$. From the given solutions, then, all cases, when any of

$$(nq)^2(n^2q - 6m)^2 - 36m^3q, \tag{1}$$

$$(3nq)^2(n^2q - 2m)^2 - 12m^3q, \tag{2}$$

$$(3mq)^2(3m^2q - 2n)^2 - 4n^3q, \tag{3}$$

may be a perfect square, are determined. In particular, if the *GCD* conditions are retained, the requirement $mnq=0$ must be met.

By fixing various of m, n, q, a large variety of Diophantine equations arise whose solutions are known in advance. For example, in set (1) let $m=1$, $n=1$. The equation

$$q(q^3 - 12q^2 + 36q - 36) = r^2$$

results. This equation has only the trivial solution $q=r=0$. Similar sets are derived from $x^2-ax+b=0$.

5. Quadratic fields. If a, b, and z are any rational integral solutions of $a^3-z^3=3ab$, there is a quadratic integer x such that $x^3+\bar{x}^3=z^3$; x and $\bar{x}$ are the roots of $x^2-ax+b=0$. W. Burnside* has shown that if the Fermat equation is soluble in a field of $\sqrt{k}$, it is soluble in the special form where x and y are conjugate and z is rational, and that any solution in the field may be obtained by multiplying the special solution by elements of the field. The solutions here obtained are integral. If the restrictions of §2 are retained, the solutions are primitive (actually the conditions should be rewritten in terms of coprimality since the *GCD* may not exist), while under the conditions of §3, there may be possible factors of q in the field.

On the other hand, if solutions exist in the field $\sqrt{k}$:

$$(\alpha + \beta\sqrt{k})^3 + (\alpha - \beta\sqrt{k})^3 = z^3,$$

it follows that $a=2\alpha$, $n=\alpha^2-k\beta^2$, and z are rational integers satisfying the basic equation of this paper.

THEOREM: *All quadratic integral solutions of $x^3+y^3=z^3$ are obtainable from the solutions of $a^3-z^3=3ab$ with $x_1=(a+\sqrt{a^2-4b})/2$, $y_1=(a-\sqrt{a^2-4b})/2$, $z_1=z$, by first removing any factor of q from x_1, y_1, and z_1, and then multiplying by any integer of the field. If the field admits unique factorization, the solutions may be obtained from the set given in* §2.

* W. Burnside, Proc. London Math Soc. (2), vol. 14, 1914, p. 1.

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College and Institute for Advanced Study

All material for this department should be sent to C. B. Allendoerfer, Institute for Advanced Study, Princeton, New Jersey.

THE RATE OF INTEREST IN INSTALLMENT PAYMENT PLANS

H. E. STELSON, Michigan State College

Three formulas (besides the *Compound Interest* formula) are commonly used in determining the rate of interest in installment payment plans. These are known as the *Constant Ratio*, *Series of Payments*, and *Interest at End* formulas.

These formulas have been independently developed from certain particular assumptions. It is the purpose of this paper to show that all three of these formulas may be derived as approximations to the *Compound Interest* formula (sometimes called the *Actuarial* formula). It is also shown that the rates as determined by the various formulas satisfy definite inequalities. A new and more accurate approximation formula is presented.

The Constant Ratio formula is derived on the assumption that each payment is composed of a principal repayment and an interest repayment in the same ratio that the original unpaid balance is to the interest. The basic assumption for the derivation of the Series of Payments formula is that the sum of the series of payments is the outstanding debt at the beginning of the installment term. The Interest at End formula is derived on the assumption that the payment should be used to repay the principal first, then after the principal has been repaid, to repay the interest.

The following symbols may be used:

R = periodic payment,
r = periodic rate,
n = number of periodic payments (not counting the down payment),
B = unpaid balance at the beginning of the credit period (cash price less the down payment, if any),
I = total carrying charge or cost of the loan,
$= Rn - B$,
h = the integral number of times R is contained in B,
$K = B - hR$.

The three formulas are written as follows:*

* One Hundred Problems in Consumer Credit. Pollak Pamphlet No. 35 by Charles H. Mergendahl and LeBaron R. Foster, Pollak Foundation for Economic Research, Newton, Mass. 56 pp. References p. 53, 1943.

Consumer Credit Cost Calculator, Household Finance Corp., 919 N. Michigan Ave., Chicago, Ill.

Constant Ratio, $$r_c = \frac{2I}{B(n+1)}.$$

Series of Payments, $$r_s = \frac{2I}{(B+I)(n+1)}.$$

Interest at End, $$r_I = \frac{2I}{(B+K)(h+1)}.$$

The formula for Compound Interest is

$$B = R\frac{1-(1+r)^{-n}}{r}. \tag{1}$$

We shall show that r_c, r_s and r_I are approximations of r as determined by (1). Since $Rn = B + I$,

$$B = \frac{I}{\dfrac{nr}{1-(1+r)^{-n}} - 1},$$

whence

$$B = \frac{2I}{r(n+1)}\left[1 - \frac{n-1}{6}r + \frac{(n-1)(n+2)}{36}r^2 + \cdots\right]. \tag{2}$$

Neglecting all but the first term of the series in the brackets, we obtain the Constant Ratio formula for r_c.

A better approximation to r could be obtained by retaining two terms in the series in the brackets. This gives

$$r_a = \frac{6I}{3B(n+1) + I(n-1)} \quad \text{or} \quad \frac{6I}{3Rn(n+1) - 2I(n+2)} \tag{3}$$

which gives a remarkably close approximation to r.

Now r_c and r_a have been determined by the relations:

$$B = \frac{2I}{r_c(n+1)} \quad \text{and} \quad B = \frac{2I}{r_a(n+1)}\left(1 - \frac{n-1}{6}r_a\right). \tag{4}$$

Hence,

$$\frac{1}{r_a} = \frac{1}{r_c} + \frac{n-1}{6}. \tag{5}$$

Relation (5) could be used to find r_a in place of (3) or it could be used to check (3).

From (2) and (4), we may obtain

$$r_c = \frac{r}{L - \dfrac{n-1}{6}r} \quad \text{and} \quad r_a = \frac{r}{L} \tag{6}$$

where

$$L = 1 + \frac{(n-1)(n+2)}{36}r^2 + \cdots .$$

By (6) $r_a < r$ since $L > 1$. Hence, if

$$r_c - r > r - r_a > 0, \tag{7}$$

then r_a is a better approximation to r than r_c. By substitution of relations (6) we see (7) is true if

$$\frac{r}{L - \dfrac{n-1}{6}r} - r > r - \frac{r}{L}$$

which in turn is true if

$$\frac{n-1}{6}r(2L-1) > 2L(L-1) \tag{8}$$

or

$$\frac{n-1}{6}r > \frac{2L(L-1)}{2L-1} > L - 1.$$

That is

$$\frac{n-1}{6}r > \frac{(n-1)(n+2)}{36}r^2$$

or

$$1 > \frac{n+2}{6}r.$$

Thus

$$r < \frac{6}{n+2}.$$

Since (8) holds for practical values of n and r, we conclude that (7) is true and r_a is a better approximation to r than r_c.

Again, if we expand (1) directly, we have

$$B = R\left(\frac{1-(1+r)^{-n}}{r}\right) = R\left[n - \frac{n(n+1)}{2}r + \cdots\right]$$

or since $B+I=Rn$

$$B = (B+I)\left[1 - \frac{n+1}{2}r + \frac{(n+1)(n+2)}{6}r^2 + \cdots\right]. \tag{9}$$

If we neglect all but the first two terms of the series in the brackets of (9) we obtain the Series of Payments formula for r_s.

Again, if we multiply both sides of formula (1) by $(1+r)^n$, we have

$$B(1+r)^n = R\frac{(1+r)^n - 1}{r}. \tag{10}$$

Direct expansion of (10) by the binomial formula gives

$$B(1+nr+\cdots) = Rn\left[1 + \frac{(n-1)}{2}r + \cdots\right]. \tag{11}$$

Neglecting all powers of r higher than the first in equation (11) gives

$$r \cong \frac{2I}{n(2B - Rn + R)} \quad \text{since} \quad Rn = B + I.$$

If $n=h+1$, we have the Interest at End formula

$$r_I = \frac{2I}{(h+1)(B+K)} \quad \text{since} \quad B = Rh + K.$$

By definition $n-h=1, 2, 3, \cdots$. Although $n-h$ may be greater than 1, the approximation r_I is obtained when $n-h=1$. This formula gives better results than formulas developed for cases $n-h>1$.

We may now obtain the relative size of the rates as they are determined by the various formulas.

By (7), $r_c>r$ and $r_a<r$.

Also $r_c<r_I$ if

$$\frac{2I}{(h+1)(B+K)} > \frac{2I}{B(n+1)}$$

which holds, since $h+1=n$, if

$$B(n+1) > (B+K)n.$$

But this inequality can be shown to be valid if $n>1$.

Again since $B(n+1)+I(n+1)>B(n+1)+(I/3)(n+1)$, it follows that

$$\frac{2I}{B(n+1)+I(n+1)}<\frac{2I}{B(n+1)+\frac{I}{3}(n-1)},$$

so that $r_s<r_a$.

The rates as computed by the various formulas may now be put in the following inequality,

$$r_s<r_a<r<r_c<r_I.$$

Some numerical illustrations follow.

B	R	n	r_s	r_a	r	r_c	r_I
189.14	10	24	.0170	.0199	.02	.0215	.027
99.54	10	12	.0262	.0299	.03	.0316	.0375
37.17	10	4	.0283	.0300	.03	.0305	.032

MATHEMATICAL VOCABULARY OF BEGINNING FRESHMEN

C. S. OGILVY, Trinity College

No teacher would attempt to take a student through a college freshman course in mathematics unless he were sure that the student understood automatically, through long familiarity, the meaning of words like multiplication and addition. But what about words like factor and term? These, to the instructor, are just as "simple," just as "automatic"; they are part of his everyday vocabulary. He uses them in class casually, expecting their meaning to be second nature to anyone who has been through high school algebra.

Unfortunately this is not so. To verify a long-standing suspicion that all was not as it should be with the freshman's mathematical vocabulary, the men in three sections were asked, at the beginning of the college year, to write down their definitions of each of five words: *polynomial*, *quotient*, *term*, *coefficient*, and *factor*. The intention was not, of course, to obtain rigorous or polished definitions. If a man showed that he understood the general meaning of the word, he was given the benefit of the doubt. Yet of 60 men quizzed, 33 did not know what a polynomial was; 11 missed quotient; 43 defined term either incorrectly or so badly that it was impossible to tell whether they knew what it was; 22 went astray on coefficient, including 9 who defined it as an exponent; and 19 were hazy on factor.

A common misconception of polynomial was that it meant an expression of more than one *variable*. Others said it was an expression involving several *factors*, which made it hard to tell whether they were misconstruing the meaning of polynomial or of factor (most of this group also gave incorrect definitions of

factor). Another wrote "a large or many-figured number." Another, "a mixed number." One even said "a figure with more than three sides."

The distinction between factor and term caused the most trouble. Many of the definitions for term covered, or rather uncovered, a multitude of sins: "a part of an equation"; "one of any two or more mathematical expressions which are related to each other"; "a number or a letter representing a constant or variable"; "a specific number or quantity"; "a number in a problem"; "any integer, unknown, or combination of them"; "anything known in a problem"; "any unknown values"; and so on. It was quite obvious that these men had been considering the word, term, in its loosest possible sense of "*any* expression," rather than the usual meaning assigned to it in algebra.

Of the 60 men sampled, 40 were pre-science majors and 20 pre-arts majors. As was to be expected, the arts men fared the worst. Yet the quotations above were all taken from papers of the science group.

A more elementary vocabulary test could scarcely be devised. It is feared that too many teachers take for granted that their freshmen are above these things. The original blame of course must be placed squarely on poor high school instruction. But if students come to us malnourished in mathematical words, we must feed them a basic diet first. The men who did so badly on this little quiz were not stupid. Some of them were even good mathematics students. They simply needed to be given some clear, correct definitions once for all.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 861. *Proposed by Vern Hoggatt, College of Puget Sound, and Leo Moser, University of Manitoba*

Let a be any positive number different from 1 and let p be any integer greater 3. Show that every integer may be expressed by using p a's and a finite number of operator symbols used in high school texts.

E 862. *Proposed by R. E. Horton, Los Angeles City College*

Find the rectangles of greatest and least area which can be circumscribed about a given parallelogram.

E 863. *Proposed by W. O. Pennell, Exeter, N. H.*

If $\alpha>0$, $\beta>0$, $\alpha+\beta<\pi$, $0<k<1$, then $\alpha=\beta$ is a necessary and sufficient condition for

$$\sin\alpha\sin(k\alpha+\beta)=\sin\beta\sin(k\beta+\alpha).$$

E 864. *Proposed by N. S. Mendelsohn, University of Manitoba*

Prove that

$$\sum_{n=1}^{r}1/n=\sum_{n=1}^{r}(-1)^{n+1}(1/n)\binom{r}{n}.$$

E 865. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find a point such that planes drawn through this point parallel to the faces of a tetrahedron cut the opposite trihedrals in equivalent triangles. Express the common area of these triangles in terms of the areas of the faces of the tetrahedron.

SOLUTIONS

Modified Harmonic Series

E 824 [1948, 365]. *Proposed by E. P. Starke, Rutgers University*

We modify the harmonic series by taking the first term positive, the next two negative, the next three positive, *etc.* Show that this modified series is convergent.

I. *Solution by R. P. Stephens, University of Georgia.* Consider the related series

$$u_1-u_2+u_3-\cdots,\tag{1}$$

formed by bracketing terms of one sign without changing order. Then

$$\begin{aligned}u_n&=2/(n^2-n+2)+\cdots+2/n(n+1),\\u_{n+1}&=2/(n^2+n+2)+\cdots+2/n(n+3)+2/(n+1)(n+2).\end{aligned}$$

Therefore

$$0<u_n<2n/(n^2-n+2)$$

whence

$$\lim_{n\to\infty}u_n=0.$$

Also

$$\begin{aligned}u_n-u_{n+1}&>n[2/n(n+1)-2/n(n+3)]-2/(n+1)(n+2)\\&=2/(n+2)(n+3)>0.\end{aligned}$$

Thus series (1) converges.

This implies convergence of the given series. For let S_p be any partial sum of the given series. Then, for suitable n, $S_p = U_n + r_n$, where U_n is the sum of n terms of (1) and r_n is a partial term of (1). As $p \to \infty$, $n \to \infty$, and U_n approaches a limit. Also $|r_n| < u_{n+1}$, which approaches zero. Therefore S_p approaches a limit and the given series converges.

II. *Solution by Fritz Herzog, Michigan State College.* The statement of the problem can be generalized as follows:

Let $0 = a_0 < a_1 < a_2 < \cdots$ be a given sequence of integers. Form the series

$$\sum_{k=1}^{\infty} (-1)^n/k, \tag{1}$$

where n is determined uniquely for each k by the inequality

$$a_n < k \leqq a_{n+1}.$$

Then the series (1) converges if and only if

$$\sum_{n=1}^{\infty} (-1)^n \log (a_{n+1}/a_n) \tag{2}$$

converges

Proof. Let $A_n = 1/(a_n+1) + 1/(a_n+2) + \cdots + 1/a_{n+1}$. Obviously, (1) converges if and only if

$$\sum_{n=0}^{\infty} (-1)^n A_n \tag{3}$$

converges. Comparing A_n with the integral of $1/x$, we obtain easily

$$\log [(a_{n+1} + 1)/(a_n + 1)] < A_n < \log (a_{n+1}/a_n), \qquad n \geqq 1. \tag{4}$$

Since $\log (1+t) < t$ for $t > 0$ we have

$$\begin{aligned} \log (a_{n+1}/a_n) - \log [(a_{n+1} + 1)/(a_n + 1)] &= \log [1 + (a_{n+1} - a_n)/a_n(a_{n+1} + 1) \\ &< (a_{n+1} - a_n)/a_n a_{n+1} \end{aligned}$$

and hence, by (4),

$$| (-1)^n \log (a_{n+1}/a_n) - (-1)^n A_n | < 1/a_n - 1/a_{n+1}. \tag{5}$$

Since $\sum_n (1/a_n - 1/a_{n+1})$ converges we conclude from (5) that the series (2) and (3) either both converge or both diverge. This completes the proof.

The statement proved above implies that (1) certainly converges if a_{n+1}/a_n is non-increasing for large n and $\lim (a_{n+1}/a_n) = 1$. This sufficient condition is satisfied in the proposed problem, in which $a_n = n(n+1)/2$ and hence $a_{n+1}/a_n = (n+2)/n$.

Also solved by Michael Aissen, F. Bagemihl, Barney Bissinger, R. C. Buck, L. J. Burton, Ragnar Dybvik, B. F. Hadnot, Harry Hochstadt, M. S. Klamkin, Roger Lessard, W. G. McGravock, Norman Miller, Leo Moser, C. S. Ogilvy, F. D. Parker, S. T. Parker, Arthur Rosenthal, R. V. Andree's engineering calculus class at the University of Wisconsin, and the proposer.

Editorial Note. Most solvers, after showing the convergence of the related series (1) of solution I above, failed to show that this implies convergence of the given series. A general theorem useful here is the supplementary theorem on p. 133 of Knopp's *Theory and Application of Infinite Series.*

Bagemihl and Klamkin located the problem as ex. 8 on p. 391 of Hardy's *A Course of Pure Mathematics*, 9th ed.

In addition to Herzog's result above, generalizations of the given problem were furnished by Buck and Miller. Buck used a general convergence criterion, based on Abel's partial summation, similar to the theorem on p. 314 of Knopp. Miller showed convergence of the harmonic series when modified by taking the first a terms positive, the next $a+d$ terms negative, the next $a+2d$ terms positive, *etc.* It is readily shown that this is a special case of Herzog's generalization.

It is easy to show that if the successive terms of the harmonic series are grouped into sets of 2^k terms, $k=0, 1, \cdots$, and the signs of the groups alternated, then the modified series diverges.

Ellipse and Sine Curve

E 826 [1948, 427]. *Proposed by C. S. Ogilvy, Trinity College*

Find the equation of the ellipse with foci at $(-1, 0)$ and $(1, 0)$ and with semi-perimeter equal to the length of one arch of the sine curve, $y=\sin x$.

I. *Solution by Robert Buschman, Reed College.* The semi-perimeter, S, of the ellipse

$$x = a \cos t, \qquad y = b \sin t$$

is

$$S = 2\int_0^{\pi/2} (a^2 \sin^2 t + b^2 \cos^2 t)^{1/2} dt.$$

If $a^2=1+b^2$, then

$$S = 2\int_0^{\pi/2} (b^2 + \sin^2 t)^{1/2} dt.$$

The length, L, of one arch of the sine curve is

$$L = 2\int_0^{\pi/2} (1 + \cos^2 x)^{1/2} dx = 2\int_0^{\pi/2} (1 + \sin^2 x)^{1/2} dx.$$

Since $(v+\sin^2 u)^{1/2}$ is an increasing function of v, $S=L$ if and only if $b^2=1$.

Hence the required equation is

$$x^2 + 2y^2 = 2.$$

II. *Solution by Leo Moser, University of Manitoba.* The ellipse of intersection of the cylinder $x^2+y^2=1$ and the plane $x=z$ would have, if properly placed in the xy-plane, the equation $x^2+2y^2=2$, and foci at $(-1, 0)$ and $(1, 0)$. On the other hand, it is easy to show that if the cylinder is unrolled into a plane, the intersection becomes one period of the sine curve. Hence $x^2+2y^2=2$ is the required equation.

Also solved by R. V. Andree, L. J. Burton, A. S. Day, M. B. Haslam, M. S. Klamkin, Roger Lessard, D. C. B. Marsh, Jr., B. E. Meserve, L. A. Ringenberg, Alex Rosenberg, P. D. Thomas, Alan Wayne, and the proposer.

Lessard pointed out that this problem appears as No. 1060 in *Intermédiaire des Recherches Mathématiques*, Jan. 1948.

Thomas also showed that the ellipse with semi-minor axis b and with perimeter equal to that of the four-leaved rose $\rho=b \sin 2\theta$ has its major axis twice its minor axis.

Reciprocals

E 827 [1948, 427]. *Proposed by Leo Moser, University of Manitoba*

Show that the reciprocal of every integer greater than 1 is the sum of a finite number of consecutive terms of the infinite series

$$\sum_{j=1}^{\infty} 1/j(j+1).$$

Solution by Michael Aissen, Stanford University. Since

$$(1) \qquad 1/j(j+1) = 1/j - 1/(j+1)$$

we see that

$$(2) \qquad \sum_{j=a}^{b-1} 1/j(j+1) = 1/a - 1/b.$$

Thus the problem is equivalent to that of finding positive integers a, b such that

$$(3) \qquad 1/a - 1/b = 1/m$$

for fixed integer $m>1$. From (1) it is obvious that a solution is

$$a = m - 1, \qquad b = m(m-1).$$

Therefore, if $m>1$,

$$(4) \qquad 1/m = \sum_{m-1}^{m(m-1)-1} 1/j(j+1),$$

and the theorem is proved.

It is not difficult to determine the number of distinct representations of $1/m$ in the form (3). We have:

If $m>1$ *and* $H(m)$ *denotes the number of distinct positive integral solutions of* (3), *then*

$$H(m) = \tfrac{1}{2}\{d(m^2) - 1\},$$

where $d(n)$ *as usual indicates the number of divisors of* n.

Since $1/a>1/m$, we have $a<m$. Let $a=m-c$. Substituting in (3) and solving for b we obtain

$$b = m^2/c - m.$$

For each value of c satisfying

$$c \mid m^2 \quad \text{and} \quad 1 \leqq c < m$$

there is one and only one pair of values a, b which satisfy (3). Also, since $a<b$, it is clear that none of these are duplicates. Now consider all of the $d(m^2)-1$ divisors, not equal to m, of m^2. If c is one of them, one and only one of the pair c, m^2/c is less than m. Therefore

$$H(m) = \tfrac{1}{2}\{d(m^2) - 1\}.$$

This result also furnishes an algorithm for determining each of the possible expressions of the form (3) for a given m. We have

Procedure: (a) List all divisors of m^2 which are less than m. Let c be such a divisor. (b) Choose a and b as $a=m-c$, $b=m^2/c-m$. (c) Then $1/a-1/b=1/m$.

In particular, if m is a prime, we see that $H(m)=1$, and in this case (4) gives the only solution to the given problem.

As another example suppose $m=pq$, $p<q$, p and q primes. Then $d(m^2)=9$ and $H(m)=4$. We have

c	a	b
1	$pq-1$	$pq(pq-1)$
p	$p(q-1)$	$pq(q-1)$
q	$q(p-1)$	$pq(p-1)$
p^2	$p(q-p)$	$q(q-p)$

Also solved by F. W. Ballantine, Murray Barbour, A. W. Boldyreff, J. C. Brixey, D. H. Browne, L. J. Burton, P. L. Chessin, A. S. Day, Roy Dubisch, Daniel Finkel, W. Fulks, H. M. Gehman, M. S. Klamkin, H. D. Larsen, Roger Lessard, Julius Lieblein, D. C. B. Marsh, Jr., B. E. Meserve, F. L. Miksa, F. D. Parker, L. A. Ringenberg, Joseph Rosenbaum, Alex Rosenberg, C. M. Sandwick, B. D. Smith, W. R. Talbot, P. D. Thomas, C. W. Trigg, E. W. Trost, W. R. Van Voorhis, and the proposer.

Rosenbaum also found the number of distinct solutions to the problem. As allied material he remarked that the reciprocal of every integer of the form

$2(m^2+m+1)$ is the sum of a finite number of consecutive terms of the series

$$\sum_{j=1}^{\infty} 1/j(j+1)(j+2),$$

and that

$$1/j(j+1)(j+2)(j+3) + 1/(j+2)(j+3)(j+4)$$

is always the reciprocal of an integer.

Editorial Note. It is easy to show that the shortest sequence for the given problem is obtained by choosing the maximum allowable c in the above procedure.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4341. *Proposed by D. H. Browne, Buffalo, New York*

The sequence $\{a\}=3, 7, 47, 2207, 4870847, \cdots$, used for the determination of primality of the Mersenne numbers, is usually defined by

$$a_{n+1} = a_n^2 - 2.$$

Show that it may also be defined by

$$a_k = f_{2^{k+1}}/f_{2^k},$$

where the f's are the Fibonacci numbers, 1, 1, 2, 3, 5, 8, $\cdots$.

4342. *Proposed by R. J. Walker, Cornell University*

A spherical planet whose density at any point P is a function only of the distance of P from the center of the planet has the following property. If a

straight frictionless tunnel is bored between two points of the planet's surface the time required for an object to slide from one of these points to the other is independent of the positions of the points. Prove that the planet has constant density.

4343. *Proposed by C. D. Olds, San Jose State College, California*

Find an approximation to $(x^2+y^2)^{1/2}$ by a linear function $\alpha x+\beta y$ where $ax \leqq y \leqq bx$, and $0 \leqq a \leqq b$, such that the absolute value of the relative error

$$[(x^2 + y^2)^{1/2} - (\alpha x + \beta y)]/(x^2 + y^2)^{1/2}$$

shall be as small as possible.

4344. *Proposed by Victor Thébault, Tennie, Sarthe, France*

(1) If, in a triangle, one of the angles is 120° (or 60°), two of the Feuerbach points are diametrically opposite on the nine point circle, and conversely. (2) If the triangle is scalene and if the circle through the feet of the interior bisectors (or one interior and two exterior bisectors) passes through one of the vertices, three of the Feuerbach points form an isosceles triangle, and conversely.

4345. *Proposed by Irving Kaplansky, Institute for Advanced Study*

An element x in a ring is said to be right quasi-regular if there exists an element y with $x+y+xy=0$. It is evident that in a division ring, every element except -1 is right quasi-regular. Prove the converse: if every element in a ring A is right quasi-regular, with exactly one exception, then A is a division ring.

SOLUTIONS

Triangles Inscribed in a Triangle

4260 [1947, 418]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a triangle ABC inscribe two triangles $A_1B_1C_1$ and $A_2B_2C_2$ whose sides are parallel to the medians. Show that (1) the triangles ABC, $A_1B_1C_1$, $A_2B_2C_2$ have the same centroid and the same Brocard angle; (2) the triangles $A_1B_1C_1$, $A_2B_2C_2$ are inscribed in an ellipse concentric and homothetic to the inscribed Steiner ellipse, the ratio of homothety being $1/\sqrt{3}$.

Solution by Ou Li, Yenching University, Peiping, China. We prove the generalization: In a triangle ABC inscribe two triangles $A_1B_1C_1$ and $A_2B_2C_2$ whose vertices divide the sides in the ratio $\lambda:\mu$ and $\mu:\lambda$ respectively, where $\lambda \neq \pm\mu \neq 0$. The conclusions of the proposed problem still hold save that the ratio of homothety is $(\lambda^2-\lambda\mu+\mu^2)^{1/2}/(\lambda+\mu)$. (For the Proposer's problem we have $\lambda:\mu=1:2$.)

For (1), proofs have been given in R. A. Johnson, *Modern Geometry*, arts. 276 and 476.

For (2), we use areal coördinates with ABC as the triangle of reference.

The coördinates of A_i, B_i, C_i $(i=1, 2)$ are easily found to be (r, s, t) where r, s, t are distinct permutations of $0, \lambda, \mu$. Evidently all six points lie on the ellipse whose areal equation is

$$S_1 \equiv \lambda\mu(x^2 + y^2 + z^2) - (\lambda^2 + \mu^2)(yz + zx + xy) = 0.$$

Moreover we have the equation of the Steiner ellipse*

$$S_2 \equiv yz + zx + xy = 0.$$

Thus

$$S_1 = - (\lambda + \mu)^2 S_2 + \lambda\mu u^2 - (\lambda + \mu)^2(S_2 - ku^2),$$

where $u = x+y+z$. Hence S_1, S_2 are homothetic and concentric with the centroid G as their homothetic center.† To find the ratio of homothety, H, let S_2 divide the line joining G to A in ratio $m:n$. We have

$$\frac{m}{n} = \frac{(\lambda^2 - \lambda\mu + \mu^2) \pm (\lambda + \mu)\sqrt{\lambda^2 - \lambda\mu + \mu^2}}{3\lambda\mu}$$

and

$$H = \frac{m}{m+n} = \frac{\sqrt{\lambda^2 - \lambda\mu + \mu^2}}{\lambda + \mu}.$$

When $\lambda:\mu = 1:2$, then $H = 1/\sqrt{3}$.

We note that as the ratio $\lambda:\mu$ varies, the equation S_1 represents a system of homothetic and concentric ellipses with the centroid G as their homothetic center. Since the ratio $\lambda:\mu$ and $\mu:\lambda$ are simultaneously treated in our solution, we need only consider the ratio $\lambda:\mu$ in the interval $(-1, 1)$. Limiting cases arise (i) when $\lambda:\mu = 1$, giving the maximum inscribed ellipse;‡ (ii) when $\lambda = 0$, giving the minimum circumscribed ellipse (Steiner ellipse); and (iii) when $\lambda:\mu = -1$, giving the line at infinity. It is also of interest that for real values of λ and μ, the system of ellipses fills the plane except for the interior of the maximum inscribed ellipse.

Also solved by L. M. Kelly and R. Goormaghtigh.

Rectifiable Plane Curves

4262 [1947, 418]. *Proposed by L. A. Santaló, Rosario, Argentina*

Let C be a rectifiable plane curve of length L, contained within a given circle of radius R. Prove that there is a circle of radius $\rho \geqq R$ which cuts C in n points, where

$$n \geqq L/\pi R. \tag{1}$$

* D. M. Y. Sommerville, *Analytical Conics*, London, 1924, p. 191.

† D. M. Y. Sommerville, loc. cit., p. 205.

‡ D. M. Y. Sommerville, loc. cit., p. 181.

In particular there is a line which cuts C in n points, where n satisfies (1). If $\rho < R$, the inequality (1) must be replaced by

$$n \geqq \frac{4L\rho}{\pi(R+\rho)^2}. \tag{2}$$

See L. A. Santaló, A theorem and an inequality referring to rectifiable curves, *American Journal of Mathematics*, 1941, p. 635.

Solution by the Proposer. Let (x, y) be the coördinates of the variable center of a circle of constant radius ρ. Let $N \equiv N(x, y)$ be the number of common points of this circle and the curve C for each position of (x, y). Then the Proposer has shown, in the paper already cited, that the following integral formula holds;

$$\int\int N dxdy = 4L\rho.$$

On the other hand, if $\rho \geqq R$, the area covered by the points (x, y) which are centers of circles of radius ρ and which cut the given circle of radius R, has the value

$$\pi(\rho + R)^2 - \pi(\rho - R)^2 = 4\pi R\rho.$$

Consequently the mean value of N is

$$\overline{N} = \frac{\int\int N dxdy}{\int\int dxdy} = \frac{L}{\pi R}.$$

As the mean value of a function is not greater than its maximum value, the inequality (1) is established.

If $\rho < R$, the area covered by the centers (x, y) of circles of radius ρ which cut the given circle of radius R or are contained in its interior is $\pi(\rho+R)^2$. Consequently $\overline{N} = 4L\rho/\pi(\rho+R)^2$ and inequality (2) holds.

The Continuum Hypothesis

4263 [1947, 419]. *Proposed by Howard Eves, Oregon State College, and Paul Halmos, Syracuse University*

Criticize the following alleged proof of the continuum hypothesis.

Let X be the set of all infinite sequences of 0's and 1's, and let E be an arbitrary uncountable subset of X. Corresponding to any finite sequence, $\{a_1, \cdots, a_k\}$, of 0's and 1's, write $E(a_1, \cdots, a_k)$ for the set of all sequences $\{x_n\}$ which belong to E and begin with $\{a_1, \cdots, a_k\}$. Since $E = E(0) + E(1)$, at least one of the two sets $E(0)$ and $E(1)$ is uncountable; write $a_1 = 0$ or 1 according as $E(0)$ is or is not uncountable. Then, in either case, $E(a_1)$ is uncountable. If a_i has already been defined for $i = 1, \cdots, k$, so that $E(a_1, \cdots, a_k)$ is uncountable, then write $a_{k+1} = 0$ or 1 according as $E(a_1, \cdots, a_k, 0)$ is or is not uncountable. The resulting infinite sequence $\{a_1, a_2, a_3, \cdots\}$ has the property

that for any value of k, it is true that $E(a_1, \cdots, a_k)$ is uncountable. Write E^* for the union of all $E(a_1, \cdots, a_k)$, for $k=1, 2, 3, \cdots$; then E^* is a subset (in fact an uncountable subset) of E.

For certain positive integers k it is true that both $E(a_1, \cdots, a_k, 0)$ and $E(a_1, \cdots, a_k, 1)$ are uncountable; in fact this must happen for an infinite number of k's. (Otherwise, for a sufficiently large k, $E(a_1, \cdots, a_k)$ would not be uncountable, contrary to its construction.) Let $k_1, k_2, k_3, \cdots$ be the integers for which this is true, and write, for any $\{x_1, x_2, x_3, \cdots\}$ in E^*, $y_n = x_{k_n+1}$; then $\{y_1, y_2, y_3, \cdots\}$ is an infinite sequence of 0's and 1's. From the way in which the k_n are defined it follows that every possible sequence of 0's and 1's occurs as a y sequence, and that consequently the sequences $\{x_1, x_2, x_3, \cdots\}$ in E^* correspond (in possibly a many to one manner) to a set (viz. the set of all y sequences) having the power of the continuum c. It follows that the cardinal number of E^* (and hence of E) cannot be less than c. In other words it has been proved that every uncountable subset of a set of power c also has power c.

I. *Solution by J. W. Gaddum, University of Missouri.* The assertion, "From the way in which the k_n are defined it follows that every possible sequence of 0's and 1's occurs as a y sequence, $\cdots$" is incorrect, as the following counter example shows.

Let E be the set of all sequences of X whose second and third terms are not both 1. Then

$$a_1 = a_2 = \cdots = 0,$$
$$k_n = n, \quad (n = 1, 2, \cdots), \qquad y_n = x_{n+1},$$
$$E^* = E(0).$$

For a y sequence of the form $(1, 1, y_3, y_4, \cdots)$ to occur, there would have to be an x sequence of the form $(0, 1, 1, x_4, x_5, \cdots)$ in E^*. This is not the case and hence not every possible sequence of 0's and 1's occurs as a y sequence.

II. *Solution by Fritz Herzog, Michigan State College.* It is not necessarily true that the set of all y sequences contains every possible sequence of 0's and 1's. Let $F(b_1, b_2, \cdots, b_m)$ denote the set of all those y sequences that begin with $b_1, b_2, \cdots, b_m$. Then the construction of the y sequences guarantees merely that both $F(b_1, b_2, \cdots, b_r, 0)$ and $F(b_1, b_2, \cdots, b_r, 1)$ are uncountable provided that $b_n = a_{k_n}$ for $n=1, 2, \cdots, r$; but it leaves the possibility open that, if $b_n \neq a_{k_n}$ for at least one of these values of n, one or even both of the two above sets is denumerable or less.

It is, in fact, possible to exhibit an uncountable set E for which the set of all y sequences is denumerable. Let E_p, for any prime p, consist of all sequences $\{x_n\}$ of 0's and 1's such that $x_p = 1$; $x_{2p}, x_{3p}, x_{4p}, \cdots = 0$, or 1 (independently); $x_n = 0$ when $n \not\equiv 0 \pmod{p}$. Let E be the union of all E_p, $p = 2, 3, 5, 7, \cdots$.

In the first place, the sets $E(0, 0, \cdots, 0)$ are all uncountable; hence $a_1 = a_2 = a_3 = \cdots = 0$ and $E^* = E(0) = E$. Secondly the sets $E(a_1, a_2, \cdots, a_k, 1)$

$=E(0, 0, \cdots, 0, 1)$ are uncountable or empty, according as $k+1$ is prime or not. Consequently, $k_n = p_n - 1$ for all n, where p_n denotes the nth prime. Thus $y_n = x_{p_n}$ and it is clear from the definition of E that, for any sequence $\{x_n\}$ belonging to $E^* = E$, the corresponding y sequence (viz., the sequence $\{x_{p_n}\}$) contains exactly one 1. Hence the set of all y sequences is denumerably infinite.

Also solved by J. Barlaz, William Gustin, and the Proposers.

Upper Bound for a Definite Integral

4264 [1947, 479]. *Proposed by G. Polya, Stanford University*

Given $a > 0$, $b > 0$, and given that $f(x)$ is a non-linear function such that $f(0) = 0$, $f(a) = b$ and that

$$f(x) \geqq 0, \qquad f''(x) \geqq 0, \qquad 0 \leqq x \leqq a,$$

give an analytic proof that

$$2\pi \int_0^a f(x)[1 + (f'(x))^2]^{1/2} dx < \pi b(a^2 + b^2)^{1/2}.$$

(The inequality becomes intuitive when both sides are interpreted as areas of curved surfaces.)

Solution by N. J. Fine, University of Pennsylvania. Define

$$\begin{aligned} F(x) &= f(x)\{x^2 + [f(x)]^2\}^{1/2}, \\ G(x) &= 2f(x)\{1 + [f'(x)]^2\}^{1/2}, \\ \Delta(x) &= \{x^2 + [f(x)]^2\}\{[F'(x)]^2 - [G(x)]^2\}. \end{aligned}$$

It is easily verified that

$$\begin{aligned} \Delta(x) = {} & x^2\{xf'(x)\}^2 + 2f(x)\{2[f(x)]^2 + x^2\}\{xf'(x)\} \\ & - [f(x)]^2\{3x^2 + 4[f(x)]^2\}. \end{aligned}$$

Clearly $f'(x)$ is non-negative and non-decreasing in the given interval. Hence for all x in $(0, a)$

$$f(x) = \int_0^x f'(t)dt \leqq x \cdot \max_{0 \leqq t \leqq x} f'(t) = xf'(x),$$

and the inequality must hold in some sub-interval. It follows that $\Delta(x) \geqq 0$. Since $F'(x)$ is positive, this implies that $F'(x) \geqq G(x)$, and again the inequality holds in some interval. Integrating,

$$F(a) > \int_0^a G(x)dx$$

and the theorem is proved.

Also solved by J. F. Locke and the Proposer.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 W. 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

Fundamentals of Statistics. By J. B. Scarborough and R. W. Wagner. Boston, Ginn and Company, 1948. 7+145 pages. $2.40.

The book under review was originally designed for a brief course in statistics at the United States Naval Academy. While this project has had to be deferred, the authors felt that their work might be useful to others. This will certainly be so. To quote from the preface, "It is intended for students who have completed the usual courses in mathematics through elementary calculus, and it should meet the needs of those desiring a relatively brief but fairly rigorous treatment of statistical methods." By "relatively brief" they mean approximately twenty lessons.

About half of the text is descriptive (i.e., non-probabilistic) statistics. The student is shown how to present data in tables and charts; to compute averages, measures of dispersion, "shape coefficients" (third and fourth moments) and formulas of two-dimensional regression analysis. Most of this is done briefly and clearly, though the student might wonder just why certain quantities are important. The purity of the descriptive approach is somewhat adulterated in Chapter V, "Correlation." One reads (page 38): "*Simple correlation . . . is a mutual variation between two variables which is not accidental.*" This apparent promise of stochastic developments is followed instead by straight analytic geometry—the least-squares fitting of a line to a finite set of points in the plane. The equations and formulas are properly derived; there is the proper final warning against identification of "correlation" with "casuality," but the gap between "non-accidental variation" and "least squares" remains unbridged.

Chapter VI, "Probability Functions," opens with brief discussions of discrete and continuous random variables and of mathematical expectation. It closes with Tchebycheff's elegant proof of the Bernoulli theorem. Chapter VII, "The Normal Curve and a Generalization," contains a difference-to differential-equation derivation of the Gaussian frequency function as the limit of the binomial; the generalization is the Gram-Charlier Type A series (or, rather, its first five terms). A special case of the central limit theorem is stated. Chapter VIII, "Sampling," is limited to large-sample theory. Some of the current words and phrases are to be found here, but even "parameter" and "statistic" are defined only in a footnote.

There are three appendices. The first is on elementary probability, and might be reviewed before Chapter VI is begun. The second contains tables of the normal frequency function, its integral, and its second, third and fourth deriva-

tives. The third is a short list of books on statistics and probability.

A beginning statistical text as brief as this one poses difficult questions in the choice of topics. For the authors' original purpose, their selection would seem to be a good one. For students of experimental science, the omission of small-sample theory is serious; fortunately, the authors do not leave the impression that pre-1908 methods constitute the final word. Those whose interest lies more in questions of inference than in techniques for displaying and summarizing data could wish for more probability, sooner, so that difficulties of motivation and of justification might have been put on a different level.

Only one misprint was noticed (on page 103, line below equation (8.9), "page 72" should read " page 78"). Format and cover are pleasing.

A. T. LONSETH

The Royal Society Newton Tercentenary Celebrations. Cambridge, at the University Press; New York, The Macmillan Company, 1947. 16+92 pages. Six Plates. $3.00.

This magnificent volume is the record of the international celebrations sponsored by the Royal Society of London on July 15–19, 1946, in honor of the three hundredth anniversary of Newton's birth, the war having prevented an international gathering of distinguished scholars for this occasion in 1942. The Royal Society extended invitations to the National Academies of all countries to send delegates for participation in the celebrations; thirty-three nations, the Vatican, and the British Colonies were represented by the 118 official delegates who succeeded in attending personally.

Printed in this volume are the eight invited lectures, which were delivered at the Royal Institution, and two official addresses of welcome, all in English. These are: "Newton," by E. N. da C. Andrade; "Newton, the Man," by Lord Keynes; "Newton and the Infinitesimal Calculus," by J. Hadamard; "Newton and the Atomic Theory," by S. I. Vavilov; "Newton's Principles and Modern Atomic Mechanics," by N. Bohr; "Newton: the Algebraist and Geometer," by H. W. Turnbull; "Newton's Contributions to Observational Astronomy," by W. S. Adams; "Newton and Fluid Mechanics," by J. C. Hunsaker; and the Addresses of Welcome by Sir Robert Robinson, President of the Royal Society, and by Dr. G. M. Trevelyan, Master of Trinity College.

In addition to these ten interesting and instructive papers there are six excellently reproduced plate illustrations: three portraits of Newton by C. Jervas (1703), G. Kneller (1689), and J. Vanderbank (1725); the northeast corner of D. Loggan's print of Trinity College showing Newton's rooms; H. Fletcher's sketch of Woolsthorpe Manor; and a holograph letter from Newton in Kensington to Halley dated "March 1st, 1724/5."

This volume will take its place alongside the 1927 British Mathematical Association's Memorial Volume *Isaac Newton*, edited by W. J. Greenstreet (G. Bell and Sons, London, 1927). (The latter book is referred to below by the

letters *B.M.A.*). The present volume is a colorfully bound, beautifully printed volume of large format.

In view of the wide interest in this subject, certain supplementary and critical remarks regarding individual items in this book may be permitted here. We refer below to four of the illustrations alphabetically and to the ten essays numerically.

(a) The frontispiece is the Charles Jervas portrait which Newton himself presented to the Royal Society in 1717. It seems to have been heretofore less widely known than it deserves. It is, for example, not mentioned by D. E. Smith in his list of commonly reproduced portraits of Newton in his *B.M.A.* essay, p. 171.

(b) The northeast corner of David Loggan's print of Trinity College is reproduced from his meticulously accurate *Cantabrigia Illustrata* (1688/90). On the left is the Great Gate started in 1518 as an isolated and independent structure, and on the right is the Chapel which was begun a few years later. The chambers between the two were built before the Chapel was finished and there Newton lived for many years in the first floor rooms next to the Gate, but for how long is not known, although he was there when the *Principia* was published. The wooden stairway which Newton used as a location for his own observatory is shown, with the attractive formal garden in the foreground, facing eastward. The only description in the present volume of this plate is contained in a short paragraph of Lord Keynes' essay (pp. 29–30). It would have been interesting to have had here from Dr. Trevelyan a more elaborate description; but fortunately he has already done a thing of this kind in an attractive, inexpensive little book to which the interested reader may be referred: G. M. Trevelyan, *Trinity College* (*An Historical Sketch*), Cambridge Univ. Press (Macmillan, N. Y.), 1943, (Cf. especially pp. 7, 19, 40–41, 48). There is also Professor G. N. Watson's essay bearing on this subject in *B.M.A.*, p. 144.

(c) Hanslip Fletcher's sketch of Woolsthorpe Manor may be compared with the reproductions of photographs of the house in the *B.M.A.*, opposite p. 45. In the latter (p. 141) is a brief description by J. A. Holden of the interior of this house, and of the stone tablet which is not decipherable in Fletcher's sketch.

(d) In the very clearly reproduced holograph letter Newton requests Halley to insert in the latter's table of data for "the comet of 1680" the comet's distances from the Sun, for inclusion by Newton in the second edition of the *Principia*. For Newton's subsequent use of these data there one may consult the "Example" of Book III, Prop. 41.

(1) At the conclusion of his address of welcome (pp. 1–2) Sir Robert Robinson takes the opportunity to mention the proposed Isaac Newton Observatory. This is to contain a 100-inch reflector and is to be located on the grounds of the Royal Observatory at its new site at Herstmonceux (Sussex) where the latter is being progressively removed from Greenwich after over 270 years. A detailed account of the initial recommendations for this research memorial will be found in *Monthly Notices, Royal Astron. Soc.*, *107*, 11–19, 1947.

(2) Professor Andrade contributes (pp. 3–23) an essay on "Newton" which surveys the grand achievements of the *Opticks* and *Principia* in particular, with appropriate reference to Newton's other work, together with relevant historical and biographical facts. This essay presents a comprehensive general survey of Newton's life and accomplishments, and Professor Andrade has succeeded admirably in a difficult undertaking. However on p. 15 the author in speaking of Newton's "wonderful skill as an experimenter" makes a statement which is unfortunately worded: "As a comparatively small matter I may cite his proof that gravitational mass and inertial mass are the same, a trifle which is often overlooked." Relative only to the galaxy of Newton's brilliant achievements, as Professor Andrade doubtless means to imply, can this matter be regarded as "comparatively small" or "a trifle . . . often overlooked," and as should be clear from the context it involved no "proof." Newton's discerning pendulum experiments answering the fundamental question of the approximate equality of gravitational and inertial mass, within allowable errors of observation, were extended, in later pre-relativity days, to a refined relative accuracy of the order 10^{-8} by Bessel, Eötvös, Zeeman, and others. Crudely, the experiment is continually repeated in the innumerable gross every day work of the engineer whenever he replaces Newton's "quantity of matter" (inertial mass) by the engineer's "W/g" and comes out safely. W. de Sitter called the result "one of the best ascertained empirical facts in physics—perhaps the best." (*Univ. of Calif. Publ. in Math.*, *2*, 143, 1933).

(3) In order to round out the picture of the shape of things at Trinity College during Newton's time, some readers may wish to supplement the brief remarks by Dr. G. M. Trevelyan in his address of welcome (pp. 24–6) with Chapters 5 and 6 of his *Trinity College*, to which we have referred above.

(4) The paper "Newton, the Man" (pp. 27–34) by the late John Maynard, Lord Keynes, was read posthumously at these celebrations, and according to the editor's prefatory note "must be regarded as unfinished." When Newton departed from Cambridge for London he took with him a great mass of manuscript which he never published nor destroyed. In 1888 the mathematical portion of these "Portsmouth Papers" came into the possession of the Cambridge University library. In 1936 the rest were dispersed at public auction and Keynes "managed gradually to reassemble about half of them, including nearly the whole of the biographical portion, that is, the 'Conduitt Papers,' in order to bring them to Cambridge . . . ." Besides the mathematical parts, and the biographical "Conduitt" section, the Portsmouth Papers appear to be largely concerned with theological questions, commentaries on apocalyptic writings, alchemical matters . . . Keynes gives no quotations from "these queer collections" nor indeed anything essentially new, but on the other hand he is not hesitant with broad estimates of the implications of these manuscripts. It is to be regretted that Keynes was denied opportunity to verify his impressions.

In the essays of both Andrade (p. 5) and Keynes (pp. 28–9) reference is made to the fundamental problem of the "central" attraction of the solid sphere,

which Newton does not seem to have resolved until as late as 1685. Keynes' reference suggested Professor J. E. Littlewood's recent article on this subject in *Math. Gaz.*, *32*, 179, 1948.

(5) In his essay "Newton and the Infinitesimal Calculus" (pp. 35–42) Professor Hadamard has given an interesting account of the differences in approach, comprehension, and explicit enunciation of the essential problems of the calculus which distinguished Newton's work. According to Hadamard, Newton possessed a realizable appreciation of the Heraclitian philosophy, ignored by antiquity and the Middle Ages, that " . . . you cannot understand the state of being if you do not watch . . . 'le devenir,' that is, the continuous change which constantly occurs in that state." This consideration of a problem in its "devenir" is carefully examined with respect to the development of both the differential and integral calculus. And here too in the midst of the "marvelous sagacity" of the pioneers in these developments Hadamard finds again how "It rather commonly happens that one overlooks a most striking result when one has just found it in a scarcely different form," a matter of importance which Hadamard examined at greater length in his well-known book on *The Psychology of Invention in the Mathematical Field* (Princeton Univ. Press, 1945).

(6) The essay "Newton and the Atomic Theory" by Academician S. I. Vavilov (pp. 43–55) is a carefully written and meticulously referenced study, with particular attention to Newton's *Lectiones Opticae*, *Opticks* and synoptic memoir *De Natura Acidorum*. With regard to this last, Vavilov adds an appendix to his essay pointing out the liberties which Harris took in the English translation of the original memoir which he himself first published in 1710. Vavilov finds "sufficient grounds to believe that Newton had a good idea of the complexity of the chemical atom and even conjectured the existence of a tiny, exceedingly stable atomic nucleus," being in this sense "a predecessor of Rutherford." Moreover the author comes "to the conviction that Newton had conjectured everything in the atomic field that could possibly be conjectured at the time, on the basis of the experimental material evidence then available. Not even in one of his fundamental points was he mistaken . . . ." The evidence which Vavilov lays before the reader in substantiation of these claims is quite convincing.

(7) In a brief but closely written essay on "Newton's Principles and Modern Atomic Physics" (pp. 56–61) Professor Niels Bohr seeks to convey "an impression of the living inspiration which Newton's work still exerts on all endeavors aiming at the progress of science in its widest sense." For one thing, treating the broad epistemological questions involved, he is led to comment (p. 60) that "Especially in the study of psychical experience we are confronted with an observational problem which exhibits a deep analogy with that in atomic physics . . . . More concretely speaking, the use of words like 'thoughts' and 'emotions' exhibits striking analogy to the complementary application of kinematical and dynamical variables in quantum mechanics . . . ." His striking remarks in this regard will doubtless be of great interest to psychologists,

physicists, and the cyberneticists.

(8) Professor H. W. Turnbull's essay (pp. 62–72) on "Newton: The Algebraist and Geometer" presents a series of snapshots of some of Newton's algebraic interests and results, with a glance, in the direction of geometry, at the *Enumeratio Linearum Tertium Ordinis* and the projective transformation regarding conics contained in Lemma 22, Book I, of the *Principia*. Professor Turnbull refers in most detail to Newton's work on the harmonic, binomial, and angular section series, and to "the master theorem" of interpolation theory. With regard to the latter Professor Turnbull cites D. C. Fraser's important discovery in the Portsmouth Papers of two sheets of foolscap indirectly labeled *Regula Differentiarum*, which explains the ideas behind Newton's typically vague contemporary (1676) letter to Oldenburg. With reference to this and Newton's other writings on interpolation there is D. C. Fraser's own valuable essay in the *B.M.A.*, page 45. Professor Turnbull also cites and illustrates Newton's rule of signs for the possible number of complex roots of an algebraic equation, which was, in Sylvester's words, "so long the wonder and opprobrium of algebraists" (*Math. Papers*, *2*, 493). It may be recalled that Sylvester's syllabus of his 1865 lectures on this subject was the very first paper to appear in the newly founded *Proc. London Math. Soc.* (cf., *Math. Papers*, *2*, 498).

(9) Dr. W. S. Adams devotes his essay (pp. 73–81) to "Newton's Contributions to Observational Astronomy," with reference to "the recognition and explanation of the spectrum, and the invention of the reflecting telescope." In particular the author writes of some of the exciting problems of interstellar matter, its composition, density and distribution. These modern astrophysical studies had their origin in Huggins' paper read before the Royal Society in 1864 on the discovery of emission lines in the spectra of certain luminous diffuse galactic nebulae, and also, particularly, in Hartmann's discovery in 1904 of the "detached" H and K lines of ionized calcium in the spectroscopic binary δ Orionis. Many of the recent discoveries and identifications of interstellar lines of atomic and molecular origin have been possible as a result of Dr. Adams' own observations. In regard to these matters, with their great cosmogonic import, the reader may wish to consult the excellent report by C. S. Beals in *Monthly Notices, R. A. S.*, *102*, 96, 1942 and the report of the recent Harvard symposium on the subject published in *Centennial Symposia*, *Harvard Obs. Mon.*, No. 7 (Cambridge, Mass., 1948), both of which contain extensive bibliographies.

(10) Professor J. C. Hunsaker's essay (pp. 82–90) on "Newton and Fluid Mechanics" refers to Newton's contributions to fluid mechanics and gives an instructive summary of the accomplishments of present day aerodynamical theory, experiment, and airfoil design for subsonic flow. The "pedigree of the low-drag wing" is traced in detail, starting with Newton and culminating in the now complete analysis of the subsonic aspects of the problem. The transonic and supersonic problems, which are in need of mathematical resolution, remain a challenge.

S. G. HACKER

Rinehart Mathematical Tables. By H. D. Larsen. New York, Rinehart and Co., 1948. 8+264 pages. $1.50.

This set of tables which is to serve as a handbook for mathematics students and for other computers in engineering, physics and allied fields, meets the demands and requirements for such work in more than satisfactory fashion.

The well designed handbook is divided into two separate sections. Part I consists of twenty-seven different tables. As the result of an extensive survey not only are all the usual tables present, but also several new tables appear for the first time in any such handbook. For example, those in actuarial work have access to two new ordinary mortality tables, and for those working in the field of statistics a table of values for F and t is given, and a χ^2 probability scale appears. In order to guarantee accuracy the page proofs of all the tables were checked independently against three sources.

Part II is made up of worthwhile miscellaneous material, formulas from the different branches of mathematics, a rather complete, alphabetically ordered, collection of the standard curves (with their graphs) met in elementary mathematics, forty-three formulas for differentiation, a list of four hundred and thirty indefinite integrals, sixty-three definite integrals, and fifty-two different series.

The compiler states in his preface that there are two main requirements for a satisfactory set of tables, namely, accuracy and an attractive format. The reviewer feels that the author has taken special care to present as accurate a table as possible, and that the publisher has produced a pleasing design and appearance. The total result is a handbook which will be more than well received by students, teachers and computers alike.

E. P. VANCE

Basic Mathematics: A Workbook. By M. W. Keller and J. H. Zant. Boston, Houghton Mifflin Co., 1948. 4+253 pages. $1.50.

This is a combined text and workbook covering the elementary principles of arithmetic, algebra, some geometry, and numerical trigonometry. A suitable diagnostic test is placed at the beginning of each of the three main topics of the book. At appropriate intervals the student encounters comprehensive tests which measure his progress. Each page can easily be torn out so as to be graded or marked as the instructor desires. The arrangement of the book is such that it can profitably be used however, without the aid of an instructor.

The arithmetic section includes a study of whole numbers and fractions, denominate numbers, percentage, square root, and cube root. Among the algebraic topics are signed numbers, factoring, linear and simultaneous linear equations, fractions and fractional equations, exponents, radicals, quadratic equations, proportion, and graphs. The third main division of the book concerns itself with logarithms and solution of right and oblique triangles. Four place logarithmic and trigonometric tables are available.

There is a pronounced need for such a book as this among college students, nowadays.

W. R. HUTCHERSON

College Algebra. By Gordon Fuller. New York, D. Van Nostrand Co., 1948. 5+255 pages.$2.85

This neat little college algebra book accomplishes a sensible review of fundamental operations in the first chapter. Freshmen often think that 2-x and x-2 are different prime factors. The author's definition, on page 19, of a prime factor as "being expressible as a product only as one times itself, or minus one times its negative" is a timely help.

In Chapter IV (*Equations*), the identity symbol is used (page 45) when discussing the identity. This is a wise move, enabling the equation and the identity to really take on new meaning to the reader.

In Chapter V (*Functions*), it is pleasing to find the use of the y function of x for a list of problems on page 62 instead of the f function of x. This should cause students to sense the meaning of a function quicker. Chapter VIII (*Exponents and Radicals*) makes a genuine effort to enable the student to conquer this material which possibly gives more trouble to the average student than any other part of college algebra. The example at the bottom of page 95 shows the wayward mathematical sinner the usual wrong solution before the correct one is illustrated.

The exhibits on page 112 seem so practical and appropriate that they should be found in all discussions on logarithms. Even though $(2+i\sqrt{11})/3$ looks involved, yet the student is shown on page 129 that this expression is a root of the equation $3x^2-4x+5=0$. This should increase the faith of the immature seeker after mathematical truth. The quadratic graph on page 167 is a timely illustration for exhibiting the idea of the inequality, of Chapter XIII. The two exhibits on page 192 for approximating irrational roots are commendable.

Such a book will find a ready place in the average college or university. Unfortunately, more students need this gradual approach to their first college mathematics than was true two decades ago.

W. R. HUTCHERSON

NEW BOOKS RECEIVED

An Introduction to the Algebra of Vectors and Matrices. By T. L. Wade. Tallahassee, 1949. (Obtainable directly from the author). 6+97 pages, lithoprinted. $2.25, plus carriage.

A Concise History of Mathematics. Vol. 1 and 2. (Dover Series in Mathematics and Physics). By D. J. Struik. New York, Dover Publications, 1948. 18+299 pages. $1.50 per volume.

Elementary Statistical Analysis. By S. S. Wilks. Princeton University Press, 1948. 11+284 pages. $2.50.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS, 1947–48

Kappa Mu Epsilon, Chicago Teachers College

Papers heard by the *Illinois Gamma* Chapter of *Kappa Mu Epsilon* were:
Quadratic curves and applications, by Prof. J. J. Urbancek
Mathematics and our knowledge of nature, by Dr. Arturo Fallico
Magic squares, by Miss Edna Boedeker
Mathematics in business, by R. E. Olson
History of weights and measures, by Wm. J. Coyne
Probabilities of winning in gambling, by J. J. Urbancek
Roots: Budan's and Sturm's methods, by J. J. Urbancek
'*Much ado about nothing*,' by Wm. Coyne and John Kelly.

In addition to several meetings which were of a social nature, the members witnessed four mathematical movies.

The officers elected are: President, Kathryn Graham; Vice-President, Sam Altshuler; Secretary, June Machovec; Treasurer, Ramona Goldblatt; Sponsor, Prof. J. J. Urbancek.

Pi Mu Epsilon, Oklahoma A. and M. College

The *Oklahoma Beta* Chapter of *Pi Mu Epsilon* resumed an active status early in 1948. The papers heard during the remainder of the year included:

Mathematics of genetics, a series of lectures by Dr. Hilda Geiringer of Wheaton College

Sampling, with application to the Greek plebiscite, by Mr. Carl Marshall, the college statistician

Let Charlie do it, an address emphasizing world peace, by Dr. Clark Dunn, Director of the Engineering Experiment Station.

New pledges prepared and operated the mathematics exhibit in the School of Engineering annual Open House.

Officers elected were: Director, Cole Downing; Vice-Director, E. W. Lewis; Secretary-Treasurer, P. C. Gross; Faculty Sponsor, Dr. James Zant.

Kappa Mu Epsilon, University of New Mexico

The *New Mexico Alpha* Chapter of *Kappa Mu Epsilon* at the University of New Mexico began the years activities with an initiation banquet at which 45 students and faculty members were initiated. During the year the following papers were given:

Recent developments in communication, by Prof. R. E. Allen

Formulas for the solution of the generalized Pascal triangle, by Prof. H. P. Rogers

Use of mathematics in high accuracy surveying, by Prof. Marvin May
Methods of research in modern physics, by Prof. Victor Regener.

The following officers were elected for 1949: President, Gregory Durand; Vice-President, Ross Schmidt; Secretary, Philip Barnhart; Treasurer, Prof. H. P. Rogers; Faculty Advisor, Prof. M. S. Hendrickson.

Mathematics Club, Immaculate Heart College

The following papers were presented by student members of the club:
Non-Euclidean geometry, by Sue Toolan
Trigonometry without angles, by Dolores Barenberg
The abacus, by Patsy Lang
Unsolved and unsolvable problems, by Marie Kreuper
Mathematics of installment buying, by Mary Claire Dominguez
History and transcendence of Pi, by Sue Toolan
Prime numbers, by Phyllis Beerling
Methods of solving the cubic equation and their histories, by Marion Synder.

Following each paper a period was devoted to mathematical recreations after which refreshments were served.

Officers for 1947–48 were: President, Joan Pfisterer; Vice-President, Mary Claire Dominguez; Secretary-Treasurer, Patsy Lang.

Kappa Mu Epsilon, Southwest Missouri State College

The *Missouri Alpha* Chapter of *Kappa Mu Epsilon* reports the largest number of initiates in its history. Student discussions were given on:
Golden section, by James Check
Solution and generalization of Problem 3, appearing in Fall issue, 1947, of *The Pentagon*, by Robert Hogan.

Officers for 1947–48 are: President, Philip Sneed; Vice-President, Dorothy House; Secretary, Shirley Mullins; Treasurer, Richard Kay; Corresponding Secretary, Prof. Carl Fronabarger.

Pi Mu Epsilon, University of Alabama

In addition to a Christmas party and spring picnic, five meetings were held at which the following papers were presented:
The Bailey recursion formula for roots of numbers, by Prof. C. L. Seebeck, Jr.
The theory of relativity, by Prof. Eric Rodgers
Generalized trigonometries, by Prof. M. O. Gonzales
Life of Descartes, by Miss Betty Murphree
The Bernoulli family, by Robert Whithurst.

The chapter started the year with 37 members and initiated 44 new members during the year.

The officers for the year 1948–49 are: Director, Robert Whitehurst; Vice-director, Ann Lutz; Treasurer, Ferdinand Mitchell; Secretary, Haskell Cohen; Publicity chairman, Ella Jones; Librarian, J. D. Mancill; Social chairman, Charlotte Evans.

Mathematics Club, Swarthmore College

The following topics were discussed during the spring semester:

A topic in kinematics, by Dr. H. W. Brinkmann

Differential calculator, by Mr. Garrahan

Vectors and matrices, by William Lichten

Tensor analysis, by Louis Howard

Conditionally convergent series, by Dr. I. J. Schoenberg

Graphical calculations, by Prof. A. Dresden.

Other activities included participation in William Lowell Putnam Mathematics contest and in problem contests.

New officers are: President, William Lichten; Vice-President, Robert Norman; Secretary, Elizabeth Urey; Treasurer, Daniel Beshers.

Mathematics Club, Harvard University

The following papers were presented before the Harvard Mathematics Club during 1947–48:

An introduction to Hilbert space, by L. J. Burton

The fundamental group of an algebraic curve, by Prof. Oscar Zariski

Transfinite numbers, by R. B. Dawson, Jr.

Topology, by Dr. Paul Olum

Aesthetic measure, by R. J. Herman

The generalized functions of Laurent Schwartz, by Prof. G. W. Mackey

Cofton's formula, by Prof. Herbert Federer, Brown University

Godel's undecidability theorem, by Prof. L. H. Loomis

The geometry of radio propagation, by W. T. Fishback

A problem in number theory, by Dr. L. I. Schoenfeld

Elliptic modular functions, by Harvey Cohn

The short-cut problem, by Chandler Davis—published in this MONTHLY, March, 1948.

First and second awards of the Rogers prize, given annually for the best papers contributed by students, went to Messrs. Cohn and Davis, respectively.

Officers for 1948–49 are: President, W. T. Fishback; Vice-president, John Wermer; Secretary, W. F. Stinespring; Treasurer, J. J. Newman; and Advisor, A. M. Gleason.

Pi Mu Epsilon, University of Delaware

The *Delaware Alpha* Chapter of *Pi Mu Epsilon* reports a symposium on the *Mathematical aspects of heat transfer*, conducted by Professors R. L. Pigford, G. M. Dusinberre and H. E. Goheen.

Officers for 1948–49 are: Director, G. Cuthbert Webber; President, A. Carl Nelson; Secretary, Patricia Spraberry; Treasurer, Ralph Jones; Program Chairman, Gilbert Kaskey; Social Chairman, Harry Smith.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

RESEARCH FELLOWSHIPS OF THE NATIONAL BUREAU OF STANDARDS

The Institute for Numerical Analysis of the National Bureau of Standards, located at the University of California, Los Angeles, offers a number of research fellowships during the summer of 1949, and the academic year 1949–50, to qualified graduate students in mathematics and mathematical physics. Fellows must be enrolled in an accredited college or university. Research work performed at the Institute may be applied toward a thesis for an advanced academic degree.

Fellows will work at the Institute and will be expected to perform mathematical research aimed at methods for advancing the applications of high speed automatic digital computing machinery. Individual work schedules may be arranged. Stipends will be based on full-time annual salaries of $2,294 for master's degree candidates, and $3,727 for doctoral candidates.

Inquiries and requests for application forms should be addressed to the Chief, Institute for Numerical Analysis, 405 Hilgard Avenue, Los Angeles 24, California.

CONFERENCE ON THE TEACHING OF MATHEMATICS

The annual Conference on the Teaching of Mathematics will be held at Illinois State Normal University, Normal, April 23, 1949. This conference is sponsored by the Department of Mathematics of the University.

There will be discussion groups centered around the problems of teaching mathematics in elementary and secondary schools. Particular attention will be given to participation in the Illinois secondary school curriculum program sponsored by the Office of the Superintendent of Public Instruction.

The speaker for the elementary session will be Dr. H. Van Engen, head of the Department of Mathematics, Iowa State Teachers College, Cedar Falls. Speaker for the secondary session will be Dr. H. P. Fawcett, chairman of the Department of Education, Ohio State University, Columbus.

INSTITUTE FOR TEACHERS OF MATHEMATICS

An Institute for Teachers of Mathematics, sponsored by the Association of Teachers of Mathematics in New England, will be held at Wellesley College, Wellesley, Massachusetts, on August 23–30, 1949. The program will include lectures on the latest developments in pure mathematics and on the applications of mathematics by mathematicians from college faculties, business, research, government agencies and industry. There will be discussion groups on methods of teaching; also, an organized program of trips and other recreations.

For details write to: R. F. Ward, Director of Mathematics, Brookline High

School, Brookline, Massachusetts; or, H. W. Syer, School of Education, Boston University, Boston, Massachusetts.

RESOLUTIONS OF THE NATIONAL COUNCIL OF THE TEACHERS OF MATHEMATICS

At the ninth Christmas Conference of the National Council of Teachers of Mathematics, which was held in Columbus, Ohio, on December 29–30, 1948, Discussion Group IV considered the question: "How can we provide better coordination between our high school and college mathematics programs?"

The resolutions adopted by this group should be of interest to all college teachers of mathematics. This group recommends: (1) the publication of a guidance pamphlet on the college level at the earliest possible moment by the National Council of Teachers of Mathematics and the Mathematical Association of America; (2) the promotion of contests similar to those sponsored by the Delta chapter of Pi Mu Epsilon at Washington Square College of New York University; (3) the appointment of a joint committee to study the problems of testing, guiding, and sectioning entering freshmen and to make recommendations to the Mathematical Association of America and the National Council of Teachers of Mathematics; (4) the appointment of a joint committee to make detailed recommendations concerning the philosophy, content, and extent of courses which are to be used to educate prospective teachers of mathematics.

To implement these recommendations the group suggests that a Workshop be instituted to be financed as follows: (**A**) The Mathematical Association of America and the National Council of Teachers of Mathematics each to contribute \$500; (**B**) a number of teachers colleges, colleges, and universities each to contribute the time of one mathematics instructor for a period of 2–3 weeks, and \$100 towards the expense of the Workshop.

HEAT TRANSFER AND FLUID MECHANICS INSTITUTE

The second Annual Meeting of the Heat Transfer and Fluid Mechanics Institute will be held June 22–24, 1949 at the University of California, Berkeley. The conference will be presented by California engineering colleges and engineering and scientific societies. For further information, write: Department of Institutes, University of California Extension, Berkeley, California.

SUMMER COURSES

The following institutions announce advanced courses in mathematics for the summer of 1949:

Cornell University. July 5 to August 13: Dr. Yood, higher analytic geometry.

Teachers College, Columbia University. July 5 to August 12: Professor Bradley, field work in mathematics, applications of mathematics; Professor Clark, teaching arithmetic in the elementary school; Professor Fehr, professionalized subject matter in advanced secondary school mathematics, current problems in teaching secondary school mathematics; Dr. Lazar, logic for teachers of mathematics, diagnostic and remedial procedures in arithmetic; Mr. Mirick, teaching

algebra in secondary schools; Professor Shuster, business mathematics, teaching geometry in secondary schools. In addition, beginning July 7, there will be special lectures and discussions pertaining to the reorganization and teaching of mathematics.

University of Buffalo. May 31 to July 2: Professor Pound, calculus of variations. July 5 to August 13: Professor Gehman, foundations of mathematics; Professor Montague, rings and ideals.

University of Detroit. June 20 to July 29: Professor Mehlenbacher, foundations and point sets; Professor McCarthy, solid analytic geometry; Professor Smith, differential equations, theory of matrices; Reverend Hausmann, theory of functions of a real variable.

University of Minnesota. (College of Science, Literature and the Arts). June 13 to July 22: Professor Carlson, solid analytic geometry, projective geometry; Professor Graves, foundations of calculus, Fourier series and orthogonal functions; Professor Hatfield, theory of equations, advanced calculus I; Professor Loud, differential equations, Laplace transforms. July 25 to August 27: Professor Cameron, probability, seminar in integration in function space; Professor Gelbaum, special functions, topics in topology; Professor Gibbens, advanced calculus II; Professor Nering, Fourier series and orthogonal functions II, non-euclidean geometry; Professor Hatfield, intermediate calculus. (Institute of Technology) June 13 to July 22: Professor Koehler, advanced calculus; Professor Warschawski, vector analysis, mathematical theory of flow. July 25 to August 27: Professor Polansky, advanced calculus; Professor Munro, vectors and dyadics.

University of North Carolina. June 9 to July 19. Professor Whyburn, foundations of geometry; Professor Mackie, theory of equations; Professor Linker, differential equations; Professor Cameron, introduction to modern algebra; advanced calculus; Professor Winsor, college geometry. July 20 to August 27: Professor Mackie, theory of equations; advanced calculus; Professor Lasley, analytic projective geometry; Professor Brauer, elementary theory of numbers; Professor Puckett, general topology.

The Institute of Statistics of The University of North Carolina announces a statistics summer session. Intensive statistical instruction will be offered for the benefit of (1) students working toward a degree in applied or theoretical statistics, (2) those preparing to teach statistics or to develop statistical theory, (3) statistical consultants in various fields, and (4) research scholars in other sciences who want a practical working knowledge of statistical theory. The instructional staff consists of the following professors: G. W. Snedecor, for fifteen years Director of the Statistical Laboratory at Iowa State College; D. J. Finney, Lecturer in the Design and Analysis of Scientific Experiment, University of Oxford, England; J. Wolfowitz, Associate Professor, Department of Mathematical Statistics, Columbia University; and three members of the staff of the Institute of Statistics, R. C. Bose, Herbert Robbins, and Gertrude M. Cox. An

announcement of the statistics summer session may be secured by writing to Director, Institute of Statistics, The University of North Carolina, Box 168, Chapel Hill, North Carolina.

University of Oklahoma. July 10 to August 10: Professor Court, college geometry; Professor Brixey, mathematical statistics; Professor Hassler, ordinary differential equations, fundamental concepts and teaching methods; Dr. Huff, ordinary and partial differential equations; Mr. LaFon, advanced calculus; Dr. Grau, theory of groups; Professor Springer, partial differential equations; Professor Goffman, integral equations.

University of Tennessee. June 13 to August 26: theory of equations; differential equations and advanced calculus for engineers; elementary theory of numbers; introduction to symbolic logic; ordinary differential equations; higher algebra; foundations of analysis; length, area, and measure.

West Virginia University. June 3 to July 14: Professor Reynolds, combinatorial topology; Professor Vehse, advanced calculus and Fourier series and partial differential equations; Professor Peters, theory of equations. July 16 to August 25: Professor Stewart, advanced calculus and higher plane curves; Professor Vest, operational methods in partial differential equations; Professor Cunningham, modern geometry.

PERSONAL ITEMS

Brooklyn College announces: Associate Professor Edward Fleisher has been promoted to a professorship; Assistant Professor Samuel Borofsky has been promoted to an associate professorship; Dr. Jennie P. Kormes has been promoted to an assistant professorship; Mr. J. B. Secrist, Jr. has been appointed Instructor.

Brown University reports: Professor Lamberto Cesari of the University of Bologna spoke on "Area and Representations of Surfaces" at the Mathematics Colloquium on January 21; his lecture was designated as a Tamarkin Memorial Lecture.

Teachers College, Columbia University, reports: Professor H. F. Fehr has been promoted to the position of Head of the Department of the Teaching of Mathematics, effective July 1, 1949; Dr. A. D. Bradley of Hunter College has been appointed to a part-time position for the Summer Session of 1949; Mr. Paul Clifford of State Teachers College, Montclair, New Jersey, has been appointed to a part-time position for the Winter Session, 1949–50.

Dr. F. L. Alt, who has been Deputy Chief of the Computing Laboratory of the Ballistic Research Laboratories, Aberdeen Proving Ground, has been appointed Assistant and Acting Chief of the Computation Laboratory of the National Bureau of Standards.

Associate Professor Holmes Boynton of Northern Michigan College of Education has been appointed to the position of Professor and Head of the Department of Mathematics.

Dr. L. A. Colquitt has been appointed to an assistant professorship at Texas Christian University.

Professor H. S. M. Coxeter of the University of Toronto has been named Visiting Professor at Barnard College, Columbia University, for the Spring Session of 1949.

Mr. B. K. Dickerson has been appointed to an instructorship at the University of the South.

Professor D. D. Kosambi of Tata Institute for Fundamental Research, Bombay, India, has received an appointment as Visiting Professor at the University of Chicago for the Winter Quarter.

Professor L. S. Laws of the Institute of Technology, University of Minnesota, is an sabbatical furlough and is studying at Michigan State College.

Professor P. H. Linehan of the College of the City of New York has retired with the title of Professor Emeritus.

Mr. F. F. Otis has been appointed Assistant Professor and Chairman of the Mathematics Department of the Sault Sainte Marie Branch of Michigan College of Mining and Technology.

Professor R. M. Pinkerton, formerly acting head of the Department of Mathematics of Texas Agricultural & Mechanical College, has accepted an appointment as aeronautical research scientist with the National Advisory Committee for Aeronautics at Langley Field, Virginia.

Mr. Arthur Porges of Occidental College has been appointed to an assistant professorship at De Paul University.

Professor T. G. Room of the University of Sydney has been appointed Visiting Professor at the University of Tennessee during the Winter term.

Mr. J. S. Mikesh, formerly chairman of the Department of Mathematics at the Lawrenceville School, died on January 29, 1949.

Professor Maximilian Philip, retired chairman of the Department of Mathematics of the College of the City of New York, died on January 17, 1949 at the age of seventy-one.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

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REPORT OF THE TREASURER FOR THE YEAR 1948

The following is a summary of the report of Professor H. M. Gehman as Treasurer of the Association for the year 1948. This report has been approved by the Finance Committee and accepted by vote of the Board of Governors. Any member of the Association who wishes the complete report of the Treasurer may obtain it by writing to the office of the Association.

I. Total Funds of the Association on January 1, 1948

First National Bank, Ithaca.....	$ 3,609.48	Current Fund.................	$ 3,609.48
Ithaca Savings Bank...........	1,045.82	Savings Account...............	1,045.82
Cleveland Trust Company......	199.30	Carus Fund...................	11,058.30
Securities......................	0,635.00	Chace Fund..................	9,059.59
		Houck Fund..................	9,300.12
		Chauvenet Fund..............	659.79
		Life Membership Fund.........	598.60
		General Fund.................	30,157.90
	$65,489.60		$65,489.60

II. Current Fund

Balance, January 1, 1948.......	$ 3,609.48	Monthly	
From Savings Account.........	1,045.82	Publication.................	$10,127.14
Dues.........................	17,488.00	Reprints (net)..............	304.65
Initiation fees.................	942.00	Editor's office...............	520.98
Subscriptions.................	4,075.38	Register....................	1,164.17
Sales of back numbers (net).....	695.16	Secretary-Treasurer's Office	
Advertisements................	2,049.00	Clerical help................	4,190.50
Contributions for publication of		Furniture...................	404.37
the Monthly...............	200.00	Office expenses..............	1,424.79
Income: Hardy Fund..........	120.00	Bank fee...................	100.00
Sale of exchange periodicals.....	48.80	Auditing fee................	100.00
Sale of Archibald's Outline		Board of Governors............	1,131.78
(Fifth Edition)..............	55.55	Meetings....................	422.71
Interest on General Fund.......	1,113.84	Committees..................	50.00
		Representatives..............	160.81
		Subventions..................	550.00
		Transfer to General Fund......	3,485.06
		Balance, December 31, 1948....	7,306.07

III. Carus Fund

Balance, January 1, 1948.......	$11,058.30	Honorarium, 8th Monograph....	300.00
Sale of Monographs...........	1,820.67	Printing, 8th Monograph.......	3,437.69
Interest......................	400.99	Reprinting, 6th Monograph.....	1,326.12
		Decrease in value of securities...	134.89
		Balance, December 31, 1948....	8,081.26

IV. Chace Fund

Balance, January 1, 1948.......	$ 9,059.59	Honorarium, 1st Slaught Paper..	86.00
Sale of Papyrus...............	185.00	Decrease in value of securities....	112.42
Sale of Slaught Papers.........	65.00	Balance, December 31, 1948....	9,776.25
Refund Eudemus appropriation.	330.93		
Interest......................	334.15		

V. Houck Fund

Balance, January 1, 1948.......	9,300.12	Decrease in value of securities...	119.91
Interest......................	356.43	Balance, December 31, 1948....	9,536.64

VI. Chauvenet Fund

Balance, January 1, 1948	$ 659.79	Award of Chauvenet Prize	50.00
Interest	22.28	Decrease in value of securities	7.49
		Balance, December 31, 1948	624.58

VII. General Fund

Balance, January 1, 1948	$30,157.90	Decrease in value of securities	$ 374.73
From Life Membership Fund	598.60	Balance, December 31, 1948	33,866.83
From Current Fund	3,485.06		

VIII. Total Funds of the Association on December 31, 1948

Current Fund	$ 7,306.07	M & T Trust Company, Buffalo, Checking Account	$ 7,306.07
Carus Fund	8,081.26	Securities	61,885.56
Chace Fund	9,776.25		
Houck Fund	9,536.64		
Chauvenet Fund	624.58		
General Fund	33,866.83		
	$69,191.63		$69,191.63

CALENDAR OF FUTURE MEETINGS

Joint Meeting with American Society for Engineering Education, Troy, New York, June 20–21, 1949.

Thirty-first Summer Meeting, Boulder, Colorado, August 29–30, 1949.

Thirty-third Annual Meeting, New York City, December 30, 1949.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

Allegheny Mountain, West Virginia University, Morgantown, May 7, 1949
Illinois, Bradley University, Peoria, May 13–14, 1949
Indiana, University of Notre Dame, May 7, 1949
Iowa, Drake University, Des Moines, April 15–16, 1949
Kansas
Kentucky, Centre College, Danville, May 14, 1949
Louisiana-Mississippi
Maryland—District of Columbia—Virginia, University of Virginia, Charlottesville, May 14, 1949
Metropolitan New York
Michigan
Minnesota, Gustavus Adolphus College, St. Peter, May 7, 1949
Missouri
Nebraska, Lincoln, May 7, 1949
Northern California
Ohio
Oklahoma
Pacific Northwest
Philadelphia, Haverford College, November 26, 1949
Rocky Mountain, Colorado School of Mines, Golden, April 22–23, 1949
Southeastern
Southern California
Southwestern
Texas
Upper New York State, University of Buffalo, April 30, 1949
Wisconsin, Lawrence College, Appleton, May 14, 1949

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CONTENTS

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The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

CARROLL V. NEWSOM, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. V. NEWSOM, State Education Building, Albany 1, N. Y.

ADVERTISING CORRESPONDENCE should be addressed to J. F. RANDOLPH, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Secretary-Treasurer, H. M. GEHMAN, University of Buffalo, Buffalo 14, N. Y. Change of address must reach the Secretary-Treasurer about six weeks before the change can become effective.

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MODERN OPERATIONAL CALCULUS FOR UNDERGRADUATES*

J. R. BRITTON, University of Michigan**

1. Introduction. For the past several years I have had the pleasure of teaching a course called Modern Operational Calculus at the University of Colorado, and, this year, Modern Operational Mathematics at the University of Michigan. It is an honor to have the privilege of describing this work to the members and friends of the Association.

Most of my remarks will be restricted to the work as it is given at the University of Colorado. There the course is set up as a two-quarter sequence, three hours credit per quarter. I have insisted on a short course in ordinary differential equations along with some background in physics and elementary mechanics as minimum prerequisites. My classes have usually run about twenty-five students, of whom most met only the minimum requirement in mathematics.

I feel that not enough publicity has been given to the worthwhile body of material in operational calculus and related fields that can be presented to students at the level described. Naturally, some topics from advanced calculus will have to be taken as part of the work. However, these topics are fairly elementary and are, moreover, given a motivation that is lacking in the usual advanced calculus course.

2. Introducing operational calculus. Perhaps one of the simplest ways of introducing the operational calculus based on the Laplace transformation is by means of the following initial value problem:

Determine $Y(t)$ if $Y''(t)+Y(t)=1$, $Y(0)=0$, and $Y'(0)=2$.

Drawing upon our experience with elementary differential equations, we can employ the "classical" method of first writing down the complete solution:

$$Y(t) = A \cos t + B \sin t + 1.$$

Next, it is easy to find that the values $A=-1$, $B=2$ will satisfy the initial conditions, so that the required determination is

$$Y(t) = 1 - \cos t + 2 \sin t.$$

It is worthwhile to write out in complete detail the work of finding this solution so that a comparison with the operational method will be available.

We now return to the original differential equation, multiply both members by $e^{-st}dt$ and integrate from zero to "infinity."† The variable s is a parameter

* This paper is based upon an address given by the author at the thirty-second annual meeting of the Association, Columbus Ohio, December 31, 1948.

** On leave, University of Colorado.

† Professor R. V. Churchill has suggested that the plausibility of this procedure can be made evident by considering the problem of finding an integral operator that will "annihilate" the nth derivative.

independent of t. Thus,

$$\int_0^\infty e^{-st}Y''(t)dt + \int_0^\infty e^{-st}Y(t)dt = \int_0^\infty e^{-st}dt.$$

A formal integration by parts applied to the first term on the left side, with the assumption that $e^{-st}Y(t)$ and $e^{-st}Y'(t)$ both approach zero as t increases indefinitely, gives

$$s^2\int_0^\infty e^{-st}Y(t)dt - 2 + \int_0^\infty e^{-st}Y(t)dt = \frac{1}{s}\cdot$$

Thus, we find

$$\int_0^\infty e^{-st}Y(t)dt = \frac{1}{s(s^2+1)} + \frac{2}{s^2+1},$$

or

$$\int_0^\infty e^{-st}Y(t)dt = \frac{1}{s} - \frac{s}{s^2+1} + \frac{2}{s^2+1}\cdot$$

Notice that the initial values $Y(0)$ and $Y'(0)$ have been used in integrating the first term. We now have an integral equation for the determination of the function $Y(t)$.

For the present purpose, we use the solution obtained by the classical method as a guide. We can identify each of the terms on the right-hand side as follows:

$$\frac{1}{s} = \int_0^\infty e^{-st}\cdot 1\, dt;$$

$$\frac{s}{s^2+1} = \int_0^\infty e^{-st}\cos t\, dt;$$

$$\frac{2}{s^2+1} = 2\int_0^\infty e^{-st}\sin t\, dt.$$

We therefore write

$$\int_0^\infty e^{-st}Y(t)dt = \int_0^\infty e^{-st}(1 - \cos t + 2\sin t)dt,$$

which suggests that

$$Y(t) = 1 - \cos t + 2\sin t.$$

The procedure which we have initiated in this example already introduces a number of important questions and the answers to these equations will see us well on the way into the methods of operational calculus. At the elementary level

from which we are starting, it is wise to begin with a discussion of "infinite" or improper integrals of the type

$$\int_0^\infty e^{-st}F(t)dt.$$

We say that the integral converges whenever

$$\lim_{T\to\infty}\int_0^T e^{-st}F(t)dt$$

exists, and we take the limit as the value of the integral. It is then not difficult to prove

THEOREM 1. *Let $F(t)$ be sectionally continuous in every finite positive interval and let $F(t)$ be such that, as $t\to\infty$,*

$$|F(t)| \leqq Me^{at},$$

where M is a positive constant and a is a constant. (We shall say that such functions are sectionally continuous and of exponential order.) *Then*

$$\int_0^\infty e^{-st}F(t)dt$$

converges for $s>a$.

The discussion of this theorem will necessitate a definition of sectional continuity which is of itself a useful concept in applied mathematics. Incidentally it appears from the proof that, for the type of function described in the theorem, the integral has a value numerically not greater than $M/(s-a)$ for $s>a$. Thus, as $s\to\infty$, the value of the integral approaches zero.

3. Introducing the Laplace transform. It is convenient next to introduce a more efficient notation and terminology.

The transformation of $F(t)$ by multiplying it by $e^{-st}dt$ and integrating from zero to infinity is called the *Laplace transformation of* $F(t)$. The result of making this transformation is clearly a function of the new variable s. This function of s is called the *Laplace transform* of $F(t)$. We shall use the notation

$$L\{F(t)\} = f(s).$$

For example, we have seen previously that

$$L\{\sin t\} = \frac{1}{s^2+1},$$

so that if

$$F(t) = \sin t, \qquad f(s) = \frac{1}{s^2+1}\cdot$$

We also speak of $F(t)$ as the *inverse Laplace transform* of $f(s)$ and write

$$L^{-1}\{f(s)\} = F(t).$$

It is necessary at this level to take without proof the theorem that the inverse transform is essentially unique.*

From the definition of the transformation, it is clear that we are dealing with a linear transformation, that is,

$$L\{AF(t) + BG(t)\} = Af(s) + Bg(s),$$

if A and B are constants. This result also holds, of course, for the inverse transformation.

With no more than the concepts thus far presented, we can prove

THEOREM 2. *Let $F(t)$ be continuous and of exponential order and let $F'(t)$ be sectionally continuous. Then*

$$L\{F'(t)\} = sf(s) - F(+0),$$

where $F(+0) = \lim_{t\to 0^+} F(t)$.

By repeated application of Theorem 2, we can derive the more general formula:

$$L\{F^{(n)}(t)\} = s^n f(s) - s^{n-1}F(+0) - s^{n-2}F'(+0) - \cdots - F^{(n-1)}(+0).$$

The students will usually be able to supply the simple sufficient conditions which correspond to those in Theorem 2.

The preceding formula enables us to write at once the transform of a linear differential equation with constant coefficients, say

$$\sum_{k=0}^{n} a_k Y^{(n-k)}(t) = F(t),$$

in the form

$$q(s)y(s) - p(s) = f(s),$$

where $q(s)$ and $p(s)$ are polynomials in s of degree n and $n-1$, respectively. Thus

$$y(s) = \frac{p(s)}{q(s)} + \frac{f(s)}{q(s)}\cdot$$

If $F(t)$ is one of the functions t^n, $n=0, 1, 2, \cdots$, e^{kt}, cosh kt, sinh kt, cos kt or sin kt, $f(s)$ can be found easily by direct evaluation of the Laplace integral or by application of the preceding results for $L\{F^{(n)}(t)\}$. In these cases, $f(s)$ is a rational fraction in s, with the numerator of lower degree than the denominator. For these simple instances, we have

* Two functions $F(t)$ and $G(t)$, which have the same transform, can differ only by a null function, that is, a function $N(t)$ such that $\int_0^T N(t)dt = 0$ for all positive T.

$$y(s) = \frac{m(s)}{n(s)},$$

where $m(s)$ and $n(s)$ are polynomials with no common factors. (Such factors may be divided out if they occur.) It remains only to find $L^{-1}\{y(s)\}$.

The handling of inverse transforms will be essentially a matter of constructing and using a table of transforms in much the same way as a table of integrals is used. Although a brief table can be constructed by means of the methods previously mentioned, the range of the table can be expanded greatly by a consideration of the question: What operation on the function $F(t)$ corresponds to a given operation on the transform $f(s)$?

Some of the more easily accessible correspondences of this type are displayed in the following self-explanatory table:

	$f(s)$	$F(t)$
1.	$f(s-a)$	$e^{at}F(t)$
2.	$f(bs)$	$\frac{1}{b}F\left(\frac{t}{b}\right)$
3.	$f^{(n)}(s)$	$(-t)^nF(t)$
4.	$f(s)g(s)$	$\int_0^t F(x)G(t-x)dx$

Items 1 and 2 involve no more than a change of variable in the Laplace integral. However, Item 3 will necessitate differentiation under the integral sign in an improper integral, and Item 4 will need a change of variable in a double integral. Both the latter topics can be included in the discussion of functions defined by infinite integrals.

The outline thus far presented covers enough material to make a good start on the type of initial-value problems encountered in electrical and mechanical vibration analysis. Many other topics suggest themselves in the course of the work, for example, solution of polynomial equations of degree higher than the second (Graeffe's method), stability problems, linear differential equations with polynomial coefficients, step functions, formal solution of boundary-value problems involving linear partial differential equations, and so on.

4. Conclusion. The application of the Laplace transformation to initial-value problems involving linear ordinary differential equations with constant coefficients serves as ample illustration of the convenience and desirability of the operational method. Boundary-value problems involving linear partial differential equations will illustrate the essential need for this method.

The references I have found most useful for a course of this kind are listed at

the end of this paper. The list is not intended to be exhaustive; a fairly extensive bibliography will be found in the third book.

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ON THE CONVERSE OF FERMAT'S THEOREM II

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1. **Introduction.** The feasibility of many investigations in the theory of numbers depends on the ability of the research worker to identify numbers as primes in a rapid and positive manner. In fact, this is the main reason for publishing lists of primes and factor tables [1]. For numbers beyond the limits of such tables one must depend on some positive test for primality. The ideal test is a positive characteristic of primes which is easily applicable. Unfortunately such criteria exist only for numbers of special form [2]. For the general number one may ask for a little less. One may ask for a criterion which is almost characteristic of primes and then tabulate those relatively few composite numbers which also satisfy this condition. According to the famous theorem of Fermat, 2^n-2 is divisible by n if n is a prime. In a matter of minutes one may decide whether or not a given number n divides 2^n-2. Unfortunately, there are also composite numbers n which divide 2^n-2. To make this a true test for primality one may list all these exceptional composite numbers or at least all interesting ones in a given range. An uninteresting composite number would be one which contains a small factor less than the limit to which one would prudently search before testing the divisibility of 2^n-2.

More than a decade ago the writer published [3] a list of all such composite numbers between 10^7 and 10^8 whose least factor exceeds 313. Soon afterwards P. Poulet [4] published a list of all composite numbers $n<10^8$ dividing 2^n-2. With either of these lists the arithmetician may determine whether an 8-digit number is prime or not in only a few minutes of computing time. This method has been so useful that the writer has often had occasion to consider the possibility of extending its range. One of the practical difficulties in the way of such an extension has been the inadequacy of existing tables of the so-called exponent of 2 modulo p on which such an extension would have to be based. The largest table of exponents published by M. Kraitchik and extending to $p<300000$ is inadequate not only in extent but also in accuracy. Thus it was apparent that some 10 or 15 years of recomputing and extending this table would have to precede any substantial extension.

This difficulty, however, has been overcome by the development of elec-

tronic computing. The writer happened to be connected with the early operation of the Army Ordnance's ENIAC [6], the first (and thus far only) all-electronic digital computer. During a holiday weekend, as a sort of test problem, the writer used the ENIAC to compute the requisite exponents of 2 modulo p. Some account of this remarkable computation is given in §2. A later comparison of these results with the table of Kraitchik revealed the latter to be seriously unreliable. A careful check of each discrepancy showed that the ENIAC was always right. A list of these errors has since been published [7]. One of the by-products of this weekend's work was the discovery of a new list of 85 factors of $2^k \pm 1$ for $k \leqq 500$. These have been given elsewhere [8] without indication of the methods used. The list of exponents furnished by the ENIAC is sufficient for the extension of the table of composite numbers n dividing $2^n - 2$ to 10^9 and beyond.

However the list presented herewith extends only from 10^8 to $2 \cdot 10^8$. The labor of producing these composite numbers, given the table of exponents of 2, is still considerable. This is especially true in the case of such composite numbers which are products of three primes. The list previously given contains no such entry. The present list contains only seven products of three primes, namely

$$113589601 = 331 \cdot 571 \cdot 601$$
$$122941981 = 337 \cdot 491 \cdot 743$$
$$139487041 = 331 \cdot 617 \cdot 683$$
$$150966901 = 337 \cdot 373 \cdot 1201$$
$$162771337 = 337 \cdot 547 \cdot 883$$
$$172028053 = 337 \cdot 457 \cdot 1117$$
$$173405233 = 397 \cdot 577 \cdot 757$$

In §4 we prove that there is an unlimited number of numbers n dividing $2^n - 2$ which are products of three primes.

The writer takes this opportunity to thank Dr. J. W. Mauchly for several helpful suggestions in setting up the ENIAC and for his share of "visiting" the ENIAC during its non-stop run. All the subsequent computing was done by Emma Lehmer.

2. Description of the ENIAC setup. The method used by the ENIAC to find the exponent of 2 modulo p differs greatly from the one used by human computers. It will be recalled that the exponent e of 2 modulo the prime p is the least value of n such that $2^n \equiv 1 \pmod{p}$, and that e is some divisor of $p-1 = ef$. The method used heretofore has been to examine as possible values of e the various divisors of $p-1$. Some of these may be eliminated wholesale by the theory of quadratic and higher residues but there may remain quite a number of divisors of $p-1$ which must be tried as possible values of e. In the majority of cases, however, the value of e is too large to be of use to our problem. In fact

it is sufficient to know that $e>2000$ in order to reject p. (For $300\ 000<p<1\ 000\ 000$ this limit was reduced to 1000 and for $10^6<p<4.5\cdot 10^6$ it was set at 300).

In the ENIAC method we try as possible values of e not the half dozen or so suitable divisors of $p-1$, but simply the natural numbers $1, 2, 3, \cdots, 2000$. At first sight this would seem to be a very crude way of looking for e. But some further considerations, in which one takes into account the incredible rapidity of the ENIAC, show that, even in the worst case, all possible values of e can be tried in less than 2.4 seconds, less time than it takes to copy down the value of p. The more sophisticated method using divisors of $p-1$ would require too much outside information via punched cards, information which would have to be prepared by hand in advance.

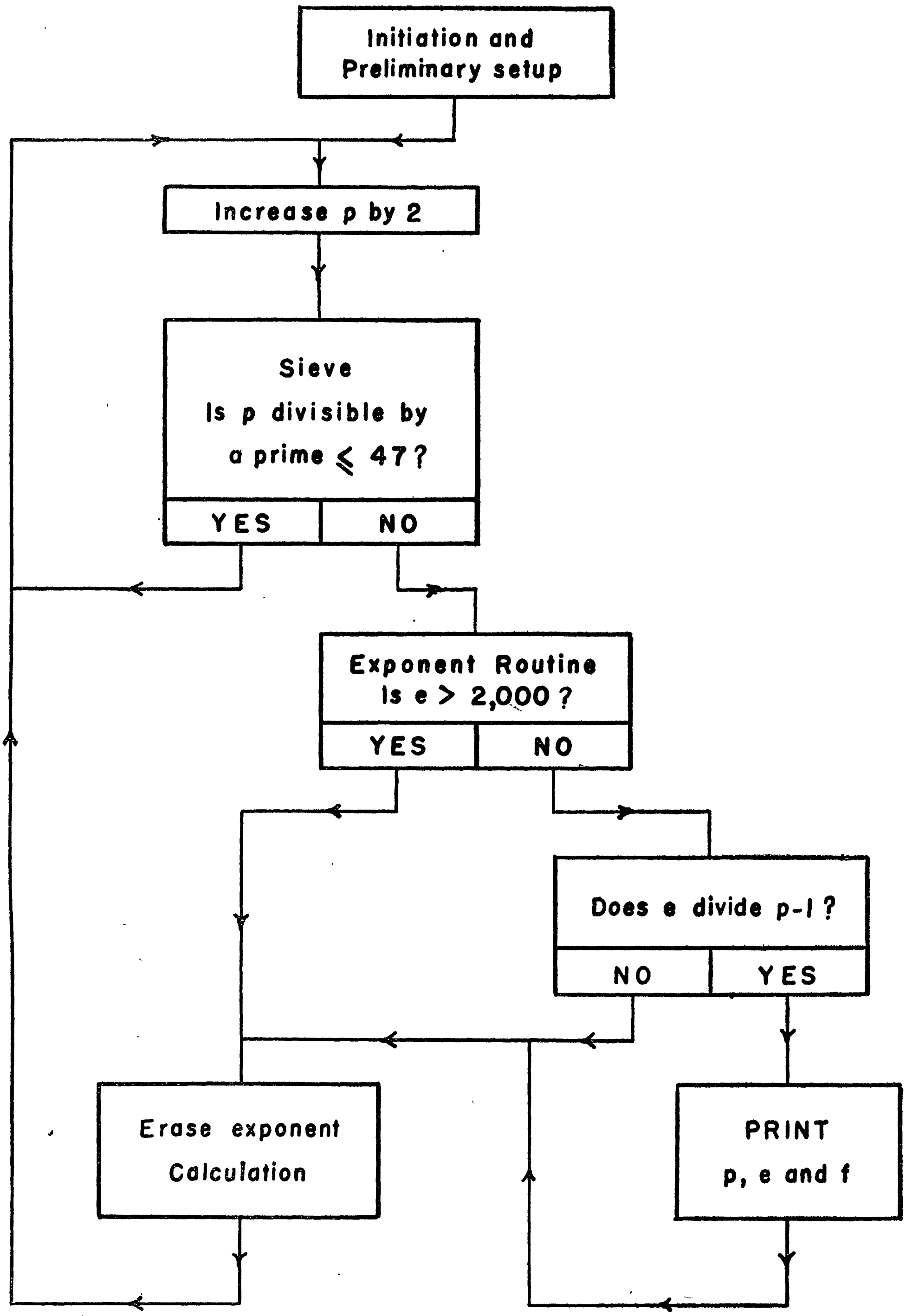
A little more detail on the actual set up may be of interest. In computing the exponent e a sequence of positive integers r_k is built up by means of the recursive definition

$$r_1 = 2, \qquad r_{k+1} = \begin{cases} 2r_k & \text{if } 2r_k < p, \\ 2r_k - p & \text{if } 2r_k > p. \end{cases}$$

Clearly r_k is nothing but the remainder on division of 2^k by p. Only in the second case, $2\ r_k>p$ is there a chance that $r_{k+1}=1$. Each time, then, that r_{k+1} is found by the formula $2r_k-p$, the ENIAC is programmed to ask itself: Is $r_{k+1}-2$ negative? If the reply is yes, then $r_{k+1}=1$, and $e=k+1$. If, on the other hand, the reply is no, then the ENIAC asks: Is $k+1=2001$? If not, then r_{k+2} is next computed. If $k+1=2001$, the ENIAC is instructed to give up the search for e and try the next value of p.

The "next value of p" presents an interesting problem to the ENIAC. One of the requirements of the problem, dictated by the circumstances under which the problem was run, was that the ENIAC works for hours without attention. This alone prevented the introduction into the ENIAC of a list of primes p via punched cards. There were at least three other good reasons for not doing this. This meant that the ENIAC should somehow compute its own values of p. To this effect a "sieve" was set up which screened out all numbers having a prime factor $\leqq 47$. Thus about 86 percent of the integers were eliminated. For the range $100000<p<300000$, for example, only 27741 values of p were used; of these, 16405 are actually primes and 11336 (or 41 percent) are composite. However, a large percentage of these cases have small exponents of 2. To prevent the punching of a card in almost all these cases, the number p and its exponent e were required to pass a further test; namely, $p-1$ must be divisible be e. This further requirement is so strict that, for example, only 25 of the 11336 composite numbers mentioned above succeeded in meeting it. The output of the ENIAC was then a set of cards each punched with a value of p (having an exponent $e\leqq 2000$) and the corresponding values of e and f. These were "tabulated" and the composite p's eliminated by comparison with Lehmer's list of primes.

A block diagram of the ENIAC set up is shown on page 303.


Initiation and
Preliminary setup
Increase p by 2
Sieve
Is p divisible by
a prime ≤ 47?
YES
NO
Exponent Routine
Is e > 2,000 ?
YES
NO
Does e divide p-1 ?
NO
YES
Erase exponent
Calculation
PRINT
p, e and f

3. Description of the table. The subjoined table* gives all composite numbers n between 10^8 and $2\cdot 10^8$ which divide 2^n-2 and whose least prime factor exceeds 313. With each n is given its least prime factor p. (In case n is a product of three primes the two smaller factors of n are given.) To examine n for primality one first looks to see if n is in the table. If so, its least factor is given. If not, one examines n for factors not exceeding 313. If this search in unsuccessful, n is prime or composite according as n divides 2^n-2 or not. The practical method of performing these operations is explained and illustrated in [3]. There also the reader will find a discussion of the methods of construction of these tables. These depend on the number of prime factors of n. It is for the case of $n=pq$ that we need extensive lists of the large primes having comparatively small exponents of 2. As might be expected, errors in Kraitchik's table of exponents [5] introduced corresponding errors in the original lists [3] and [4]. Some of the other errors listed below were communicated by Poulet. The list in [3] has since been recomputed so that the following errata should be complete.

Insert	44070841	2113
Delete	68462551	5851
Insert	70541099	4643
Insert	71079661	3187
Insert	74705401	3529
Insert	74874869	3533
Delete	76839733	1019
Insert	92438581	3331
Insert	96135601	881

The present list has been compared with a manuscript of Poulet for products of two primes below $1.5\cdot 10^8$. There were very few discrepancies. The second half of the present list has been computed twice.

4. Products of three primes. In [3] we showed that infinitely many numbers n dividing 2^n-2 are products of two primes. In this section we show that this is true also of products of three primes. First, it is convenient to establish three lemmas.

LEMMA 1. *Let $\phi(N)$ denote the number of numbers $\leqq N$ and prime to N. Then $\phi(N)>8\log N$ for all sufficiently large N.*

The function $\phi(N)$ is, of course, of order greater than $\log N$. In fact ([9]) there exists a constant A such that $\phi(N)\log\log N>An$. Hence $\phi(N)/\log N$ tends to infinity with N and therefore exceeds 8 for all sufficiently large N.

LEMMA 2. *Let N be an integer divisible by a prime of the form $4x+1$. The number of numbers $<N/4$ and prime to N is $\phi(N)/4$.*

This lemma is a special case of a theorem of van der Corput and Kluyver

* This list contains 329 entries; the previous list has 526.

[10]. Their conclusions for a general n are not quite correct. For our purposes, however, Lemma 2 is sufficient. It may be proved easily as follows. Let $\psi(m)$ denote the least positive residue of m modulo 4 so that

$$\psi(m) = m - 4[m/4].$$

Let $\phi'(N)$ denote the number of numbers $<N/4$ and prime to N. Then by a theorem of Legendre [11]

$$\phi(N) = \sum_{\delta \mid N} N\mu(\delta)/\delta$$

and

$$\phi'(N) = \sum_{\delta \mid N} [N/(4\delta)]\mu(\delta)$$

where μ is the Möbius' function and δ ranges over the divisors of N. Hence the function

$$F(N) = \sum \psi(N/\delta)\mu(\delta) = \phi(N) - 4\phi'(N).$$

To prove the lemma it is sufficient to show that in case N is divisible by a prime $\omega = 4x+1$, then $F(N)=0$. The divisors of N are of three types: (a) Those divisible by ω^2, (b) those divisible by only the first power of ω, and (c) those not divisible by ω. The divisors in (b) and (c) are in one to one correspondence, the divisors d of type (c) corresponding to ωd of type (b). Since ω is of the form $4x+1$,

$$\psi(N/d) = \psi(N/(\omega d)),$$

the arguments being congruent modulo 4. However

$$\mu(\omega d) = \mu(\omega)\mu(d) = -\mu(d).$$

Hence corresponding divisors of types (b) and (c) together contribute nothing to $F(N)$. Neither do divisors of type (a) since for them $\mu(\delta)=0$. Hence $F(N)=0$.

LEMMA 3. *Let*

$$Q_N(x) = x^{\phi} + \cdots + 1 \qquad (\phi = \phi(N))$$

denote the polynomial whose roots are the primitive Nth roots of unity. If N is an odd multiple of a prime $\omega = 4x+1$ *and sufficiently large, then*

$$Q_N(-2) > N(2N+1).$$

Proof. Let ν range over the numbers $\leqq N$ and prime to N so that

$$Q_N(x) = \prod_{\nu} (x - \exp(2\pi i\nu/N)).$$

Since ν and $N-\nu$ are distinct and both prime to N, we have

$$Q_N(-2) = \prod_{\nu<N/2} (5 + 4 \cos (2\pi\nu/N)) > \prod_{\nu<N/4} 5 = 5^{\phi/4},$$

by Lemma 2. By Lemma 1, if N is sufficiently large,

$$Q_N(-2) > 5^{2 \log N} = N^{\log 25} > N^3 > N(2N + 1).$$

We are now in a position to prove the following:

THEOREM. *The congruence*

$$2^n \equiv 2 (\text{mod } n) \tag{1}$$

has an infinity of solutions n that are products of three primes.

Proof. Let $p=40m+11=2N+1$ be a prime for which

$$N = (p - 1)/2 = 5(4m + 1),$$

and which is sufficiently large for Lemma 3 to hold, with $\omega=5$. Let q be any primitive prime factor of 2^N-1. This is a divisor which divides no number of the form 2^s-1 for $0<s<N$. That such a q exists for every $N>6$ follows from a theorem of Bang [**12**]. Clearly the exponent of 2 modulo q is N. Since this odd exponent divides $q-1$, we have $q=2jN+1$. Now q divides

$$2^N - 1 = 2\cdot(2^{10m+2})^2 - 1.$$

Hence 2 is a quadratic residue of q and therefore $q\equiv \pm 1$ (mod 8). But $p\equiv 3$ (mod 8). Hence $p\neq q$. We now separate two cases.

Case I. $2^{2N}-1$ has a primitive prime factor different from p. Let this factor be r. Then the exponent of 2 modulo r is $2N$ so that $q\neq r$ and $r=2kN+1$. Now take $pqr=n$. We may show that (1) holds for n as follows. The exponent of 2 modulo p being some divisor of $2N$, and since p, q and r are relatively prime in pairs it follows that the exponent of 2 modulo n is precisely $2N$. But

$$n - 1 = pqr - 1 = (2N + 1)(2jN + 1)(2kN + 1) - 1 \equiv 0 \ (\text{mod } 2N).$$

Hence $n-1$ is divisible by the exponent of 2 modulo n. Therefore (1) holds in Case I.

Case II. $2^{2N}-1$ has no primitive factor different from p. It is known ([**13**]) that $Q_N(-2)$ contains only the primitive factors of $2^{2N}-1$ together with a possible extrinsic factor E dividing N. In Case II therefore

$$Q_N(-2) = Ep^\alpha \qquad (\alpha \geqq 0). \tag{2}$$

We show that if p is sufficiently large, $\alpha\geqq 2$. In fact, if we suppose the contrary we would have

$$Q_N(-2) \leqq Ep \leqq N(2N + 1),$$

a result which would then be in contradiction with Lemma 3. Hence we must have $\alpha\geqq 2$. Since $Q_N(-2)$ divides $2^{2N}-1$, it follows from (2) that

$$2^{2N} \equiv 1 \pmod{p^2}.$$

That is, the exponent of 2 modulo p^2 is some divisor of $2N$. Now take $n=p^2q$. Then the exponent of 2 modulo n will be either N or $2N$. But

$$n - 1 = p^2q - 1 \equiv (2N+1)^2(2jN+1) - 1 \equiv 0 \pmod{2N}.$$

Hence (1) holds for $n=p^2q$ in Case II. This completes the proof of the theorem.

TABLE OF COMPOSITE SOLUTIONS n OF FERMAT'S CONGRUENCE $2^n \equiv 2 \pmod{n}$ AND THEIR SMALLEST PRIME FACTOR p

n	p	n	p	n	p
100463443	7577	312773	3541	558011	6449
618933	4729	413333	6067	940853	503
860997	9649	495083	1987	120296677	229
907047	5023	717861	1013	517021	2341
943201	5801	111202297	5273	838609	433
101152133	5807	370141	883	121062001	1201
158093	3673	654401	6101	128361	6961
218921	8713	112032001	4001	374241	6361
270251	9001	402981	3061	121472359	4409
276579	6163	828801	6133	122166307	739
954077	1597	844131	3067	396737	2857
102004421	2381	113359321	761	941981	337·491
443749	4049	589601	331·571	123330371	691
678031	3583	605201	7537	481777	3881
690677	2069	730481	433	559837	4177
690901	5851	892589	919	671671	9631
103022551	6121	114305441	6173	886003	1187
301633	7873	329881	7561	987793	709
104078857	6679	469073	3089	124071977	2089
233141	2441	701341	1229	145473	397
524421	5903	842677	2459	793521	4561
105007549	1033	115085701	1801	818601	2281
305443	2833	174681	773	125284141	4231
919633	4603	804501	5381	686241	6473
941851	1051	873801	1051	848577	2897
106485121	7297	116090081	6221	126132553	5023
622353	433	151661	7621	886447	6793
743073	1699	321617	5393	127050067	5347
107360641	2161	617289	2357	710563	9787
543333	4889	696161	2161	128027831	11161
108596953	7369	998669	1459	079409	5437
870961	2609	117246949	1597	124151	2311
109052113	4993	445987	5419	468957	2927
231229	2699	959221	2053	536561	8017
316593	3697	987841	7681	665319	2383
437751	5231	118466401	1249	987429	4637
541461	6043	119118121	2729	129205781	6563
879837	2707	204809	2383	256273	739
110135821	3967	261113	4657	461617	10177
139499	6427	378351	911	524669	2939

n	p	n	p	n	p
130513429	709	272901	6983	304001	7333
556329	2857	884393	3833	369101	601
693393	5113	147028001	7001	423377	6353
766239	1279	287141	2861	498681	9281
944133	6607	148109473	5443	162026869	5197
131023201	881	171769	4657	067441	853
567929	6221	392781	3517	690481	1861
821747	659	910653	8629	771337	337·547
132332201	2003	149069989	5821	776041	5209
338881	3469	389633	7057	163021423	479
440521	2713	150260893	7753	759753	9049
575071	2879	379693	907	164111281	1777
133216381	541	960239	4759	165061909	1123
427449	10169	966901	337·373	224321	3181
467517	9433	988753	8689	538447	3391
496221	661	151533377	8059	938653	9109
134384069	4733	589881	1249	166082309	10333
696801	6701	152255611	4363	339057	6449
767153	8209	716537	5527	406561	3041
868029	2999	153384661	2341	444181	11173
135263269	6217	393667	3917	166827943	4567
296053	2239	754873	2039	167579497	6473
308881	4397	928133	5849	692141	9157
437129	1433	154195801	7393	881121	7481
969401	9521	287451	6211	168566501	10601
136043641	997	513633	7177	169655641	3613
545067	5843	910869	997	930549	2557
661201	401	944533	7187	170782921	9241
137415821	6421	155203361	7193	856533	7547
763037	5869	840777	10193	171149749	2069
138012733	6553	156114061	3061	567481	2089
030721	4441	532799	5711	747577	6553
336661	8317	157069189	1117	823693	6529
403981	4447	368661	7243	172028053	337·457
736153	8329	405249	3137	116181	9277
828821	6803	725829	7489	272187	6563
139295701	3559	158068153	5623	436713	4153
319293	3733	192317	2713	173401621	7603
363927	5903	397247	11257	405233	397·577
487041	331·617	496911	10903	174479729	5393
710421	8761	895281	11117	638419	2203
141574219	3259	159874021	8941	175484291	4721
142525333	1459	160348189	8009	656601	1297
922413	4519	378861	4787	747457	641
143071601	1777	491329	1889	176030977	4423
106133	6907	587841	1933	571089	3617
168581	8461	672201	3121	597821	9397
145206361	8521	730389	3823	609441	7673
334821	5581	161184013	4799	977921	7681
348529	1579	216021	7331	177167233	5953
146156617	5407	289649	1873	254533	7687

n	p	n	p	n	p
349147	6659	653333	7867	913297	2213
927641	881	186183469	1181	191191933	4373
951973	12577	654241	8641	233813	5897
179083601	8761	739057	6833	648161	7993
285137	1283	846301	11161	981609	5657
820257	8779	983521	3793	192346153	3847
180497633	7757	187050529	2017	857761	6211
703451	1163	155383	8831	193330237	11353
801253	1013	667969	10211	949641	3863
181285537	1217	761241	2741	194556451	4027
542601	1801	188382487	6863	195412621	4421
647497	4493	821951	6871	475351	6991
182383111	911	985961	9721	196035001	7001
183554407	10711	189714193	1399	049701	9901
677341	4649	738361	1531	197466361	4057
788161	641	190212181	7963	198712079	9967
184411567	2593	382161	1861	982759	3527
185206757	5821	824817	8737	199674721	4261

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BIOGRAPHIES AND COLLECTED WORKS OF MATHEMATICIANS—ADDENDA

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In the decade 1935–1945 the author endeavored—as a satisfying and profitable avocation—to seek out, to read and to record all *book-length* biographies (individual and collected) in English of physicists and astronomers, mathematicians, chemists, and engineers, metallurgists and industrialists. Subsequently, the titles of pertinent items—obtained by search of (i) the stacks and card catalogs of the important public, university and technical libraries located in the East and Middle West; (ii) the accumulated catalogs of the principal American and British publishers of technical and scientific books; (iii) the lists of offerings, over a decade, of the larger American and British dealers in used and rare technical and scientific works; (iv) much relevant miscellaneous bibliographical reference works: book review journals, printed catalogs of American and British private, public and national libraries, and kindred aids—were published in a series of four bibliographies [1–4].

Subsequently, reprints of each of these bibliographies were sent to certain major libraries (in both America and Great Britain) which the author had not been able to visit in person, together with a request for the titles of additional items, if any, contained in the library. A limited number of titles of rather obscure items stemmed from these requests. Several others were contributed by interested American and British readers of the published biographies. These titles, together with those of recently published items, comprise a series of four short addenda to appear—it is hoped—in those periodicals containing the corresponding bibliographies.

In consideration of the manner of compilation it is believed that practically all significant English-written book-length biographies of mathematicians are encompassed in the original bibliography [2] or in the following addendum. In consequence of this definitive character, these listings are of obvious worth to all who are professionally interested in the history of pure and applied mathematics or, more broadly, in the history of science in general. In particular, they can be utilized very usefully in preparing the biographical content of a course of study utilizing the historical approach in the manner delineated by Dr. James B. Conant in his recent well received and widely discussed book. [5]

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4. Higgins, T. J., Book-length biographies of engineers and metallurgists, Bulletin of Bibliography, vol. 18, 1946, pp. 207–210, 235–239; vol. 19, 1947, pp. 10–12, 32.

5. Conant, J. B., On Understanding Science; An Historical Approach, Yale University Press, New Haven, 1947, 145 pp.

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The Religious Opinions of Milton, Locke and Newton. By H. McLachlan. Manchester, University Press, 1941. 221 pp.

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Mathematical Physics

The Early Work of Willard Gibbs in Applied Mechanics. By L. P. Wheeler, E. O. Waters, and S. W. Dudley. New York, Henry Schuman, Inc., 1947. 78 pp.

A SPECIAL TETRAHEDRON*

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1. Definitions. (a). Let $(T)=DABC$ be a tetrahedron, (O) its circumsphere, and D_0 the diametric opposite on (O) of one vertex, say, D of (T).

If D_0 lies in the face ABC opposite the vertex D, we shall say that the tetrahedron is "special" and designate it by (σ). The vertex D and the face ABC will be referred to as the "special vertex" and the "special face" of (σ).

(b). Let D_h be the foot of the altitude DD_h of (σ), and O_d the center of the circumcircle (O_d) of the triangle ABC.

The points D, D_0 are symmetrical with respect to the center O of the circumsphere (O), hence their projections D_h, D_0 upon the plane ABC are symmetrical with respect to O_d; that is, *in a special tetrahedron* (σ) *the foot of the altitude issued from the special vertex lies on the circumsphere of* (σ).

* Read before the Mathematical Association of America, Oklahoma Section, Feb. 10, 1939, Tulsa, Okla.

(c). Conversely, *if in a tetrahedron $DABC$ the foot of the altitude DD_h lies on the circumsphere, then the tetrahedron is a special tetrahedron* (σ).

Indeed, D_h lies, by assumption, on the circumcircle (O_d) of the triangle ABC. Now the diametric opposite D_0 of D_h on the circle (O_d) is also the diametric opposite of D on the sphere (O).

(d). In the triangle DD_hD_0 we have DD_h is twice OO_d. Thus: *In a special tetrahedron (σ) the length of the altitude issued from the special vertex is equal to twice the distance of the circumcenter from the special face of* (σ).

(e). Conversely, *If in a tetrahedron an altitude is equal to twice the distance of the circumcenter from the corresponding face, the tetrahedron is a special tetrahedron* (σ).

Indeed, if DD_h is twice OO_d, then O is the mid-point of DD_0. Now D lies on the circumsphere (O) of the tetrahedron, hence D_0 is the diametric opposite of D on (O); and since D_0 lies in the plane ABC, by assumption, the proposition follows.

2. A Simson line. (a). If DD_h is the altitude of a tetrahedron $(T)=DABC$ issued from D, and DX is the altitude of the triangle DBC issued from D, the line D_hX is perpendicular to BC; that is, the point X coincides with the projection of the point D_h upon BC. A similar condition exists for the analogous points Y, Z on the edges CA, AB, respectively. Now if (T) is a special tetrahedron (σ), D its special vertex, the point D_h lies on the circumcircle (O_d) of the triangle ABC [art. 1(b)]; hence the points X, Y, Z lie on the Simson line of D_h for the triangle ABC. Thus: *In a special tetrahedron (σ) the feet of the perpendiculars from the special vertex upon the three edges of the special face are collinear.*

(b). Conversely, *if in a tetrahedron the projections of a vertex upon the three edges of the opposite face are collinear, the tetrahedron is a special tetrahedron* (σ).

Indeed, the feet X, Y, Z of the altitudes DX, DY, DZ of the faces DBC, DCA, DAB of a tetrahedron $DABC$ are the projections upon BC, CA, AB of the foot D_h of the altitude DD_h of the tetrahedron. Now if X, Y, Z are collinear, the point D_h lies on the circumcircle of the triangle ABC, by the converse of Simson's theorem; hence we have the proposition [art. 1(c)].

3. The Monge point.† (a). Let O, G be the circumcenter and the centroid of a tetrahedron $DABC$, O_d the circumcenter of the triangle ABC, and G' the projection of G upon ABC. If K is the trace in ABC of the Euler line OG of the tetrahedron, we have

$$OK:GK = OO_d:GG'. \tag{1}$$

If G_d is the centroid of the triangle ABC, and DD_h is the altitude of the tetrahedron issued from D, we have:

$$DD_h:GG' = DG_d:GG_d = 4:1. \tag{2}$$

† The Monge point of a tetrahedron (T) is the symmetric M of the circumcenter O with respect to the centroid G of (T).

Now if we assume that $DABC$ is a special tetrahedron (σ), we also have [art. 1(d)]:

$$DD_h = \text{twice } OO_d. \tag{3}$$

From (1), (2), (3) we obtain

$$OK:GK = 2:1. \tag{4}$$

That is, the point K coincides with the Monge point of (σ). Thus: In a special tetrahedron (σ) the Monge point lies in the special face of (σ).

(b). Conversely, *if the Monge point of a tetrahedron lies in a face, the tetrahedron is a special tetrahedron* (σ).

Indeed, if the trace K of OG in ABC is the Monge point of the tetrahedron, the relation (4) holds, and from (4), (1), (2) we obtain (3); hence we have the proposition [art. 1(e)].

(c). Since in a special tetrahedron (σ) the Monge point M lies in the special face ABC, it coincides therefore with the mid-point of the segment determined by the orthocenter H_d of the triangle ABC and the foot of the altitude DD_h.** Moreover, the mid-point M of the segment joining the point D_h of the circumcircle of the triangle ABC [art. 1(b)] to the orthocenter H_d lies on the nine-point circle of ABC and on the Simson line of D_h for ABC.† Thus: *In a special tetrahedron* (σ) *the Monge point:* i. *Bisects the segments joining the orthocenter of the special face to the foot of the altitude upon that face*; ii. *lies on the nine-point circle of the special face*; iii. *lies on the Simson line, for the triangle of the special face, of the foot of the altitude upon that face*; iv. *is collinear with the projections of the special vertex upon the edges of the opposite face* [Art. 2(a)].

(d). The line joining the Monge point of a special tetrahedron (σ) to the diametric opposite of the special vertex on the circumsphere of (σ) passes through the centroid of the special face and is trisected by that point.

(e). In a special tetrahedron (σ) the foot of the altitude upon the special face is a center of similitude of the nine-point circle of that face and the circle along which that face cuts the twelve-point sphere of (σ).

The proofs of the last two propositions [3(d) and 3(e)] are left to the reader.

4. A doubly special tetrahedron. (a). A tetrahedron in which the feet of two altitudes, say, DD_h, AA_h lie on the circumcircles of the respectively opposite faces will be denoted by $(\sigma\sigma)$.

(b). *In a tetrahedron* $(\sigma\sigma) = DABC$, *the Monge point* M *lies on the edge* BC [art. 3(a)].

(c). Conversely, *if in a tetrahedron the Monge point lies on an edge the tetrahedron is* $(\sigma\sigma)$ [art. 3(b)].

5. A pair of rectangular opposite edges of a tetrahedron. (a). If the Monge

** Nathan Altshiller-Court, Modern pure solid geometry, p. 70, art. 233, New York, 1935. This book will be referred to as MPSG.

† Nathan Altshiller-Court, College geometry, pp. 116–117.

point M of a doubly special tetrahedron $(\sigma\sigma)=DABC$ lies on the edge BC, the lines DM, AM are each perpendicular to the edge BC [art. 2]. Thus the edge BC is perpendicular to the plane ADM. Hence: *If the Monge point M of a tetrahedron lies on an edge*, i. *that edge is perpendicular to the opposite edge*: ii. *M lies on the common perpendicular to those two edges.*

(b). *In a tetrahedron $(\sigma\sigma)$ the segment joining the feet of the two special altitudes is equal and parallel to the segment joining the orthocenters of the two special faces.*

Indeed, the feet A_h, D_h of the altitudes AA_h, DD_h lie on the lines DM, AM, and the Monge point M bisects the two segments D_hH_d, A_hH_a [art. 3(c)].

6. The trirectangular tetrahedron. (a). If the Monge point of a tetrahedron $(T)=DABC$ coincides with a vertex, say, D of (T), the edges DA, DB, DC are respectively orthogonal to the opposite edges BC, CA, AB of (T) [art. 5(a)], and (T) is a trirectangular tetrahedron with D as the vertex of the right angle [art. 5(a)].

(b). Conversely, in a trirectangular tetrahedron the vertex of the right angle is the Monge point of the tetrahedron (*MPSG.*, p. 93, Art. 288).

FUNCTIONS OF SEVERAL COMPLEX VARIABLES AND MULTIHARMONIC FUNCTIONS

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1. Polygenic functions of n complex variables. This paper presents a brief introduction to the theory of polygenic functions of several complex variables. The mean and phase derivatives (4) are shown to be important in connection with the study of the Kasner clocks. We discuss how analytic polygenic functions can be extended to spaces of double the original number of dimensions. Applications are made to the theory of monogenic functions. It is proved that if a multiharmonic function (the real or imaginary part of a monogenic function) is rational, algebraic, or entire, then the associated monogenic function is rational, algebraic, or entire. A characterization of multiharmonic functions is studied. In the final part of the paper, we discuss several polygenic functions of several complex variables. An expression for the jacobian is obtained in terms of the mean and phase derivatives. Application is made to the pseudo-conformal group which may be defined as the group preserving the pseudo-angle (35) of Kasner.

A complex function

$$
\begin{aligned}
w = U &= X + iY = F(u_\alpha) = F(u_1, \cdots, u_n) \\
&= \phi(x_1, \cdots, x_n; y_1, \cdots, y_n) + i\psi(x_1, \cdots, x_n; y_1, \cdots, y_n) \qquad (1)\\
&= \phi(x_\alpha; y_\alpha) + i\psi(x_\alpha; y_\alpha),
\end{aligned}
$$

where ϕ and ψ are real continuous functions possessing continuous partial deriva-

tives of first order with respect to the $2n$ real variables $(x_\alpha; y_\alpha) = (x_1, \cdots, x_n; y_1, \cdots, y_n)$ in a certain region R of $2n$ dimensional space, is termed a polygenic function [1] of the n complex variables $u_\alpha = x_\alpha + iy_\alpha$, where $\alpha = 1, \cdots, n$. For each fixed α, the point (x_α, y_α) varies over a region R_α of the (x_α, y_α)-plane. For $\alpha = 1, \cdots, n$, these n planes are a certain selected set of n coördinate planes of the $2n$ coördinate planes which are defined by the coördinate system (x_α, y_α).

We shall use the complex variables v_α, conjugate to u_α, defined by the obvious relations

$$
\begin{aligned}
u_\alpha &= x_\alpha + iy_\alpha, & v_\alpha &= x_\alpha - iy_\alpha, \\
x_\alpha &= \frac{u_\alpha + v_\alpha}{2}, & y_\alpha &= \frac{u_\alpha - v_\alpha}{2i}.
\end{aligned} \tag{2}
$$

The pair (u_α, v_α) for each fixed α are called minimal or isotropic coördinates of a point (x_α, y_α) in the (x_α, y_α)-coördinate plane. For any such plane, the square of the differential of arc length is $ds_\alpha^2 = dx_\alpha^2 + dy_\alpha^2 = du_\alpha dv_\alpha$ and the inclination θ_α of a direction in this plane to the x_α-axis is given by $dv_\alpha/du_\alpha = e^{-2i\theta_\alpha}$.

The polygenic function w may be considered to be a function of the n complex variables u_α, or the n complex variables v_α, or the $2n$ non-independent complex variables u_α and v_α.

2. Partial derivatives $\delta w/\delta u_\alpha$ of a polygenic function w. The partial derivatives of a polygenic function w with respect to the complex variable u_α for a fixed α, is

$$
\begin{aligned}
\frac{\delta w}{\delta u_\alpha} &= \frac{\dfrac{\partial w}{\partial x_\alpha} dx_\alpha + \dfrac{\partial w}{\partial y_\alpha} dy_\alpha}{dx_\alpha + i dy_\alpha} \\
&= \frac{1}{2}\left(\frac{\partial w}{\partial x_\alpha} - i\frac{\partial w}{\partial y_\alpha}\right) + \frac{1}{2}\left(\frac{\partial w}{\partial x_\alpha} + i\frac{\partial w}{\partial y_\alpha}\right) e^{-2i\theta_\alpha}.
\end{aligned} \tag{3}
$$

It is seen that this partial derivative depends not only on the point $(x_1, \cdots, x_n; y_1, \cdots, y_n)$ but also on the inclination θ_α in the (x_α, y_α)-coördinate plane.

Noting that $\delta w/\delta u_\alpha$ represents a single complex number, it can be plotted as a point in a new plane, which may be called the u_α-partial derivative plane. Upon fixing the point $(x_1, \cdots, x_n; y_1, \cdots, y_n)$, it is found that $\delta w/\delta u_\alpha$ in this new plane represents a Kasner clock [2]. That is, as θ_α in the (x_α, y_α)-plane changes, the points (3) describe a circle in the u_α-partial derivative plane at double the rate of θ_α and in the opposite direction. The center of the clock is $\frac{1}{2}(\partial w/\partial x_\alpha - i\partial w/\partial y_\alpha)$ and the radius is $\frac{1}{2}|\partial w/\partial x_\alpha + i\partial w/\partial y_\alpha|$.

In connection with the partial derivatives $\delta w/\delta u_\alpha$ of (3), it is seen that the following linear operators are important, namely

$$
\frac{\partial}{\partial u_\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x_\alpha} - i\frac{\partial}{\partial y_\alpha}\right), \qquad \frac{\partial}{\partial v_\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x_\alpha} + i\frac{\partial}{\partial y_\alpha}\right), \tag{4}
$$

where $\alpha=1, \cdots, n$. The first set of operators are called the mean derivatives, and the second set are termed the phase derivatives. It is remarked that the symbols $\partial/\partial u_\alpha$ and $\partial/\partial v_\alpha$ merely denote the application of the linear operators (4).

Of course the linear operators are commutative. We have

$$
\begin{aligned}
\frac{\partial^2}{\partial u_\alpha \partial v_\beta} &= \frac{1}{4}\left[\left(\frac{\partial^2}{\partial x_\alpha \partial x_\beta}+\frac{\partial^2}{\partial y_\alpha \partial y_\beta}\right)+i\left(\frac{\partial^2}{\partial x_\alpha \partial y_\beta}-\frac{\partial^2}{\partial x_\beta \partial y_\alpha}\right)\right],\\
\frac{\partial^2}{\partial u_\alpha \partial u_\beta} &= \frac{1}{4}\left[\left(\frac{\partial^2}{\partial x_\alpha \partial x_\beta}-\frac{\partial^2}{\partial y_\alpha \partial y_\beta}\right)-i\left(\frac{\partial^2}{\partial x_\alpha \partial y_\beta}+\frac{\partial^2}{\partial x_\beta \partial y_\alpha}\right)\right],\\
\frac{\partial^2}{\partial v_\alpha \partial v_\beta} &= \frac{1}{4}\left[\left(\frac{\partial^2}{\partial x_\alpha \partial x_\beta}-\frac{\partial^2}{\partial y_\alpha \partial y_\beta}\right)+i\left(\frac{\partial^2}{\partial x_\alpha \partial y_\beta}+\frac{\partial^2}{\partial x_\beta \partial y_\alpha}\right)\right].
\end{aligned}
\tag{5}
$$

3. Prolongation of analytic polygenic functions into $4n$ dimensional space. We shall term w an analytic polygenic function over a region R of $2n$ dimensional space if both of the components ϕ and ψ are expansible into Taylor series about any point $x_\alpha=x_\alpha^0$; $y_\alpha=y_\alpha^0$ of R. Now consider the analytic polygenic function $w=\phi+i\psi$ with the $2n$ positive radii of convergence $|x_\alpha-x_\alpha^0|<r_\alpha$, $|y_\alpha-y_\alpha^0|<s_\alpha$. The point $(x_\alpha^0; y_\alpha^0)$ is called the center of this analytic power series. If in such a function, we replace each x_α and y_α by a complex variable, that is, set $x_\alpha=x_{1\alpha}+ix_{2\alpha}$, $y_\alpha=y_{1\alpha}+iy_{2\alpha}$ where $(x_{1\alpha}, x_{2\alpha}; y_{1\alpha}, y_{2\alpha})$ are real numbers, the polygenic function w is still convergent for all complex variables $(x_\alpha; y_\alpha)$ such that $|x_\alpha-x_\alpha^0|<r_\alpha$; $|y_\alpha-y_\alpha^0|<s_\alpha$. This follows from a theorem in analysis which states that if a multiple power series converges for at least one point $(x_\alpha^1; y_\alpha^1)$, not its center $(x_\alpha^0; y_\alpha^0)$, then the multiple power series converges absolutely for all complex variables $(x_\alpha; y_\alpha)$ such that $|x_\alpha-x_\alpha^0|<|x_\alpha^1-x_\alpha^0|$; $|y_\alpha-y_\alpha^0|<|y_\alpha^1-y_\alpha^0|$. The convergence is uniform in every $4n$-dimensional domain whose closure lies in that set. Thus an analytic polygenic function of $2n$ real variables may be extended to an analytic polygenic function of $2n$ complex variables so that it is defined over a certain region of $4n$-dimensional space. Moreover this extended polygenic function is unique.

Let $0<t_\alpha\leqq\min(r_\alpha, s_\alpha)$. Since an analytic polygenic function w can be considered as a function of $2n$ complex variables $(x_\alpha; y_\alpha)$, it is seen by (2) that w is an analytic polygenic function of the $2n$ independent complex variables $(u_\alpha; v_\alpha)$ where $|u_\alpha-u_\alpha^0|<t_\alpha$ and $|v_\alpha-v_\alpha^0|<t_\alpha$. The complex variables (u_α, v_α) for fixed α, are conjugate if and only if (x_α, y_α) are real. In this case, the linear operators (4) denote the formal partial derivatives of w with respect to u_α and v_α [**3**].

4. Monogenic functions. Now we return to the real variables $(x_\alpha; y_\alpha)$ and we assume merely that ϕ and ψ are real continuous functions possessing continuous partial derivatives of first order with respect to $(x_\alpha; y_\alpha)$ over a region R of $2n$ dimensional space. The partial derivatives $\delta w/\delta u_\alpha$ for $\alpha=1, \cdots, n$, as defined by (3) will be independent of the directions θ_α if and only if

$$\frac{\partial w}{\partial v_\alpha} = \frac{1}{2}\left(\frac{\partial w}{\partial x_\alpha} + i\frac{\partial w}{\partial y_\alpha}\right) = 0 \qquad \text{for } \alpha = 1, \cdots, n. \tag{6}$$

In this event, the Kasner clocks reduce to points. Upon placing $w=\phi+i\psi$ and remembering that ϕ and ψ are real, we obtain the Cauchy-Riemann equations for n complex variables

$$\frac{\partial \phi}{\partial x_\alpha} - \frac{\partial \psi}{\partial y_\alpha} = 0; \qquad \frac{\partial \phi}{\partial y_\alpha} + \frac{\partial \psi}{\partial x_\alpha} = 0. \tag{7}$$

There are 2n such partial differential equations of first order.

If these are valid at every point of the region R, the function w is said to be monogenic, or holomorphic, or regular, or analytic over the region R. In that event, we have the actual partial derivatives of w with respect to u_α, as it follows by (6) and (7),

$$\frac{\partial w}{\partial u_\alpha} = \frac{1}{2}\left(\frac{\partial w}{\partial x_\alpha} - i\frac{\partial w}{\partial y_\alpha}\right) = \frac{\partial w}{\partial x_\alpha} = -i\frac{\partial w}{\partial y_\alpha}. \tag{8}$$

This is the same as

$$\frac{\partial w}{\partial u_\alpha} = \frac{\partial \phi}{\partial x_\alpha} + i\frac{\partial \psi}{\partial x_\alpha} = \frac{\partial \psi}{\partial y_\alpha} - i\frac{\partial \phi}{\partial y_\alpha} = \frac{\partial \phi}{\partial x_\alpha} - i\frac{\partial \phi}{\partial y_\alpha} = \frac{\partial \psi}{\partial y_\alpha} + i\frac{\partial \psi}{\partial x_\alpha}. \tag{9}$$

For monogenic functions over a region R, the Cauchy Integral Theorem is valid [4]. The single, double, $\cdots$, n-tuple integral of a monogenic function $w=F(u_1, \cdots, u_n)$ along a closed curve, surface, $\cdots$, n dimensional manifold, which do not have double points, is zero. Let C_α, a closed rectifiable curve without double points, be in the interior of a region R_α of the (x_α, y_α)-coördinate plane. Then Cauchy's Integral Formula for a monogenic function $F(u_1, \cdots, u_n)$ is

$$F(u_1, \cdots, u_n) = \frac{1}{(2\pi i)^n}\int_{C_n} dz_n \int_{C_{n-1}} dz_{n-1} \cdots \int_{C_1} \frac{F(z_1, \cdots, z_n)}{(z_1 - u_1)\cdots(z_n - u_n)}\, dz_1. \tag{10}$$

It is deduced from this that a monogenic function $F(u_1, \cdots, u_n)$ possesses partial derivatives of all orders with respect to u_α and hence with respect to (x_α, y_α) for $\alpha=1, \cdots, n$. Also it can be proved that a monogenic function can be expanded into a Taylor series about any point in the interior of the given region R over which the function is monogenic. Thus the monogenic functions are special cases of analytic polygenic functions.

5. Multiharmonic functions. A function is said to be multiharmonic if it is the real or imaginary part of a monogenic function. Two functions ϕ and ψ are said to be conjugate multiharmonic if they are the real and imaginary parts of a given monogenic function. Thus two real continuous functions ϕ and ψ,

which possess continuous first order partial derivatives, are conjugate multiharmonic if and only if they obey the Cauchy-Riemann equations (7). Of course, all the multiharmonic functions are analytic polygenic functions.

Upon eliminating ψ from the Cauchy-Riemann equations (7), it is seen that *any multiharmonic function ϕ obeys the system of n^2 Poincaré partial differential equations of second order*

$$\frac{\partial^2\phi}{\partial x_\alpha \partial x_\beta} + \frac{\partial^2\phi}{\partial y_\alpha \partial y_\beta} = 0, \qquad \frac{\partial^2\phi}{\partial x_\alpha \partial y_\beta} - \frac{\partial^2\phi}{\partial x_\beta \partial y_\alpha} = 0. \tag{11}$$

These are the extensions of the Laplace equation which is the special case $n=1$. Of course, the function ψ also obeys this system of n^2 partial equations.

A multiharmonic function ψ conjugate to a given multiharmonic function ϕ may be found in the following way. Since ϕ and ψ are conjugate multiharmonic, they obey the Cauchy-Riemann equations (7). Hence

$$d\psi = \sum_{\alpha=1}^{n}\left(\frac{\partial\psi}{\partial x_\alpha} dx_\alpha + \frac{\partial\psi}{\partial y_\alpha} dy_\alpha\right) = \sum_{\alpha=1}^{n}\left(-\frac{\partial\phi}{\partial y_\alpha} dx_\alpha + \frac{\partial\phi}{\partial x_\alpha} dy_\alpha\right). \tag{12}$$

The last written Pfaffian is exact since ϕ obeys the Poincaré system (11). Hence ψ is found and any two such functions ψ will differ by a real constant. Having found ψ, the monogenic function $w=\phi+i\psi$ can be constructed.

It is deduced that *any solution ϕ of the Poincaré partial differential equations* (11) *is analytic.* For the conjugate function ψ is found by (12). Since the monogenic function $w=\phi+i\psi$ is analytic, it follows that both ϕ and ψ are analytic.

6. Conjugate multiharmonic functions in minimal coördinates [5]. Since a multiharmonic function ϕ is analytic, it follows by the remarks in Section 3, that ϕ may be extended to $2n$ complex variables and hence ϕ can be written as an analytic function of $2n$ independent complex variables $(u_\alpha; v_\alpha)$ for $\alpha=1, \cdots, n$. The function ϕ is real if and only if the conjugate of any term in ϕ is also in ϕ when u_α and v_α for every α are considered to be conjugate complex variables. The system of n^2 Poincaré partial differential equations (11) may be written in minimal coördinates by (5) as follows

$$\frac{\partial^2\phi}{\partial u_\alpha \partial v_\beta} = 0. \tag{13}$$

This shows that *any multiharmonic function ϕ is of the form*

$$2\phi = F(u_1, \cdots, u_n) + G(v_1, \cdots, v_n), \tag{14}$$

where F is monogenic in $(u_1, \cdots, u_n)$ and G is monogenic in $(v_1, \cdots, v_n)$ such that the coefficients of the terms in G are the conjugates of the coefficients of the corresponding terms in F.

If ψ is a multiharmonic function conjugate to the function ϕ of (14), then,

$$\phi + i\psi = \lambda(u_1, \cdots, u_n), \qquad \phi - i\psi = \mu(v_1, \cdots, v_n), \tag{15}$$

where λ and μ are related to each other in the same manner as the F and G of (14) above. In order to determine ψ, it is only necessary to find the value of λ and hence μ. Evidently by addition, we have

$$2\phi = \lambda + \mu = F + G. \tag{16}$$

This identity demonstrates that $\lambda = F + ic$ and $\mu = G - ic$, where c is a real constant. Therefore by subtracting equations (15), we have

$$2i\psi = \lambda - \mu = F - G + 2ic. \tag{17}$$

Thus if a function $\phi(x_\alpha; y_\alpha)$ is multiharmonic, we find

$$\begin{aligned} \phi(x_\alpha; y_\alpha) &= \phi\left(\frac{u_\alpha + v_\alpha}{2}; \frac{u_\alpha - v_\alpha}{2i}\right) \\ &= \frac{1}{2}\left[F(u_1, \cdots, u_n) + G(v_1, \cdots, v_n)\right]. \end{aligned} \tag{18}$$

The conjugate multiharmonic function ψ is

$$\psi(x_\alpha; y_\alpha) = -\frac{i}{2}\left[F(u_1, \cdots, u_n) - G(v_1, \cdots, v_n)\right] + c, \tag{19}$$

where c is an arbitrary real constant. The associated monogenic w is

$$w = \phi + i\psi = F(u_1, \cdots, u_n) + ic. \tag{20}$$

7. **Application of the preceding results** [6]. It is evident that if ϕ is a rational multiharmonic function of $(x_\alpha; y_\alpha)$, then by (18), F is a rational monogenic function of $(u_1, \cdots, u_n)$, and similarly G is a rational monogenic function of $(v_1, \cdots, v_n)$. By (19), the conjugate multiharmonic function ψ is also rational.

The conjugate of any rational multiharmonic function is also rational multiharmonic. Therefore a monogenic function $w = \phi + i\psi = F(u_\alpha)$ is rational if at least one component is a rational function of $(x_\alpha; y_\alpha)$.

Of course, the corresponding result for rational integral multiharmonic functions is obvious.

If ϕ is an algebraic multiharmonic function, there exists a polynomial P in the $(2n+1)$ variables $(\phi; x_1, \cdots, x_n; y_1, \cdots, y_n)$ such that

$$P(\phi; x_1, \cdots, x_n; y_1, \cdots, y_n) = 0. \tag{21}$$

At any ordinary point $(x_\alpha^0; y_\alpha^0)$, it is seen that the multipower series expansion for ϕ obeys this equation identically. Converting this into minimal coördinates, we obtain the identity

$$P\left[\frac{F(u_\alpha) + G(v_\alpha)}{2}; \frac{u_\alpha + v_\alpha}{2}; \frac{u_\alpha - v_\alpha}{2i}\right] = 0. \tag{22}$$

This is a polynomial in the letters F, G; u_α; v_α, which vanishes identically when

F is replaced by $F(u_\alpha)$ and G by $G(v_\alpha)$. Hence take $v_\alpha = v_\alpha^0 = x_\alpha^0 - iy_\alpha^0$. The identity (22) proves that the monogenic function $F(u_\alpha)$ is algebraic, that is, $F(u_\alpha)$ obeys an equation of the form

$$Q[F(u_\alpha); u_\alpha] = 0 \tag{23}$$

where Q is a polynomial in the $(n+1)$ variables $(F; u_1, \cdots, u_n)$. Placing $F = \phi + i\psi$ and $u_\alpha = x_\alpha + iy_\alpha$ into (23), and putting the real and imaginary parts of (23) equal to zero, we obtain two polynomial equations in the $(2n+1)$ variables $(\phi, \psi, x_1, \cdots, x_n; y_1, \cdots, y_n)$. Eliminating ϕ from these two equations, we find a polynomial equation in the $(2n+1)$ variables $(\psi; x_1, \cdots, x_n; y_1, \cdots, y_n)$. This proves that ψ is algebraic.

The conjugate of any algebraic multiharmonic function is also algebraic multiharmonic. A monogenic function $w = \phi + i\psi = F(u_\alpha)$ is algebraic if only one component is known to be an algebraic function of $(x_\alpha; y_\alpha)$.

In a very similar fashion, the following result may be proved.

The conjugate of any entire multiharmonic function is also entire multiharmonic. Thus a monogenic function $w = \phi + i\psi = F(u_\alpha)$ is entire if only one component is known to be an entire function of $(x_\alpha; y_\alpha)$.

By definition, $\phi(x_\alpha; y_\alpha)$ is entire if $\phi(x_\alpha; y_\alpha)$ is a multipower series convergent for all finite values of $(x_\alpha; y_\alpha)$ where $\alpha = 1, \cdots, n$. By (18), $F(u_\alpha)$ is convergent for all finite values of u_α, and also $G(v_\alpha)$ is convergent for all finite values of v_α. Hence the monogenic function F is entire, and by (19), ψ is convergent for all finite values of $(x_\alpha; y_\alpha)$. That is, ψ is entire.

8. A characterization of multiharmonic functions [7]. Before giving this characterization, we shall consider briefly the conformal or analytic surfaces of $2n$-dimensional space. Let

$$u_\alpha = x_\alpha + iy_\alpha = u_\alpha(u), \qquad v_\alpha = x_\alpha - iy_\alpha = v_\alpha(v), \tag{24}$$

be n monogenic functions of the single complex variable $u = x + iy$ for $\alpha = 1, \cdots, n$, where at least one of the $u_\alpha'(u)$ (and also the $v_\alpha'(v)$) does not vanish in a certain region R_0 of the (u, v)- or (x, y)-plane. In general each of these is a conformal representation of a certain domain R_α of the (x_α, y_α)-coördinate plane upon this region R_0 of the (x, y)-plane. The points of the domains R_α for $\alpha = 1, \cdots, n$, give rise to a certain region R of $2n$ dimensional space. In R, the equations (24) are the parametric equations of a surface of special type, which is called a conformal surface or an isoclineal surface or an analytic surface. It can be shown that (24) also induces a conformality between the analytic surface and the region R_0. Thus in all, we have $(n+1)$ conformal representations upon the region R_0 of the (x, y)-plane, namely, the n regions R_α of the (x_α, y_α)-coördinate planes for $\alpha = 1, \cdots, n$, and the portion of the analytic surface defined by (24).

In particular, if the equations (24) are all similitudes, that is, if

$$u_\alpha = a_\alpha u + b_\alpha, \qquad v_\alpha = A_\alpha v + B_\alpha, \tag{25}$$

where A_α is the conjugate of the complex constant a_α (not all of which are zero), and B_α is the conjugate of the complex constant b_α, the surface is a plane in special position. Such planes are called conformal planes or isoclinal planes, or more simply, isoclines. The tangent planes of a conformal surface are isoclines. The sum of the n orthogonal projections upon the (x_α, y_α)-coördinate planes of an area on an isocline is equal to the original area itself.

Now let ϕ be a function of the $2n$ variables $(x_1, \cdots, x_n; y_1, \cdots, y_n)$ which possess continuous partial derivatives of the second order over the $2n$ dimensional region R defined above. Upon substituting $x_\alpha = x_\alpha(x, y)$, $y_\alpha = y_\alpha(x, y)$ which are the real and imaginary parts of (24), or (25), into the function ϕ, it is found that ϕ becomes a function of only two variables x and y. If the equations (24) are used, this operation is called a conformal substitution on the function ϕ. Otherwise if (25) are used, the operation is called a similitude substitution on ϕ.

A function ϕ is multiharmonic if and only if ϕ is carried into a harmonic function by every conformal substitution.

This result was first stated and proved by Kasner for the case $n=2$.

Applying the mean derivative on the function ϕ, we have

$$\frac{\partial\phi}{\partial u} = \frac{1}{2}\left(\frac{\partial\phi}{\partial x} - i\frac{\partial\phi}{\partial y}\right) = \sum_{\alpha=1}^{n} \frac{\partial\phi}{\partial u_\alpha}\frac{du_\alpha}{du} = \sum_{\alpha=1}^{n} \frac{\partial\phi}{\partial u_\alpha} u'_\alpha. \tag{26}$$

Next apply the phase derivative upon this. The result is

$$\frac{\partial^2\phi}{\partial u\partial v} = \frac{1}{4}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) = \sum_{\alpha,\beta=1}^{n} \frac{\partial^2\phi}{\partial u_\alpha\partial v_\beta}\frac{du_\alpha}{du}\frac{dv_\beta}{dv} = \sum_{\alpha,\beta=1}^{n} \frac{\partial^2\phi}{\partial u_\alpha\partial v_\beta} u'_\alpha v'_\beta. \tag{27}$$

Since ϕ is harmonic in (x, y), this must vanish. As it is to be harmonic for every conformal substitution, we find that the right member of equation (27) vanishes for all values of u'_α and v'_α. Hence the coefficient of $u'_\alpha v'_\beta$ is zero, that is, $\partial^2\phi/\partial u_\alpha \partial v_\beta = 0$ for $\alpha, \beta = 1, \cdots, n$. This means that ϕ is multiharmonic.

A function ϕ is multiharmonic if and only if it is transformed into a harmonic function by every similitude substitution.

This result is proved by the same argument as above. The only change is that in equation (27), we have $u'_\alpha = a_\alpha$ and $v'_\beta = A_\beta$. The coefficients of $a_\alpha A_\beta$ being zero, we obtain $\partial^2\phi/\partial u_\alpha\partial v_\beta = 0$. Thus ϕ is multiharmonic.

Elsewhere these results have been extended to multi-isothermal systems of manifolds of $(2n-1)$ dimensions [8].

9. **The Jacobian J of n polygenic functions of n complex variables.** Consider a transformation T defined over a certain region R of $2n$-dimensional space by the $2n$ equations

$$X_\beta = X_\beta(x_1, \cdots, x_n; y_1, \cdots, y_n), \quad Y_\beta = Y_\beta(x_1, \cdots, x_n; y_1, \cdots, y_n), \tag{28}$$

where $\beta = 1, \cdots, n$. These $2n$ functions are assumed to have continuous partial derivatives of first order over R. Construct the Jacobian J of these functions.

Let the element in the row β and the column α be $\partial X_\beta/\partial x_\alpha$, the element in the row β and the column $(n+\alpha)$ be $\partial X_\beta/\partial y_\alpha$, the element in the row $(n+\beta)$ and column α be $\partial Y_\beta/\partial x_\alpha$, and finally the element in the row $(n+\beta)$ and the column $(n+\alpha)$ be $\partial Y_\beta/\partial y_\alpha$. The Jacobian J is given the expression

$$J = \begin{vmatrix} \dfrac{\partial X_\beta}{\partial x_\alpha}, & \dfrac{\partial X_\beta}{\partial y_\alpha} \\ \dfrac{\partial Y_\beta}{\partial x_\alpha}, & \dfrac{\partial Y_\beta}{\partial y_\alpha} \end{vmatrix}. \tag{29}$$

If J is not zero over the region R, it follows that the region R is mapped by the transformation T into another region S of $2n$-dimensional space such that in the neighborhood of any point of S, the transformation T has a single valued inverse T^{-1} with continuous partial derivatives of first order.

The transformation T as given by equations (28) may be defined by the n polygenic function U_β together with the n conjugate polygenic functions V_β defined as follows

$$U_\beta = X_\beta + iY_\beta, \qquad V_\beta = X_\beta - iY_\beta, \tag{30}$$

for $\beta=1, \cdots, n$.

In terms of the mean and phase partial derivatives (4), *the Jacobian J as given by* (29), *may be written in the form*

$$J = \begin{vmatrix} \dfrac{\partial U_\beta}{\partial u_\alpha}, & \dfrac{\partial U_\beta}{\partial v_\alpha} \\ \dfrac{\partial V_\beta}{\partial u_\alpha}, & \dfrac{\partial V_\beta}{\partial v_\alpha} \end{vmatrix}. \tag{31}$$

For by the elementary properties of determinants, we have

$$\begin{aligned} J &= \frac{1}{(2i)^n} \begin{vmatrix} \dfrac{\partial X_\beta}{\partial x_\alpha} - i\dfrac{\partial X_\beta}{\partial y_\alpha}, & 2i\dfrac{\partial X_\beta}{\partial y_\alpha} \\ \dfrac{\partial Y_\beta}{\partial x_\alpha} - i\dfrac{\partial Y_\beta}{\partial y_\alpha}, & 2i\dfrac{\partial Y_\beta}{\partial y_\alpha} \end{vmatrix} = \frac{2^{2n}}{(2i)^n} \begin{vmatrix} \dfrac{\partial X_\beta}{\partial u_\alpha}, & \dfrac{\partial X_\beta}{\partial v_\alpha} \\ \dfrac{\partial Y_\beta}{\partial u_\alpha}, & \dfrac{\partial Y_\beta}{\partial v_\alpha} \end{vmatrix} \\ &= \frac{2^{2n}}{(2i)^n(-2i)^n} \begin{vmatrix} \dfrac{\partial}{\partial u_\alpha}(X_\beta + iY_\beta), & \dfrac{\partial}{\partial v_\alpha}(X_\beta + iY_\beta) \\ -2i\dfrac{\partial Y_\beta}{\partial u_\alpha}, & -2i\dfrac{\partial Y_\beta}{\partial v_\alpha} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial U_\beta}{\partial u_\alpha}, & \dfrac{\partial U_\beta}{\partial v_\alpha} \\ \dfrac{\partial V_\beta}{\partial u_\alpha}, & \dfrac{\partial V_\beta}{\partial v_\alpha} \end{vmatrix}. \end{aligned} \tag{32}$$

10. The pseudo-conformal group G. If the n functions $U_\beta=X_\beta+iY_\beta$ are monogenic in $(u_1, \cdots, u_n)$ and the n functions $V_\beta=X_\beta-iY_\beta$ are monogenic in $(v_1, \cdots, v_n)$, then the transformation T is termed pseudo-conformal. Thus a

pseudo-conformal transformation T is defined by the equations

$$U_\beta = U_\beta(u_1, \cdots, u_n), \qquad V_\beta = V_\beta(v_1, \cdots, v_n), \tag{33}$$

for $\beta=1, \cdots, n$. Since $\partial U_\beta/\partial v_\alpha=0$ and $\partial V_\beta/\partial u_\alpha=0$ for $\alpha, \beta=1, \cdots, n$, it is seen by (31) that the Jacobian J of the pseudo-conformal transformation T is

$$J = \left| \frac{\partial U_\beta}{\partial u_\alpha} \right| \cdot \left| \frac{\partial V_\beta}{\partial v_\alpha} \right|. \tag{34}$$

That is, J is the product of the Jacobian of the n monogenic functions U_β with respect to the u_α and the Jacobian of the n monogenic functions V_β with respect to the v_α. The pseudo-conformal transformations T form an infinite group.

The pseudo-conformal group G of $2n$ dimensional space for $n>1$ is not the conformal group which is merely the inversive group of $(2n+1)(n+1)$ parameters.

The pseudo-conformal group G of $2n$ dimensional space is defined by the preservation of Kasner's pseudo-angle θ between a curve $C: x_\alpha = x_\alpha(t)$; $v_\alpha = y_\alpha(t)$, for $\alpha=1, \cdots, n$, and a $(2n-1)$ dimensional manifold $\sum_{2n-1}: F(x_1, \cdots, x_n; y_1, \cdots, y_n)=0$, at their common point of intersection. This pseudo-angle θ is given by the formula [9]

$$\theta = \arctan \frac{\sum_{\alpha=1}^{n}\left(\dfrac{\partial F}{\partial x_\alpha}\, dx_\alpha + \dfrac{\partial F}{\partial y_\alpha}\, dy_\alpha\right)}{\sum_{\alpha=1}^{n}\left(-\dfrac{\partial F}{\partial y_\alpha}\, dx_\alpha + \dfrac{\partial F}{\partial x_\alpha}\, dy_\alpha\right)}. \tag{35}$$

If C is an analytic curve, there is in general one and only one analytic surface S passing through it. This surface S intersects $\sum_{2n-1}$ in a curve C'. The angle on S between C and C' is the pseudo-angle θ of Kasner.

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A GENERALIZATION OF THE GEOMETRIC SERIES*

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1. Introduction. Methods for the exact evaluation of

$$K_n(x) = \sum_{k=1}^{\infty} k^n x^k, \tag{1}$$

when n is a non-negative integer and $|x| < 1$, are developed in this paper. This series is a generalization of the geometric series since

$$K_0(x) = \sum_{k=1}^{\infty} x^k$$

is a geometric series with common ratio x.

Also the problem of summing a power series in which the coefficient of x^k is a rational integral function of k reduces immediately to the summation of (1).

It may be easily shown that $K_n(x)$ converges only for $|x| < 1$.

2. Methods for evaluating $K_n(x)$. The recursion relation,

$$K_{n+1}(x) = xK_n'(x), \tag{2}$$

may be verified by performing the indicated operations on

$$K_n(x) = \sum_{k=1}^{\infty} k^n x^k.$$

Also since $K_0(x)$ is a geometric series, we have

$$K_0(x) = \frac{x}{1 - x}. \tag{3}$$

Now proceeding from (3) by repeated use of (2), we see by induction with

* This is a condensation of one chapter of a Master's thesis at Oregon State College. The author acknowledges the help of Dr. W. E. Milne and Mr. Newton Smith.

respect to n that

$$K_n(x) = \frac{xP_n(x)}{(1-x)^{n+1}}, \tag{4}$$

where $P_0(x)=1$ and $P_n(x)$ is a polynomial of degree $n-1$ in x if $n>0$.

Substituting $K_n(x)$ from (4) into (2), performing the differentiation, and removing the factor $x/(1-x)^{n+2}$, we have

$$P_{n+1}(x) = (nx+1)P_n(x) + x(1-x)P_n'(x). \tag{5}$$

By use of (5) and the known fact, $P_0(x)=1$, we may readily build up several $P_n(x)$'s:

$$\begin{aligned}
P_1(x) &= 1 \\
P_2(x) &= 1 + x \\
P_3(x) &= 1 + 4x + x^2 \\
P_4(x) &= 1 + 11x + 11x^2 + x^3 \\
P_5(x) &= 1 + 26x + 66x^2 + 26x^3 + x^4. \\
&\cdots\cdots\cdots\cdots
\end{aligned}$$

Since $P_n(x)$ is a polynomial of degree $n-1$, we may set

$$P_n(x) = \sum_{r=1}^{n} f_r(n)x^{r-1}. \tag{6}$$

If we substitute $P_n(x)$ from (6) into (5) and equate like powers of x, we obtain the recursion relation for $f_r(n)$, namely,

$$f_r(n) = (n-r+1)f_{r-1}(n-1) + rf_r(n-1). \tag{7}$$

Now from this relation, since $f_0(n)=0$, then $f_1(n)=1$, and since $f_{n+1}(n)=0$, then $f_n(n)=1$. It may be easily shown that these coefficients are symmetrical, are positive integers, and increase toward the center of the polynomials.

By use of (7) and the above initial conditions we build up a triangular array of $f_r(n)$'s:

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 4 & & 1 & & & & \\
 & & & 1 & & 11 & & 11 & & 1 & & & \\
 & & 1 & & 26 & & 66 & & 26 & & 1 & & \\
 & 1 & & 57 & & 302 & & 302 & & 57 & & 1 & \\
1 & & 120 & & 1191 & & 2416 & & 1191 & & 120 & & 1 \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots
\end{array}$$

The law of formation is evidently similar to that for Pascal's triangle. Instead of adding two elements to obtain the one below, we add certain multiples of them. A multiple is determined by the number of steps we must take from a side of the triangle to reach the particular element. The method is evident from (7). For example, $11=3\times 1+2\times 4$, and $1191=5\times 57+3\times 302$.

Three methods for the summation of $K_n(x)$ have now been developed: Repeated application of (2) starting with (3); repeated application of (5) starting with $P_1(x)=1$, and substitution in (4); and finally the most ideal method, use of (4) where the coefficients of $P_n(x)$ are obtained from the nth row of the triangle.

3. An expression for $K_n(x)$. An analytic expression for $K_n(x)$ may be developed by induction with respect to n. An expression for $f_r(n)$ is obtained and substituted in (6). Then $P_n(x)$ from (6) is substituted in (4) for $K_n(x)$.

Thus, repeated application of (7) yields

$$f_r(n) = \sum_{i=1}^{n-r+1} r^{i-1}(n-r-i+2)f_{r-1}(n-i). \tag{8}$$

With the initial fact $f_1(n)=1$ we have by repeated application of (8):

$$\begin{aligned}
f_1(n) &= 1^n \\
f_2(n) &= 2^n - (n+1)1^n \\
f_3(n) &= 3^n - (n+1)2^n + \frac{(n+1)n}{2!}1^n \\
f_4(n) &= 4^n - (n+1)3^n + \frac{(n+1)n}{2!}2^n - \frac{(n+1)n(n-1)}{3!}1^n.
\end{aligned}$$

Now an intelligent conjecture for $f_r(n)$ evidently is

$$f_r(n) = \sum_{m=1}^{r} (-1)^{m+1}\binom{n+1}{m-1}(r-m+1)^n.$$

This conjecture is proved by first showing it satisfies (7) and then showing $f_1(n)=f_n(n)=1$. Hence

$$K_n(x) = \sum_{k=1}^{\infty} k^n x^k = \frac{\sum_{r=1}^{n}\left[\sum_{m=1}^{r}(-1)^{m+1}\binom{n+1}{m-1}(r-m+1)^n\right]x^r}{(1-x)^{n+1}},$$

valid for $|x|<1$.

MATHEMATICAL NOTES

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ON A LEMMA OF STIELTJES ON MATRICES

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In an article published in Acta Mathematica in 1886,* Stieltjes gave as a lemma a theorem amounting to the following:

THEOREM 1. *If the elements off the principal diagonal of the symmetric matrix of a positive definite quadratic form are all negative, then all the elements of the inverse of that matrix are positive.*

An extension of Stieltjes' remarks to include not necessarily symmetric matrices has been made in a Cowles Commission Monograph** by Jacob L. Mosak in a lemma which amounts to the following:

THEOREM 2. *If all the principal minors of a matrix are positive and all the elements off its main diagonal are negative then all the elements of its inverse are positive.*

It is clear that Theorem 1 is included in Theorem 2. In the proof of Theorem 1, Stieltjes used the special theory of positive definite quadratic forms, while in the proof of Theorem 2, Mosak uses a purely arithmetic method founded upon complete induction. Mosak's proof is straight-forward, but is susceptible of simplification.

Consider the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \cdot \; \cdot & \cdot \; \cdot \; \cdot \; \cdot \; \cdot \\ a_{n1} & a_{n2} \cdots a_{nn} \end{bmatrix}$$

satisfying the conditions of Theorem 2. It is to be proved that its inverse has all positive elements. This will be proved if it is proved that the cofactor of a_{ij} is positive; therefore, the following method of attack will be used. It will be assumed that the cofactors of all the elements are positive for any matrix of

* T. J. Stieltjes, Sur les racines de l'équation $X_n=0$, Acta Mathematica 9, 385–400 (1886). Reprinted in Oeuvres Complètes de Thomas Jan Stieltjes vol. 2, 73–88. Groningen, P. Noordhoff (1918).

** J. L. Mosak, General Equilibrium Theory in International Trade. Cowles Commission Monograph Number Seven. Bloomington, Ind., Principia Press (1944) pp. 49–51.

degree $n-1$ satisfying the conditions. It will be proved that then the theorem is true for matrices of degree n satisfying the conditions. Inasmuch as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix},$$

the theorem is true for matrices of degree 2 satisfying the conditions, and hence it will follow that the theorem is true for all n.

Let A_{ij} be the cofactor of a_{ij} in A (with $i \neq j$). Then

$$A_{ij} = (-1)^{i+j} \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}' = (-1)^{i+j} B_{ij},$$

the prime indicating that the ith row and jth column are omitted from the determinant of A to yield the determinant B_{ij}.

Consider $B^{(j)}$, a matrix of degree $n-1$ which is obtained from A by omitting the jth row and jth column of A. This matrix satisfies the conditions of Theorem 2; and by hypothesis, since it is of degree $n-1$, the cofactors of all its elements are positive.

In $B^{(j)}$ consider the minors of the elements in the row whose elements in the notation used for the elements of the matrix A have first subscript i. If $i>j$ this is the $(i-1)$st row of $B^{(j)}$; if $i<j$ it is the ith row of $B^{(j)}$. The cofactors of the elements of this row are by hypothesis all positive. If the minor of the kth element in this row is denoted by M_k ,we have

$$\operatorname{sgn} (-1)^{i+k} \ M_k = 1, \qquad \text{if} \quad i < j;$$
$$\operatorname{sgn} (-1)^{i+k-1} M_k = 1, \qquad \text{if} \quad i > j.$$

M_k is the $(n-2)$-rowed minor obtained by suppressing the ith and jth rows and the jth and kth columns of A.

In B_{ij} consider the row whose elements have first subscript j. If $i<j$ this is the $(j-1)$st row while if $i>j$ this is the jth row. The minor of a_{jk} in B_{ij} is M_k since it is the $(n-2)$-rowed minor obtained by suppressing the ith and jth rows and the jth and the kth columns of A. Consequently the sign of the cofactor of of a_{jk} in B_{ij} is $(-1)^{j+k-1}$ sgn M_k if $i<j$ and $(-1)^{j+k}$ sgn M_k if $i>j$. In the former case sgn M_k is $(-1)^{i+k}$ and in the latter $(-1)^{i+k-1}$. Hence, in all cases, the sign of the cofactor of a_{jk} in B_{ij} is $(-1)^{i+j+2k-1}$. Since $j \neq k$ for any k, sgn $a_{jk} = -1$. Hence the sign of the product of a_{jk} and its cofactor in B_{ij} is $(-1)^{i+j+2k-2} = (-1)^{i+j}$. Hence the sign of B_{ij} which is a sum of these like-signed products as can be seen by expanding it by the minors of the row which has first subscript j, is $(-1)^{i+j}$. Hence the sign of A_{ij} is $(-1)^{2i+2j}$; that is, A_{ij} is positive.

ON THE DENSEST PACKING OF SPHERICAL CAPS

L. FEJES TÓTH, Budapest, Hungary

In the present note we shall give a new proof of the following result.

THEOREM [1]. *From $n>2$ given points of the surface of the unit sphere there always can be found two, having a spherical distance*

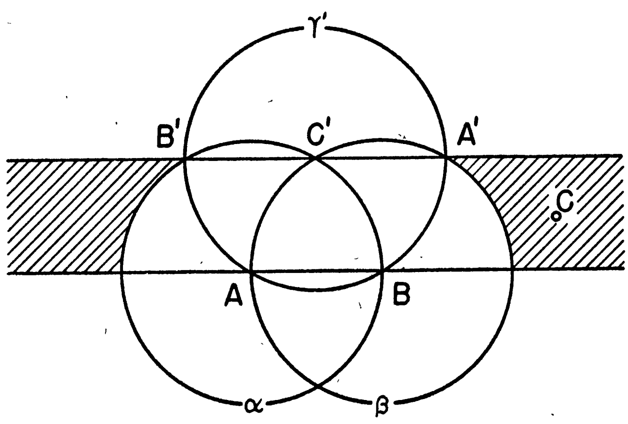
$$(1) \qquad d \leqq \arccos \frac{\cot^2 \omega - 1}{2}, \qquad \omega = \frac{n}{n-2} \frac{\pi}{6}.$$

The inequality (1) can not be improved for $n=3, 4, 6, 12$ [2] and it gives an exact asymptotic estimate for large values of n.

The following inequality, equivalent to (1), improves certain results of A. Thue [3] concerning the densest packing of circles in the plane:

Consider $n>2$ congruent spherical caps of the unit sphere such that no two of them overlap. If f denotes the area of a cap then

$$\frac{nf}{4\pi} - \frac{n}{2}\left(1 - \frac{1}{2}\sin^{-1}\omega\right) < \frac{\sqrt{3}\,\pi}{6}.$$



The proof of statement (1) rests on the following lemma.

LEMMA. *If the area of a spherical triangle ABC is less than the area of the equilateral spherical triangle ABC' drawn upon the shortest side AB of ABC then the spherical radius of the circle circumscribed to ABC is greater than AB.*

For suppose that C' lies on the same side of the great circle AB as C. Consider the congruent circles α, β, γ' of radius AB having the center, A, B, C', respectively. Let A' be the point of intersection of β and γ' different from A, and B' the point of intersection of α and γ' different from B.

Since the triangles ABA', ABB', ABC' have the same area, the arc of the circle through A', B', C' lying "above" the great circle AB is the locus of the vertices C^* of the triangles ABC^* the area of which remains invariant.

C lies, by supposition, in the domain bordered by the above Lexell-circle $A'B'C'$ and the great circle AB. Since, on the other hand, C lies outside α and β, the point C does lie without γ'. This proves the lemma.

Let us now turn to the proof of the inequality (1).

The case $n=3$ being trivial, we can restrict ourself to the case $n \geqq 4$. Obviously we may suppose that the points $P_1, P_2, \cdots, P_n$ lie not all on a hemisphere and thus the convex hull Π of the points contains the center O of the sphere S. Furthermore we may suppose that the polyhedron Π has only triangular faces since otherwise we could decompose them into triangles. The number of the faces of Π is then $2n-4$.

Consider the spherical net N arising by projection of the edges of Π from O upon S. Suppose that—contrary to (1)—

$$P_iP_j > b; \qquad i, j = 1, 2, \cdots, n; \; i \neq j,$$

where we denote by b the bound on the right in (1). Since b equals—as a simple computation shows—the length of the side of an equilateral spherical triangle of area $4\pi/(2n-4)$, the triangle $P_iP_jP_k$ having the least area among the $2n-4$ triangles determined by N satisfies the condition of our lemma.

Let us denote the circle circumscribed to $P_iP_jP_k$ by γ and the (spherical) center of γ by P_{n+1}. Since the spherical cap bordered by γ contains no point of $P_1, \cdots, P_n$, and the radius of γ is by our lemma greater than b, we can complete the point system $P_1, \cdots, P_n$ by P_{n+1} without loss of the property that any two points of the system have a spherical distance greater than b.

This proceeding can be continued. But since the number ν of the points of a system having the above property is obviously bounded [4] we arrive in a finite number of steps to a contradiction. This completes the proof.

Comments

1. See my paper Über eine Abschätzung des kürzesten Abstandes zweier Punkte eines auf einer Kugelfläche liegenden Punktsystems, Jahresbericht d. D.M.V., vol. 53 (1943), pp. 66–68. According to the kind information of Professor H. Hadwiger, he also found the inequality (1) and presented it in a lecture at the Mathematical Seminary of the University of Bern in the winter semester 1942–43.

2. The extremal distribution of the points is determined, for these values of n, by the vertices of an equilateral triangle inscribed in a great circle, of a regular tetrahedron, of an octahedron or of an icosahedron, respectively.

3. Cf., for instance, the paper of B. Segre and K. Mahler, On the densest packing of circles, this MONTHLY, vol. 51 (1944), pp. 261–270.

4. Consider the spherical caps of radius $b/2$ having the top points $P_1, P_2, \cdots, P_\nu$. From the fact that the sum of the areas of these caps is less than 4π, we obtain, for instance, the bound $\nu < \sin_\mu^2 (b/4)$.

THE ADJOINT OF EULER'S LINEAR DIFFERENTIAL OPERATOR

H. J. ZIMMERBERG, Rutgers University

Rainville [this MONTHLY, vol. 46 (1939), pp. 623–627] has defined an adjoint operator as applied to linear differential operators and has investigated its properties. In this note we consider the special case of Euler's linear differential operator and exhibit an interesting form for its adjoint. Further related results are also derived which are utilized to solve certain exercises involving differential operators.

Let $D\equiv d/dx$, $D^n\equiv d^n/dx^n$ $(n=1, 2, \cdots)$, $D^0\equiv 1$, be the customary symbols for denoting the first, nth, and 0th differential operator, respectively. A linear differential operator is then defined as any linear combination of terms of the type $p_n(x)D^n$ $(n=0, 1, 2, \cdots)$, where $p_n(x)$ is an arbitrary function of x having as many derivatives as are necessary for the purposes of our discussion. The *adjoint operator* α acting upon linear differential operators is defined as a linear operator, that is,

$$\alpha\left(\sum_s p_s(x)D^s\right) = \sum_s \alpha[p_s(x)D^s],$$

$$\alpha[c_np_n(x)D^n] = c_n\alpha[p_n(x)D^n], \qquad c_n \text{ constant},$$

such that

$$\alpha[p_n(x)D^n] = (-1)^nD^np_n(x) \qquad (n = 0, 1, 2, \cdots). \tag{1}$$

It is to be emphasized that the right-hand member of (1) above is to be interpreted as an operator and *not* as $(-1)^n$ multiplied by the nth derivative of $p_n(x)$. For example, by D^2x^2 operating on a function of x, say $F(x)$, we shall understand that the operator D^2 is acting upon $x^2F(x)$, and the result is therefore equivalent to that obtained when the operator $x^2D^2+4xD+2$ acts upon F. *A differential operator followed by a function of x is to be interpreted as an operator in this manner throughout this note.*

For Euler's differential operator $\mathfrak{D}\equiv xD\equiv x(d/dx)$ we have the familiar relations

$$x^nD^n = \mathfrak{D}(\mathfrak{D}-1)(\mathfrak{D}-2)\cdots(\mathfrak{D}-n+1) \qquad (n = 1, 2, \cdots). \tag{2}$$

The following two lemmas are readily verified by direct evaluation.

LEMMA 1. *The linear differential operator* $\mathfrak{D}+c$, *c constant, permutes with an operator of similar type*; i.e., $(\mathfrak{D}+a)$ $(\mathfrak{D}+b)=(\mathfrak{D}+b)$ $(\mathfrak{D}+a)$ *for arbitrary constants a and b.*

LEMMA 2. $\alpha[\mathfrak{D}]=-(\mathfrak{D}+1)$.

For convenience, we merely restate the main result of Rainville's paper.

THEOREM 1. *If A and B are arbitrary linear differential operators, then* $\alpha[AB] = (\alpha[B])(\alpha[A])$.

THEOREM 2. *If n is a nonnegative integer then*

$$\alpha[p(x)x^nD^n] = (-1)^n(\mathfrak{D}+1)(\mathfrak{D}+2)\cdots(\mathfrak{D}+n)p(x).$$

From the relations (2) above, the repeated application of Theorem 1 of Rainville, Lemma 2 above, and the linearity of the adjoint operator, we have

$$\begin{aligned}\alpha[p(x)x^nD^n] &= \alpha[p(x)\mathfrak{D}(\mathfrak{D}-1)\cdots(\mathfrak{D}-n+1)]\\ &= (\alpha[\mathfrak{D}-n+1])\cdots(\alpha[\mathfrak{D}-1])(\alpha[\mathfrak{D}])p(x)\\ &= (-1)^n(\mathfrak{D}+n)\cdots(\mathfrak{D}+2)(\mathfrak{D}+1)p(x).\end{aligned}$$

The desired result then follows in view of Lemma 1.

COROLLARY 1. *For any nonnegative integer n we have*

$$D^nx^n = (\mathfrak{D}+1)(\mathfrak{D}+2)\cdots(\mathfrak{D}+n).$$

The above equality of differential operators is an immediate consequence of Theorem 2 above and the definition of the adjoint (1). The next corollary then follows in view of Lemma 1 above.

COROLLARY 2. *For nonnegative integers n and r the differential operators D^nx^n and $\mathfrak{D}^r$ permute.*

This result can be employed to give a simple proof of the equality of the differential operators

$$D^n(\mathfrak{D}-n)^r = \mathfrak{D}^rD^n \qquad (n, r = 0, 1, 2, \cdots), \tag{3}$$

which is equivalent to problem 9 (iii) on page 86 of Forsythe, *A Treatise on Differential Equations*. By direct computation, we have $(\mathfrak{D}-n)x^n = x^n\mathfrak{D}$, and, by repeated application, $(\mathfrak{D}-n)^rx^n = x^n\mathfrak{D}^r$. Consequently, $D^n(\mathfrak{D}-n)^rx^n = D^nx^n\mathfrak{D}^r = \mathfrak{D}^rD^nx^n$, the latter equality holding in view of Corollary 2 above. As $D^n(\mathfrak{D}-n)^r$ operating on $F(x)$, an arbitrary function of x, is equivalent to $D^n(\mathfrak{D}-n)^rx^n$ acting upon $x^{-n}F(x)$ we have the desired relation (3).

COROLLARY 3. *For nonnegative integers m and n the differential operators D^mx^m and x^nD^n permute.*

This result follows at once from Corollary 1 above, relation (2), and Lemma 1 above. As an immediate consequence, we have that, for nonnegative integers m, n and r,

$$D^mx^{m+r}D^rx^{-m}D^{n-r} = x^rD^{m+r}x^mx^{-m}D^{n-r} = x^rD^{m+n}$$

(see problem 9 (ii) on page 86 of Forsythe). One may easily devise other problems which are readily solvable by the permutability relations above.

THEOREM 3. *If n is a nonnegative integer then*

$$\alpha[p(x)\mathcal{D}^n] = (-1)^n(\mathcal{D}+1)^n p(x).$$

This result follows at once by a repeated application of Theorem 1 of Rainville and Lemma 2 above.

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College and Institute for Advanced Study

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FINDING THE EQUATION OF THE CIRCLE THROUGH THREE POINTS

HANAN RUBIN, New York University

In an elementary course in analytic geometry, the problem of finding the equation of the circle passing through three given points is usually solved by three methods; namely:

a) Solving three linear equations for the three essential constants in the equation of a circle.

b) Finding the center of the circle by finding the intersection of the perpendicular bisectors of two of the chords determined by the three given points.

c) Using a determinant.

The first two methods involve a considerable amount of calculation for such a simple problem, and the third method gives the answer in an inconvenient form. However, the equation of the required circle can be obtained rapidly in a usable form with the help of the formula discussed below.

The equation of the circle passing through the points (x_1, y_1), (x_2, y_2), and (x_3, y_3) is

$$\begin{aligned}(x-x_1)(x-x_2)&+(y-y_1)(y-y_2)\\ &+k[(x_2-x_1)(y-y_1)-(y_2-y_1)(x-x_1)]=0\end{aligned} \tag{1}$$

where k is a constant chosen so that (x_3, y_3) should satisfy the equation. Inspection of (1) reveals trivially that it represents a circle and that (x_1, y_1) and (x_2, y_2) satisfy the equation for all values of k. It is also to be observed that it is a simple matter to apply this formula to particular examples.

The formula can be motivated as well as remembered by noting that

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0 \tag{2}$$

represents a circle through the points (x_1, y_1) and (x_2, y_2) and that

$$(x_2-x_1)(y-y_1)-(y_2-y_1)(x-x_1)=0$$

represents the straight line through the points (x_1, y_1) and (x_2, y_2). It is obvious then that (1) represents the family of circles passing through the points (x_1, y_1) and (x_2, y_2).

A special case occurs when the three points through which the circle is to pass involve only two different abscissas and two different ordinates. For example, the equation of the circle passing through the points (x_1, y_1), (x_2, y_2), and (x_1, y_2) is given by (2) or by (1) with $k=0$. In this case, the points (x_1, y_1) and (x_2, y_2) are the end-points of a diameter and the point (x_2, y_1) automatically lies on the circle.

As a generalization of the formula to solid analytic geometry, the equation of the sphere passing through the points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4) is

$$\begin{aligned}(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) \\ + k_1[(y_2 - y_1)(z - z_1) - (z_2 - z_1)(y - y_1)] \\ + k_2[(z_2 - z_1)(x - x_1) - (x_2 - x_1)(z - z_1)] \\ = 0\end{aligned}$$

where k_1 and k_2 are constants chosen so that (x_3, y_3, z_3) and (x_4, y_4, z_4) should satisfy the equation. Or, in vector notation, the equation of the sphere is

$$(X - X^1)\cdot(X - X^2) + K\cdot(X^2 - X^1) \times (X - X^1) = 0$$

where

$$\begin{aligned}X &= (x, y, z)\\ X^1 &= (x_1, y_1, z_1)\\ X^2 &= (x_2, y_2, z_2)\\ K &= (k_1, k_2, 0).\end{aligned}$$

This form of writing the equation seems to bring out in a new light the fact that X^1 and X^2 satisfy the equation for all values of K.

The formula can be generalized to n dimensions, but its value decreases rapidly as n increases since it merely reduces by two the number of constants to be determined. However, in the case of the circle, the use of the formula provides a significant saving in calculation.

"INTEGRATION BY PARTS" AS A METHOD IN THE SOLUTION OF EXACT DIFFERENTIAL EQUATIONS

C. R. PHELPS, Rutgers University

The method of "integration by parts," well-known to the student of the calculus, can be used to advantage in the solution of a large class of exact differential equations. For example, let us take the equation

$$(1) \qquad (x^3 + 4xy - 1)dy + (2x + 3x^2y + 2y^2)dx = 0,$$

which is easily shown to be exact by the usual test.* Our method consists of integrating (1) term-by-term, using "parts" whenever more than one variable is involved. Since $\int x^3dy=x^3y-\int 3x^2ydx$ and $\int 4xydy=\int(2x)(2ydy)=2xy^2-\int 2y^2dx$, equation (1) integrates to

$$x^3y-\int 3x^2ydx+2xy^2-\int 2y^2dx-y+x^2+\int 3x^2ydx+\int 2y^2dx=C;$$

the remaining integrals cancel, giving the solution

$$x^3y+2xy^2-y+x^2=C.$$

It is to be noted that this method calls upon the student's previous experience, rather than upon the (to him) hazy notion of integration leaving one of the variables fixed.

The general situation for first-order exact differential equations can be stated and proved as follows: *If the differential equation $Mdx+Ndy=0$ is exact, and if N (or, equivalently, M) is expressible as a sum $\sum_{i=1}^n a_i(x)b_i(y)$ of separated arbitrary (differentiable and integrable) functions $a_i(x)$ and $b_i(y)$, then we may integrate by parts.*

Proof: we integrate each summand of N by parts: $\int a_i(x)[b_i(y)dy]=a_i(x)h_i(y)-\int a_i'(x)h_i(y)dx$, where $h_i(y)=\int b_i(y)dy$. Also, since the equation is exact, $\partial M/\partial y=\partial N/\partial x=\sum a_i(x)b_i(y)$; thus $M=\sum a_i'(x)h_i(y)+c(x)$. Consequently, integrating the equation $Ndy+Mdx=0$ gives us $\sum a_i(x)h_i(y)-\sum\int a_i'(x)h_i(y)dx+\sum\int a_i'(x)h_i(y)dx+\int c(x)dx=C$, or $\sum a_i(x)h_i(y)+f(x)=C$, where $f(x)=\int c(x)dx$.

Specific criteria for the exactness of a second order exact differential equation may be obtained by this same method. A differential equation of the second order is defined as exact if it is formed by equating to zero the precise derivative of some function of x, y, and y'; thus it is evident that y'' can occur only linearly. We merely must assume in addition that y' occurs only to integral powers; therefore we have the general form

$$\left[\sum_{i=0}^{n} A_i(x,y)(y')^i\right]y''+\sum_{j=0}^{m} B_j(x,y)(y')^j=0. \tag{2}$$

For the first set of terms, we use integration by parts in the form $uv'=(uv)'-vu'$, letting $u_i=A_i$ and $v_i'=(y')^iy''$. Then (2) becomes

$$\left(\sum_{i=0}^{n}\frac{A_i(x,y)(y')^{i+1}}{i+1}\right)'-\sum_{i=0}^{n}\frac{(y')^{i+1}}{i+1}\left[\frac{\partial A_i}{\partial x}+\frac{\partial A_i}{\partial y}y'\right]+\sum_{j=0}^{m}B_j(x,y)(y')^j=0.$$

Since the first term is exact, the remainder must be also; but a first order exact equation is necessarily linear in y'; hence the terms of higher degree in y' must

* See any text; for example, Cohen, "*Differential Equations*," N.Y. 1933, pp. 22–29 and 162–164.

vanish, and we have remaining only the terms

$$-\frac{\partial A_0}{\partial x}y' + B_0 + B_1y' = 0. \tag{3}$$

The necessary and sufficient conditions for exactness of (2) thus consist of the exactness condition for (3) and the conditions for the vanishing of the higher powers:

$$\frac{\partial}{\partial x}\left(B_1 - \frac{\partial A_0}{\partial x}\right) = \frac{\partial B_0}{\partial y}; \tag{E_1}$$

$$\frac{1}{2}\frac{\partial A_1}{\partial x} + \frac{\partial A_0}{\partial y} = B_2; \tag{E_2}$$

$$\cdots$$

$$\frac{1}{k}\frac{\partial A_{k-1}}{\partial x} + \frac{1}{k-1}\frac{\partial A_{k-2}}{\partial y} = B_k, \qquad 2 \leqq k \leqq \max(m, n+2); \tag{E_k}$$

where we let $A_i=0$ and $B_j=0$ for $i>n$ and $j>m$.

As a practical matter, if the above *process* is carried out at the outset, both the question of exactness and the solution if exact will be determined simultaneously—the former merely by whether the higher powers of y' cancel and whether the remaining first order equation is exact (determined as in the first paragraph by a repetition of the process).

For an example, let us consider the equation

$$(x^2y + 2x^3y')y'' + 4x^2y'^2 + 2xyy' = 0. \tag{4}$$

Here $A_0=x^2y$, $A_1=2x^3$, $n=1$; $B_0=0$, $B_1=2xy$, $B_2=4x^2$, $m=2$. It is easily checked that conditions E_1, E_2, E_3 are satisfied. We have $\int(x^2y)y''dx=x^2yy'-\int(x^2y'+2xy)y'dx$ and $\int(x^3)(2y'y''dx)=x^3y'^2-\int 3x^2y'^2dx$ so that, upon multiplying by dx and integrating, (4) becomes $x^2yy'-\int x^2y'^2dx-\int 2xyy'dx+x^3y'^2-3\int x^2y'^2dx+4\int x^2y'^2dx+\int 2xyy'dx=C$ or $x^2yy'+x^3y'^2=C$. (This equation, solvable for y, has the solution $xy+C_1x+C_2=0$; see Cohen, p. 79, ex. 9.)

Two special cases of (2) seem worthy of mention. First, the equation $P_0(x)y''+P_1(x)y'+P_2(x)y=0$, treated in many texts, is exact if and only if $P_0''-P_1'+P_2=0$, a direct corollary of (E_1). Also, the equation $F(x, y)y''+G(x, y)y'+H(x, y)=0$ is exact if and only if (E_1) $\partial G/\partial x-\partial^2F/\partial x^2=\partial H/\partial y$ and (E_2) F is a function of x only!

This method of integration by parts is easily extended both to exact equations of order higher than the second, and to the total differential equation $Pdx+Qdy+Rdz=0$ when exact.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 866. *Proposed by L. J. Burton, Bryn Mawr College*

Players A and B take turns, beginning with A, each marking a previously unmarked unit line segment joining any two points with integral coordinates at a unit distance in a plane.

(a) Prove that B can prevent A from ever marking all the line segments in the perimeter of any closed polygon.

(b) Prove that if P_1: (x_1, y_1) and P_2: (x_2, y_2) are any two fixed points with integral coordinates such that $|P_1P_2| > 1$, then B can prevent A from joining P_1 and P_2 by a broken line consisting of segments marked by A.

E 867. *Proposed by Walter Fleming, Fort Hays Kansas State College*

Find

$$\lim_{n\to\infty} n^{-p} \sum_{j=1}^{n} j^{p-1}.$$

E 868. *Proposed by P. D. Thomas, Washington, D. C.*

Let P and Q be, respectively, the feet of the common perpendicular to two fixed skew lines p and q. A variable line r meets p in R and q in S. Find the locus of r if the volume of the tetrahedron $PQRS$ is constant. Also find the locus of the centroid of $PQRS$.

E 869. *Proposed by P. T. Bateman, Institute for Advanced Study*

If a polynomial $f(x)$ with integral coefficients has the property that $f(n)$ is a perfect square for all integers n, then $f(x)$ is the square of another polynomial with integral coefficients.

E 870. *Proposed by Joseph Rosenbaum, Hartford, Conn.*

Characterize quadrilaterals $A_1B_1C_1D_1$ such that if A_2, B_2, C_2, D_2 are the circumcenters of $A_1B_1C_1$, $B_1C_1D_1$, $C_1D_1A_1$, $D_1A_1B_1$, then A_1, B_1, C_1, D_1 are the circumcenters of $A_2B_2C_2$, $B_2C_2D_2$, $C_2D_2A_2$, $D_2A_2B_2$.

An Arithmetic-Geometric Series

E 837 [1948, 576]. *Proposed by Roy Dubisch, Fresno State College*

Find the sum and interval of convergence of the series

$$\sum_{n=0}^{\infty} (a + nd)x^n.$$

I. *Solution by the Proposer.* The ratio test shows that the interval of convergence is $-1<x<1$. To find the sum consider

$$S_n - xS_n = a + d(x + x^2 + \cdots + x^n) - (a + nd)x^{n+1},$$

so that

$$S_n = \frac{(a - ax + dx) - (a + d + nd)x^{n+1} + (a + nd)x^{n+2}}{(1 - x)^2}.$$

Hence

$$S = \lim_{n\to\infty} S_n = (a - ax + dx)/(1 - x)^2 \qquad \text{for } |x| < 1$$

II. *Solution by W. Fulks, University of Minnesota.* It is evident that

$$\sum_{n=0}^{\infty} (a + nd)x^n = a\sum_{n=0}^{\infty} x^n + xd\sum_{n=0}^{\infty} nx^{n-1}$$

within the common interval of convergence of the two series on the right. But

$$\sum_{n=0}^{\infty} x^n = 1/(1 - x), \qquad |x| < 1.$$

Differentiating we get

$$\sum_{n=0}^{\infty} nx^{n-1} = 1/(1 - x)^2, \qquad |x| < 1.$$

Hence, for $|x| <1$, we have

$$\sum_{n=0}^{\infty} (a + nd)x^n = a/(1 - x) + xd/(1 - x)^2 = [a + x(d - a)]/(1 - x)^2.$$

The interval of convergence is clearly $-1<x<1$, since for $x=\pm 1$ the terms of the series numerically approach infinity.

Also solved by P. R. Beesack, A. R. Brown, Jr., L. J. Burton, Richard Courter, Maurice Dunn, Ragnar Dybvik, B. K. Gold, R. T. Hood, J. M. Kingston, Frank Kocher, Roger Lessard, Julius Lieblein, H. D. Lipsich, D. C. B.

Marsh, Leo Moser, C. S. Ogilvy, H. Orlin, S. T. Parker, C. F. Pinzka, C. M. Sandwick, Robert Stalley, W. R. Talbot, and C. W. Trigg.

The formula for S_n given in solution I agrees with a result given by C. W. Trigg, Problem 1357, *School Science and Mathematics*, Jan. 1935, p. 95. An application of this formula for S_n is

$$1 + 3\cdot 2 + 5\cdot 2^2 + \cdots + (2n-1)2^{n-1} = 3 - 2^n + (n-1)2^{n+1},$$

which is Problem 2822 [1921, 284]. For an application of the formula for S see Problem 400 [1914, 158]. Moser pointed out that the formulas for both S and S_n appear in Hall and Knight, *Higher Algebra*, p. 44.

Stalley, in his Master's Thesis (note his paper in this issue of the MONTHLY), found an expression for

$$S(x, m) = \sum_{k=1}^{\infty} k^m x^k.$$

Now

$$\sum_{k=0}^{\infty} (a + kd)x^k = a\sum_{k=0}^{\infty} x^k + d\sum_{k=0}^{\infty} kx^k = a + S(x, 0) + dS(x, 1).$$

Associated Polynomial Curves

E 838 [1948, 576]. *Proposed by H. T. R. Aude, Colgate University*

By a translation of axes and a suitable choice of scale of the ordinates the general polynomial equation of the fifth degree can be taken in the form

$$y = x^5 + ax^3 + bx^2 + cx + d.$$

Assume that there exist four bend points. Find the polynomial cubic equation which is satisfied by these four points. Also find the equation of the parabola with axis parallel to the y-axis which passes through the three points of inflection.

Solution by S. T. Thompson, Tacoma, Washington. We shall use the general, and obvious, theorem: "If $f(x)$, $g(x)$, $q(x)$, $r(x)$ are polynomials such that

$$f(x) = g(x)q(x) + r(x),$$

then the curve $y = r(x)$ passes through the points on $y = f(x)$ having for abscissas the roots of $g(x)$."

Since the abscissas of the bend points of $y = f(x)$ are roots of $f'(x)$, and the flexes of $y = f(x)$ are roots of $f''(x)$, we take $f(x)$ as the given quintic and let the divisor $g(x)$ be $f'(x)$ and $f''(x)$ in turn. Calculating the corresponding remainders $y = r(x)$ we find, respectively,

$$5y = 2ax^3 + 3bx + 4cx + d$$

and

$$100y = 90bx^2 + (100c - 21a^2)x + (100d - 7ab).$$

These are the sought curves. The bend points and the flexes of $y=f(x)$ need not be real.

It is easy to obtain results applying to a polynomial $f(x)$ of general degree n.

Also solved by L. J. Burton, R. T. Hood, Roger Lessard, C. M. Sandwick, C. W. Trigg, and the proposer.

A Palindromic Expression

E 839 [1948, 576]. *Proposed by E. D. Schell, Office of the Comptroller, United States Air Forces*

Given $S_n = n\cdot 1+(n-1)2+\cdots+2(n-1)+1\cdot n$. Show that $S_n = C(n+2, 3)$.

Solution by Philip Anselone, College of Puget Sound. Evidently

$$\begin{aligned} S_n &= \sum_{k=1}^{n}(n-k+1)k = (n+1)\sum k - \sum k^2 \\ &= n(n+1)^2/2 - n(n+1)(2n+1)/6 = n(n+1)(n+2)/6 = C(n+2, 3). \end{aligned}$$

Also solved by P. R. Beesack, A. R. Brown Jr., D. H. Browne, L. J. Burton. Richard Courter, R. E. Crane, Ragnar Dybvik, W. Fulks, B. K. Gold, J. F. Heyda, R. T. Hood, S. J. Jasper, J. M. Kingston, H. D. Larsen, R. S. Lehman, Roger Lessard, Julius Lieblein, H. D. Lipsich, W. R. McEwen, Leo Moser, Z. I. Mosesson, C. R. Newell, C. S. Ogilvy, S. T. Parker, C. F. Pinzka, C. M. Sandwick, N. C. Scholomiti, Joan Snapper, W. M. Stone, W. R. Talbot, C. W. Trigg, E. W. Trost, W. R. Van Voorhis, and the proposer.

Trigg found the sum of the more general series formed from any arithmetic progression and its palindrome multiplied term by term:

$$\begin{aligned} T_n &= a[a+(n-1)d] + (a+d)[a+(n-2)d] + \cdots + [a+(n-1)d]a \\ &= adn^2 + an(a-d) + d^2C(n, 3). \end{aligned}$$

If $a=d=1$, then $T_n = S_n$.

As another generalization Trigg showed that

$$\begin{aligned} S_{n,k} &= (1)(2)\cdots(k)(n)(n-1)\cdots(n-k+1) \\ &\quad + (2)(3)\cdots(k+1)(n-1)(n-2)\cdots(n-k) + \cdots \\ &\quad + (n-k+1)\cdots(n-1)(n)(k)\cdots(2)(1) \\ &= (k!)^2C(n+k+1, 2k+1). \end{aligned}$$

Here $S_{n,1} \equiv S_n$.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results found in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4346. *Proposed by N. S. Mendelsohn, University of Manitoba*

Prove that

$$n - 1 = \sum_{r=1}^{\infty} \left[\frac{n + 2^{r-1} - 1}{2^r} \right],$$

for any positive integer n. The brackets denote, as usual, the greatest integer function.

4347. *Proposed by Paul Erdös, Syracuse University*

If m and n are integers satisfying

$$\left(1 - \frac{1}{m}\right)^n > \frac{1}{2}, \qquad \left(1 - \frac{1}{m-1}\right)^n < \frac{1}{2},$$

prove the relations

$$(m-1)^n > (m-2)^n + (m-3)^n + \cdots + 1^n.$$
$$(m+1)^n < m^n + (m-1)^n + \cdots + 1^n,$$

Show also that the inequality

$$m^n > (m-1)^n + (m-2)^n + \cdots + 1^n$$

is true in infinitely many instances, but it is also untrue in infinitely many instances.

4348. *Proposed by D. A. Darling, Rutgers University*

This problem was brought from Poland by Professor H. Steinhaus. It appears that Professor Banach was accustomed to carrying a box of matches in each of two coat pockets. To light his pipe, he would take a match from either box at random. The boxes contained originally n matches each. Banach's question is: when first a box is opened and found empty, what is the expected number of matches left in the other box?

4349. *Proposed by H. F. Sandham, Trinity College, Dublin, Ireland*

Prove that

$$\frac{2}{1}\bigg/\frac{5}{4}\bigg/\frac{8}{7}\bigg/\frac{11}{10}\cdots = \sqrt{3}.$$

SOLUTIONS

Second Lemoine Point

4218 [1946, 471]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a tetrahedron $T \equiv ABCD$, if a point L with the normal coördinates (x, y, z, t) is such that its associates $(-x, y, z, t)$, $(x, -y, z, t)$, $(x, y, -z, t)$, $(x, y, z, -t)$ are on the circumsphere, it coincides with the point whose distances to the planes of the faces BCD, CDA, DAB, ABC are proportional to the radii of the circumcircles of these faces (second Lemoine point for T), and conversely.

Solution by the Proposer.† If T is chosen as the reference tetrahedron, with $a, b, c, a', b', c', (A), (B), (C), (D)$ denoting respectively the lengths of the edges BC, CA, AB, DA, DB, DC and the areas of the faces BCD, CDA, DAB, ABC, then the equation in normal coördinates of the circumsphere of T is given by*

$$S(x, y, z, t) \equiv c^2(A)(B)xy + b^2(A)(C)xz + a'^2(A)(D)xt$$
$$+ a^2(B)(C)yz + b'^2(B)(D)yt + c'^2(C)(D)zt = 0.$$

If the point L has coördinates (x_1, y_1, z_1, t_1), the conditions imposed are:

$$S(-x_1, y_1, z_1, t_1) = S(x_1, -y_1, z_1, t_1) = S(x_1, y_1, -z_1, t_1)$$
$$= S(x_1, y_1, z_1, -t_1) = 0.$$

From the above we conclude that

$$c'^2(C)(D)z_1t_1 - c^2(A)(B)x_1y_1 = 0$$
$$a'^2(A)(D)x_1t_1 - a^2(B)(C)y_1z_1 = 0$$
$$b'^2(B)(D)y_1t_1 - b^2(A)(C)x_1z_1 = 0.$$

Eliminating x_1, t_1 we find $b^2c'^2(C)^2z_1^2 = b'^2c^2(B)^2y_1^2$, or, assuming x_1, y_1, z_1, t_1 are all positive, $bc'(C)z_1 = b'c(B)y_1$. If R_a, R_b, R_c, R_d are the radii of the circumcircles of the faces BCD, CDA, DAB, ABC, we have $(B) = a'bc'/4R_b$, $(C) = a'b'c/4R_c$. From these and similar calculations we conclude that to within a factor of proportionality (homogeneous coördinates)

$$L(x_1, y_1, z_1, t_1) \equiv (R_a, R_b, R_c, R_d).$$

L is called the second Lemoine point for T.

Conversely, the lines AL, BL, CL, DL meet the circumsphere of T again in the points $A'(-R_a, R_b, R_c, R_d)$, $B'(R_a, -R_b, R_c, R_d)$, $C'(R_a, R_b, -R_c, R_d)$,

† Translated by W. E. Byrne, Virginia Military Institute.

* Niewenglowski, *Cours de Géométrie Analytique*, t. III, 2nd ed., p. 109.

$D'(R_a, R_b, R_c, -R_d)$. This is easily proved by forming linear combinations of the coördinates of A and L, B and L, $\cdots$, and substituting them into $S(x, y, z, t)=0$.

If A_1, B_1, C_1, D_1 designate the points of intersection of AL and face BCD, BL and face CDA, CL and face DAB, DL and face ABC, respectively, we find that L and A' are harmonic conjugates with respect to A and A_1, etc. This follows immediately if we express the coördinates of A' and L as linear combinations of those of $A(R_a, 0, 0, 0)$ and $A_1(0, R_b, R_c, R_d)$.

The two tetrahedrons $ABCD$ and $A'B'C'D'$ have the same second Lemoine point. The line AA' meets the plane $B'C'D'$ in $A_1'(3R_a, R_b, R_c, R_d)$. We have only to express the coördinates of L and A as linear combinations of those of A' and A_1 to verify the above statement.

Further Property of the Second Lemoine Point

4224 [1946, 537]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a tetrahedron $ABCD$; (1) The right cones with vertices at the orthogonal projections of the second Lemoine point L on the axes of the circumcircles of the faces and with these circles as bases have the same base angle V (Brocard angle). (2) The symmedians AL, BL, CL, DL meet the circumsphere in the vertices of the tetrahedron $A'B'C'D'$ having the Brocard angle V and the same Lemoine point L as $ABCD$.

Dedicated to N. A. Court.

*Solution by R. Bouvaist, Vincelles, Saône-et-Loire, France.** [As the solution of problem 4218 by the proposer, given above, contains some of the results submitted by R. Bouvaist, only that part of the solution not involving a repetition is given here. The same notation is used as in the solution of 4218.]

Let O_a be the circumcenter of the face BCD of the tetrahedron $T\equiv ABCD$ and L_a be the orthogonal projection of the second point of Lemoine L of T on the axis of the circumcircle (O_a, R_a) of BCD. If V is the base angle of the cone of revolution $L_a(BCD)$, we have

$$L_aO_a = R_a \tan V.$$

Hence, the normal coördinates of L may be written as

$$x/R_a = y/R_b = z/R_c = t/R_d = \tan V. \tag{1}$$

The circles BCD, $B'C'D'$ are two antiparallel sections of the cone $L(BCD)$. If R_a and R_a' are the radii of these circles and δ and δ' the distances from L to their planes we have

$$\delta/R_a = \delta'/R_{a'} = \tan V.$$

By (1) the two tetrahedrons T, $T'\equiv A'B'C'D'$ have the same Brocard angle V.

It may be noted that the two tetrahedrons T, T' have the following properties:

1). The point L has the same polar plane with respect to T, T' and the com-

* Translated by W. E. Byrne, Virginia Military Institute.

mon circumsphere of these tetrahedrons.*

2). There exists a quadric (Q) tangent to the faces of T at A_1, B_1, C_1, D_1 and tangent to the faces of T' at A_1', B_1', C_1', D_1'.†

3). In a given sphere there may be inscribed an infinity of tetrahedrons with a given second Lemoine point L and a given Brocard angle V. The envelope of the faces of these tetrahedrons is an ellipsoid of revolution of axis OL.†

A Property of the Equilateral Hyperbola

4274 [1947, 550]. *Proposed by R. Bouvaist, Vincelles, Saône-et-Loire, France*

Let A, B, C, D be arbitrary points on an equilateral hyperbola (H), and let A', B', C', D' be the corresponding diametrically opposite points. (1) The isogonal conjugates of A', B', C', D' with respect to the triangles BCD, CDA, DAB, ABC, respectively, coincide in the same point P. (2) The isogonal conjugates of A, B, C, D with respect to the triangles $B'C'D'$, $C'D'A'$, $D'A'B'$, $A'B'C'$, respectively, coincide in the same point P'. (3) P and P' are diametrically opposite on (H).

Solution by Roscoe Woods, State University of Iowa. By a familiar property of the equilateral hyperbola, if A, B, R are three points on (H), and A' and B' are diametrically opposite A, B, respectively, then angle ARB and angle $B'RA'$ are equal. It is thus easily seen that, in the present problem, the lines isogonally conjugate to DA', DB', DC' with respect to the corresponding line pairs DB, DC; DA, DC; DA, DB coincide. Call this line L_4. Similarly through A, B, C there are lines L_1, L_2, L_3, respectively. Denote the point isogonally conjugate to D' with respect to the triangle ABC by P. Then the lines L_1, L_2, L_3 meet in P. Also the point isogonally conjugate to C' with respect to the triangle ABD is the same point P since it is determined by the intersection of the lines L_1, L_2, L_4; and similarly for the points A' and B'. This establishes (1).

Part (2) follows by symmetry since the entire configuration is merely reflected in the center O of (H). In regard to (3), although P and P' are symmetric with respect to O, they do not in general lie on (H). In fact, if A, B, C are held fixed, the locus of P for variable D' on (H) is a line L through the circumcenter of triangle ABC. (See Casey, *A Treatise on Analytic Geometry of the Point, Line, and Circle*, 2nd Ed., 1893, pp. 290, 291.) As D' runs over (H), P traces L. Since L cuts (H) at most twice, it happens only twice that P and P' fall on (H).

Also solved by Ou Li, and the Proposer.

Function with Prescribed Values on a Given Point Set

4275 [1947, 601]. *Proposed by Raymond Redheffer, Massachusetts Institute of Technology, Cambridge*

Let a_i represent any set of points in the complex plane, with sole limit point at infinity, while b_i are any complex numbers. Prove there exists an integral function $f(z)$ such that $f(a_i)=b_i$.

* V. Thébault, This MONTHLY, Problem 4223 [1948, 169].

† R. Bouvaist, Mathesis, t. LV, pp. 352–356.

Solution by Fritz Herzog, Michigan State College, Lansing. By Weierstrass' factor-theorem, assuming the a_i are distinct, there exists an integral function $g(z)$ which has a simple zero at each a_i. Consequently, $g'(a_i) \neq 0$ for all i. By Mittag-Leffler's partial-fractions-theorem, there exists a meromorphic function $h(z)$ which has at each a_i a simple pole with the residue $b_i/g'(a_i)$ and is regular otherwise. (In case b_i vanishes, $h(z)$ is to be regular at a_i.) The function $f(z) = g(z)\cdot h(z)$ is then an integral function for which $f(a_i) = b_i$ for all i.

Also solved by R. P. Boas, Jr., Hwang Cheng-Chung, P. Franklin, Edgar Reich, Helene Reschovsky, W. Seidel, O. Szasz, and Albert Wilansky.

Editorial Note. The theorem seems to be well known. Several references were given by solvers: K. Knopp, *Theory of Functions*, II, §3, ex. 3, and §5, ex. 4; Bieberbach, *Lehrbuch der Functiontheorie*, v. 1, (1923), pp. 290, 291; A. Pringsheim, *Vorlesungen über Zahlen- und Funktionenlehre*, II, 2 (1932), pp. 693–696; Riesz, *Les systems d'equations bineaires a un infinité des inconnus* (in connection with the discussion of Poincare's paradox); J. M. Whittaker, *Interpolatory Function Theory*, 1935, pp. 2, 3. A generalization in which the orders of the zeros are arbitrarily prescribed was given by Germay, *Bulletin de la Société Royale des Sciences de Liége* 14, 1945, pp. 476–478 (See Mathematical Reviews, v. 8, p. 508.)

Summation, Binomial Coefficients

4276 [1947, 601]. *Proposed by P. A. Pizá, San Juan, P. R.*

Let the integers ${}_nK_c$ be defined by the relations

$$ {}_nK_1 = 1; \qquad {}_nK_m = 0, \quad m > n; \qquad {}_{n+1}K_c = c({}_nK_c + {}_nK_{c-1}), \quad c > 1. $$

Prove the following summations:

$$ \text{(A)} \qquad x^n = \sum_{j=1}^{n} {}_nK_j \binom{x}{j}, $$

$$ \text{(B)} \qquad \sum_{a=1}^{x-1} a^n = \sum_{j=1}^{n} {}_nK_j \binom{x}{j+1}. $$

Solution by M. S. Klamkin, Brooklyn Polytechnic Institute, Brooklyn, N. Y. On the assumption that (A) is true, we have

$$ \begin{aligned} x^{n+1} &= \sum_{j=1}^{n} {}_nK_j \binom{x}{j} x = \sum_{j=1}^{n} {}_nK_j \left[(j+1)\binom{x}{j+1} + j\binom{x}{j} \right] \\ &= {}_nK_1 x + \sum_{j=2}^{n} \binom{x}{j} [j\cdot {}_nK_j + j\cdot {}_nK_{j-1}] + (n+1){}_nK_n \binom{x}{n+1} \\ &= {}_{n+1}K_1 x + \sum_{j=2}^{n} {}_{n+1}K_j \binom{x}{j} + {}_{n+1}K_{n+1} \binom{x}{n+1} = \sum_{j=1}^{n+1} {}_{n+1}K_j \binom{x}{j}. \end{aligned} $$

Since (A) is evidently true when $n=1$, it is true for all n by induction.

(B) follows immediately from (A) by use of the familiar relation

$$\sum_{a=j}^{x-1}\binom{a}{j}=\binom{x}{j+1}.$$

Thus

$$\sum_{a=1}^{x-1}a^n=\sum_{a=1}^{x-1}\sum_{j=1}^{n}{}_nK_j\binom{a}{j}=\sum_{j=1}^{n}{}_nK_j\sum_{a=1}^{x-1}\binom{a}{j}=\sum_{j=1}^{n}{}_nK_j\binom{x}{j+1}.$$

Solved also by H. W. Becker, Joseph Bram, E. T. Frankel, B. G. Lang, Yu-shu Luan, and Helene Reschovsky.

Frankel points out that (A) and (B) are special cases of general formulas in the calculus of finite differences which express the general term and the sum of a given number of terms of a rational integral function by means of its leading differences and binomial coefficients. (See Whittaker and Robinson, *The Calculus of Observations*, London, 1924, p. 7.) The integers ${}_nK_1$, ${}_nK_2$, $\cdots$, nK_c are the leading differences of the nth powers of the natural numbers 0^n, 1^n, 2^n, $\cdots$, c^n.

In consequence, as noted by Yu-shu Luan, we have the following explicit expression for ${}_nK_c$,

$${}_nK_c=\sum_{j=1}^{n}(-1)^j\binom{c}{j}(c-j)^n.$$

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Theory of Equations. By J. V. Uspensky. New York, McGraw-Hill Book Company, Inc., 1948. 7+353 pages. \$4.50.

When a new textbook in elementary mathematics is received, the recipient immediately wonders why the author has troubled himself with the task and whether he has actually contributed anything essentially different from the numerous texts already on the market. This book states in the preface that the book was written as "a textbook to be used in the standard American university and college courses devoted to the theory of equations. As such it is elementary in character and, with few exceptions, contains only material ordinarily included in texts of this kind. But the presentation is made so explicit that the

book can be studied by students without a teacher's help." The reviewer has tried to examine the book to find how well the above stated intention has been carried out.

Throughout, the author has been at great pains to make the exposition detailed, clear, and explicit. This copiousness of explanation accounts for the large size of the book, which is almost twice the length of the usual textbook on the theory of equations, although, for the most part, it contains only the topics ordinarily treated in such a book. Numerous illustrative problems are worked out at great length, with full explanation of each step. The lists of exercises are well selected, consisting of both drill problems and problems requiring some mathematical ingenuity. The number of exercises is more than sufficient to keep the most industrious student busy. In the opinion of this reviewer, the author has accomplished his aim of writing a book which can be studied successfully by the student of average ability without the aid of a teacher.

In chapter I complex numbers are defined as ordered pairs of real numbers, with suitable definitions of equality and the rational operations. From this beginning the usual properties of complex numbers are developed, including a discussion of binomial equations and the roots of unity. The construction of regular polygons is discussed as an application of the roots of unity, although the problem of ruler and compass constructions in general is not considered anywhere in the book. Reciprocal equations are discussed only in connection with an illustrative example. This chapter is particularly well done.

The next four chapters are concerned with the usual general theorems on polynomials in one variable and algebraic equations, the determination of rational roots, and the solution in radicals of the cubic and biquadratic equations.

Chapters VI, VII, and VIII take up the isolation of roots and their approximate evaluation. Rolle's theorem is proved for polynomials and its usefulness is emphasized by many illustrative problems and exercises. A general method of isolation of roots is based on a theorem of Vincent. An entire chapter is devoted to Sturm's theorem. Horner's method is given with the process of contraction, whereby additional decimal places can be obtained without too much labor. Newton's method is exhibited as a special case of the method of iteration. Both in Horner's method with contraction and Newton's method, a complete discussion of the limit of error in the final approximation is given. Much emphasis is placed on systematic numerical computation and the use of labor-saving devices. For this reason, the computation is carried out with complete detail in the illustrative problems.

The next chapter is devoted to determinants and the more elementary theorems of matric algebra. Determinants are defined by means of three general descriptive properties, rather than by the constructive definition used in most textbooks in the theory of equations. The author points out certain advantages in this definition, but fails to note certain disadvantages, which this reviewer believes might cause considerable trouble to the average college junior.

The last three chapters are devoted to the usual treatment of the theory and solution of systems of linear equations, the application of determinants to geometry, symmetric functions, and elimination.

The appendix contains a proof of the Fundamental Theorem of Algebra, a proof of the theorem of Vincent mentioned above, a section on equations whose roots have negative real part, a section on the frequency equation, and finally a discussion of Graeffe's root-squaring method. The proof of the Fundamental Theorem of Algebra is the fourth proof by Gauss, and is based on geometric intuition.

The format and typography are pleasing, the type easily read, and mathematical formulas are well displayed. The book seems to be singularly free from misprints and errors. In fact, the reviewer found in his somewhat brief perusal only one error, a slip in the definition of $\theta(x)$ in the illustrative problem on p. 171. The statements of the theorems are not prominently displayed; in some instances, they are not even italicized. This detracts from the value of the book for reference purposes, as one must read unnecessary material in order to find the statement of a theorem.

This is an excellent text for students who are studying the theory of equations as a tool subject. The reviewer feels, however, that students who intend to specialize in pure mathematics should be introduced to more of the concepts of modern algebra than are treated in this book.

L. E. Bush

The Theory of Mathematical Machines. Revised Edition. By F. J. Murray, New York, King's Crown Press, 1948. 9+139 pages. $3.00.

For several years Professor Murray has been interested in computing instruments, and in fact he designed an electrical linear equation solver, which was recently built and placed in operation. His interest in the subject has also led him to teach a course at Columbia University on the subject of mathematical instruments. The present book is an outgrowth both of his lectures and of his continuing interest in the subject. It is a revision of the first edition of the work and contains not only an expansion of the previous text but also several new chapters.

The subject of computing instruments became one of considerable importance during the last war, and quite vigorous efforts went into the development of new and improved machines. Since interest in this field has been steadily increasing we are fortunate indeed in having available this new edition of Professor Murray's book. He has seen fit to start collecting into a connected text, intended for the mathematical reader interested in physical apparatus, an account of the mathematical and engineering considerations entering into some types of computing instruments.

In the previous edition Professor Murray concentrated his attention mainly on the so-called analogy or measurement types of machines and discussed in detail some of the basic problems encountered in their design. In the present

edition he has added new chapters on "Electronic Digital Computers" and on "Noise, Accuracy, and Stability," and thus has striven to keep his book abreast of the times. He has also expanded Parts I and II of the book.

In over-all plan the book is somewhat arbitrarily divided into four major parts: digital machines, continuous operators, the solution of problems, and mathematical instruments.

Part I, which is quite short, is devoted mainly to a discussion of the basic principles underlying mechanical and electro-mechanical counters, adders and multipliers, and is concluded by a chapter on the punch card machines.

Part II, which is considerably longer than Part I, deals with the analogy or measurement counterparts of the devices considered in the first part. Professor Murray gives in a succinct manner a complete and careful analysis of a number of methods for adding, multiplying, and integrating by analogy means. His examples are well-chosen and exemplify not only mechanical but also electrical devices. His discussions of electrical apparatus are carefully written to make them understandable to those not well-versed in modern electrical techniques. In fact, he devotes a chapter to the subject of amplifiers, which play a fundamental role in most analogy computers. This part closes with an interesting account of methods for "remembering" in a mechanism a function of a single variable.

Part III is devoted to integrating together the background material of the two previous parts so that here the reader is shown how one can combine the various components, previously analyzed, to effect the solution of problems. The author considers not only analogy machines but also gives an account of some aspects of the newer digital instruments so that the reader can get an over-all picture of the field of modern mathematical instruments.

Part IV, the concluding quarter of the book, is concerned mainly with a number of simple yet highly important instruments such as planimeters, integrometers, integraphs, and harmonic analyzers.

The various augmentations in the present edition of Professor Murray's book have considerably enhanced its value and the reviewer is pleased to commend the book to the mathematical reader's attention. The reviewer would especially like to call attention to the excellent typography done by the King's Crown Press and to the clear illustrations in the text.

H. H. GOLDSTINE

Introduction to the Differential Equations of Physics. By L. Hopf. Translated by Walter Nef. New York, Dover Publications, 1948. 5+154 pages. $1.95.

The principal objective of this concise little book is to acquaint the reader, having a knowledge of the calculus, with the differential equations describing the more important theories of classical physics, such as, the potential, the propagation of waves, the flow of heat and fluid, and the field theory for electrodynamics, *etc.*

The requisite vector concepts such as gradient, divergence, and curl are

carefully treated and in such a manner as to make evident their physical interpretation.

The attractiveness of the book lies in the skillful manner in which the author blends elementary physical intuition with heuristic mathematics and, within these limitations, gives precise derivations of the differential equation forms of the physical theories. A harmonious balance between the physical and mathematical treatment of the considered phenomena is achieved. However, the reviewer believes that a slight amplification of the nature of the physical process would be of great interest to the reader.

In the concluding chapters the author deals briefly with some of the standard mathematical methods used for solving the derived differential equations, for instance: power series solutions of ordinary differential equations, separating of variables, characteristic values and functions, Fourier series and integrals, the fundamental solutions of the potential and heat equations, *etc.*

This book is ideal for the young prospective physicist who desires a glimpse of the fields awaiting him in classical physics, and, for the more advanced student who wants a compact compilation of the differential equations relating to the mentioned physical theories.

O. G. OWENS

Mathematical Table Makers. (*Portraits, Paintings, Busts, Monuments, Bio-Bibliographical Notes*). By R. C. Archibald. New York, Scripta Mathematica, 1948. 82 pages. $2.00.

This book is a "revised, rearranged, and somewhat extended reprint of two articles appearing in *Scripta Mathematica* in 1946, together with three additional sketches and portraits." A selection of 53 Mathematical Table Makers is presented. For each individual there is included a few biographical notes, a list of references to photographs, *etc.*, selected references to biographical information, and a list of his published tables. The bibliographies appear to be unusually complete, attesting to the author's extensive knowledge of mathematical literature. This book would be indispensable to anyone interested in collecting portraits or biographical data of Table Makers. Also the wealth of bibliographical references should be useful for many purposes, particularly for seeking the original sources of various tables. The reviewer noted a slight lacuna in the Introduction: frequent reference is made to POGGENDORFF, but a description of this particular source is lacking.

H. D. LARSEN

Plane Geometry. By D. T. Sigley and W. T. Stratton. New York, Dryden Press, 1948. 12+242 pages. $2.25.

According to the authors, this text is for the more mature student, and for those students who have a "working knowledge of measurements and informal geometry." To many teachers of plane geometry a first glance at the book may be startling, since many theorems are given without the proofs which are left as

exercises for the student, or to be supplied by the instructor. The reviewer's experience indicates that most of these proofs will have to be supplied by the instructor.

The organization of the material deviates somewhat from that of most texts. For example, similar triangles are discussed from the standpoint of parallel lines, which is a departure from the usual procedure found in most books. Similar triangles are also discussed in a chapter headed "Special Triangles." A strong point of the book is its development of an adequate mathematical vocabulary, which should be of benefit to students in their later study of college mathematics.

Other features found in the book are: a few famous theorems in plane geometry not given in most texts; a very good appendix where a table of mensuration formulas is given; a list of examination questions; and many challenging problems at various places throughout the text. In the reviewer's opinion this book would be best suited to the needs of mature students who have somewhat more than average mathematical ability.

O. J. MELBY

NEW BOOKS RECEIVED

Analytic Geometry. Revised Edition. By C. H. Sisam. New York, Henry Holt and Co., 1949. 16+304 pages. $2.40.

College Algebra. By E. A. Cameron and E. T. Browne, New York, Henry Holt and Co., 1949. 10+406 pages. $3.00.

Inside the Campus. By C. E. McAllister. New York, Revell, 1948. 248+102 pages. $5.00.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

EDITOR'S NOTE—A letter asking for a record of club activities for 1947–48 has been sent to the sponsors of the various mathematical clubs and societies. In writing the report, please follow the style as printed in this MONTHLY. They will be published in the chronological order in which they are received.

CLUB REPORTS, 1947–48

Mathematics Club, Lafayette College

The *Hall Mathematics Club* of Lafayette College, inactive during the war, was reorganized for the year 1947–1948. Papers given during the year included:

Nine-point circle, by Prof. W. M. Smith

Direction numbers in plane analytic geometry and calculus, by Prof. J. C. Smith

Algebra vectors, by Mr. J. A. Rohrer.

At the last two meetings topics from Classroom Notes taken from the MONTHLY were presented by various students, the last meeting being conducted exclusively by new mathematics majors.

A mathematics prize examination was given to sophomore students. Prizes of $75, $50, and $25 were awarded to the three best Liberal Arts students and similar prizes to the three best Engineering students. The winners were:

Liberal Arts: D. McIntyre, R. P. Barnes, and G. Veronis.

Engineers: R. A. Kudlich, J. H. Richmond, and G. W. Hoffman.

Kappa Mu Epsilon, Texas Technological College

The *Texas Alpha* chapter of *Kappa Mu Epsilon* held regular monthly meetings during 1947–48. Among the interesting programs presented by faculty members and students were:

The application of mathematics to radar, by Prof. B. E. Bennett of the Electrical Engineering staff

Applications of calculus to engineering, with emphasis on some unique methods of integration, by William L. Adair.

The outstanding meeting of the year was the initiation banquet and program at which time sixty-five new members were taken into the chapter.

The officers for 1948–49 are: President, William L. Adair; Vice-President, Ken Hancock; Secretary, Kathryn Witty; Treasurer, Allen R. Orr; Sponsor, Dr. Emmett Hazelwood; Corresponding Secretary, Mrs. Annie N. Rowland.

Pi Mu Epsilon, University of Illinois

The *Illinois Alpha* chapter of *Pi Mu Epsilon* held five open meetings during the academic year 1947–48. The following papers were presented:

Poisson distribution, by Prof. A. G. Carlton

High speed electronic calculators, by Prof. P. W. Ketchum

Inequalities of higher degree, by Dr. B. E. Meserve

Electromagnetic wave propagation in a stratified atmosphere, by Mr. B. E. Howard

Problems from the theory of elastic stability, by Dr. E. J. Scott.

At the annual initiation banquet seventy-nine new members were initiated. Prof. Leon Brillouin of Harvard University spoke on *Waves and electrons traveling together*. Mr. P. F. Conrad received the annual $25 Pi Mu Epsilon award for oustanding scholarship in mathematics while an undergraduate.

Officers for 1948–1949 are: President, William Orton; Vice-President, Evelyn Lind; Secretary, Charlene Sprankel; Treasurer, Leland Scott; Faculty Adviser, Prof. E. D. Pepper.

Pi Mu Epsilon, University of Nebraska

Eight meetings were held by the *Nebraska Alpha* chapter of *Pi Mu Epsilon* during 1947–48. The following talks were presented:

The algebra of functions, by Dr. Wm. G. Leavitt
Topologies of parallelotopes, by Dr. Edwin Halfar
Generating functions in probability, by Prof. Jorgensen
Elementary matrices, by Mr. Maurice Lamoree
Discussion of permutations, by Mr. D. M. Mesner.

The annual Freshman and Sophomore competitions were won by Mr. R. E. Kleppinger and Mr. E. C. Luschei. Seventy-six new members were initiated into the chapter.

Officers elected for the current year are: President, Marlin Kroger; Vice-President, William Bade; Secretary, J. Denny Cochran; Treasurer, Frederick Pelton; Faculty Adviser, Dr. W. G. Leavitt.

Mathematics Club, Purdue University

Topics discussed at the monthly meetings of the Purdue *Mathematics Club* were:

The development of the number system, by Prof. G. H. Graves
The irrationality of pi, by Mr. K. J. Hammerle
Magic squares, by Dan Overlade
Trisecting the angle, by R. R. Kenyon
Significant figures, by Prof. Carl Holtom.

At each meeting problems of interest to the membership were proposed and discussed.

Officers for 1947–48 were: President, A. F. Sterling; Secretary, F. VanNess; Program Chairman, W. F. Haldeman.

Mathematics Club, New York University

The activities of the *Mathematics Club* of New York University included several regular meetings, at which the following talks were given:

Stellar distances and their measurement, by Dean P. H. Graham
The role of the scientist today, by Prof. D. Jan Struik of Massachusetts Institute of Technology
Mechanical computing machines, by Dr. Alfred Leitner
Mathematics and symbolism, by Joseph Alper
Topology, by Prof. Leo Zippin of Queens College
Mathematics in radar, by William Sollfrey
Mathematical theory of binocular vision, by Dr. J. B. Keller.

Social hours were held in connection with four of the lectures.

The organization's annual publication, *Math X* contained analysis, surface geometry, classes, mathematics of investment, and other topics of general interest.

The tutoring group was active and supplied considerable help to those students who requested it.

Newly elected officers for 1948–49 are: President, Stefan Mengelberg; Vice-President, Israel Teitelbaum; Secretary, Elaine Weiss; Treasurer, Dan Fondiller.

Kappa Mu Epsilon, Southern Methodist University

The following papers were presented to the *Texas Beta* chapter of *Kappa Mu Epsilon:*

Hyper-spacial tit-tat-toe, by Grace Mitchell

Concepts of infinity, by Joseph Rice

A definition of $\sqrt{-1}$, by Gene Archer.

The discussion of the paper by Miss Mitchell, was based upon mathematical considerations (assumption of coordinates for the squares and proof of a winner by showing that the points lie on a line, etc.) Miss Mitchell ended the discussion by staging a "game" of three-dimensional and then one of four-dimensional tit-tat-toe, enlisting the participation by the audience.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

SUMMER COURSES

The following institutions announce advanced courses in mathematics for the summer of 1949:

Catholic University of America. June 22 to August 6: Professor Finan, fundamentals of mathematics; Professor Ramler, advanced Euclidean geometry, analytic projective geometry; Professor Rice, solid analytic geometry; Mr. Markowitz, differential equations, mathematical astronomy; Mr. Moller, theory of equations; Mr. Slud, partial differential equations of mathematical physics.

Columbia University. July 5 to August 12: Professor Kolchin, introduction to higher algebra; Professor Kasner, survey of mathematics; Professor Levi, foundations of projective geometry; Professor Ritt, theory of functions of a real variable; Professor Strodt, existence theorems; Professor Federer, measure and area; Professor Kasner, geometric transformations; Professor Thrall, theory of group representations.

Duke University. June 13 to July 21: Professor Carlitz, theory of numbers, thesis seminar; Professor Rankin, teaching of mathematics; Professor Roberts, solid analytic geometry; Professor Thomas, plane geometry and trigonometry from the advanced standpoint. July 22 to August 31: Professor Dressel, probability; Professor Gergen, infinite series.

Indiana University. June 17 to August 12: Dr. Hlavaty, seminar in foundations of geometry; Dr. Whaples, algebra and elementary number theory; Staff, mathematical reading and research.

Northwestern University. June 27 to August 27. Differential equations; fundamental concepts of analysis; definite integrals; theory of equations; theory of statistics; algebra of matrices and quadratic forms; introduction to the theory of numbers; geometry for teachers; the history and teaching of mathematics; independent study; functions of a complex variable; topics in linear operations; introduction to the theory of groups; independent study; engineering mathematics I; engineering mathematics III; the teaching of mathematics in the secondary school; mathematics in the upper grade; implications of recent studies for the teaching of mathematics; multi-sensory aids in the teaching of mathematics.

Ohio State University. June 20 to September 2: Professor Mann, theory of fields; Professor Helsel, introduction to the theory of functions of a complex variable, advanced geometry; Dr. Rechard, infinite series and products, advanced calculus.

Oklahoma Agricultural and Mechanical College. June 4 to August 2: Professor Allen, advanced calculus I, vector calculus; Professor Barnett, astronomy; Professor Caskey, analytic geometry of three dimensions; Professor Diamond, geometrical introduction to function theory; Professor Dresden, foundations of algebra and geometry (July 1 to July 31); Professor Hamilton, Hilbert space theory, advanced calculus II; Professor Morrison, differential equations; Professor Robison, mathematics of finance; Professor Scholz, partial differential equations, introduction to higher algebra; Professor Smith, analytic projective geometry; Professor Walsh, topics in the theory of functions (June 15 to July 15); Professor Zant, history of mathematics, the teaching of high school mathematics; Staff, research and thesis.

University of California at Berkeley. June 20 to July 30: Professor Ahlfors, conformal invariants; Professor Hurewicz, topological methods in the theory of differential equations; Professor Neyman, consistent estimates; Professor Lehmann, the theory of testing hypotheses; Professor Wishart, design of experiments, seminar in advanced design of experiments; Professor Loève, limit probability laws; Professor Lewy, difference equations and related mathematical tools of probabilistic research.

University of California at Los Angeles. June 20 to August 12: Professor Brauer, introduction to higher algebra, theory of rings; Professor Green, introductions to Fourier analysis, functions of a complex variable; Professor Hoel, probability; Professor Martin, analytic functions of several variables; Professor Sorgenfrey, advanced calculus.

University of Colorado. June 20 to July 25 and July 26 to August 26: Professor Kline, integral equations; Mr. Hunt, elementary differential equations; Professor Hutchinson, functions of a complex variable; Professor Jones, theory

of matrices; Professor Stahl, teaching of mathematics; Dr. Leveque, theory of equations; Professor Britton, vector analysis.

University of Kansas. June 10 to August 6. Professor Bell, tensor and vector calculus, differential equations; Dr. Herstein, differential equations; Professor Schatten, higher algebra; Professor Smith, modern synthetic geometry, partial differential equations; Professor Ulmer, advanced algebra, history of mathematics.

University of Kentucky. June 20 to August 13: Professor Downing, vector analysis, differential geometry; Professor Pence, solid analytic geometry, introduction to higher geometry; Professor Pulliam, vector analysis, functions of a complex variable; Professor South, mathematical statistics, solid analytic geometry.

University of Maryland. June 27 to August 5: Professor Brigham, number theory; Professor Good, theory of equations; Professor Jackson, higher geometry

University of Mississippi. June 1 to July 11: Professor Trott, modern algebra, vector analysis; Professor Bickerstaff, mathematical theory of statistics; Professor Miller, non-Euclidean geometry, history of mathematics; elementary differential equations; real variables. July 13 to August 20: Professor Trott, modern algebra, matrices; Professor Bickerstaff, mathematical theory of statistics; Professor Miller, advanced geometry, fundamental concepts of algebra and geometry; intermediate differential equations, complex variables.

University of Missouri. June 9 to August 3: Professor Betz, advanced calculus, differential equations; Professor Blumenthal, distance geometry; Professor Burcham, theory of infinite series and summability; Professor Utz, theory of equations.

University of Oregon. June 21 to August 12: Professors Morsund and Niven, selected topics (pure mathematics); Professor Civin, Fourier series; Professor Massey, selected topics (applied mathematics), statistics; Professor Ghent, algebra and geometry (for high school teachers).

University of South Carolina. June 15 to August 17: Professor Hedberg, theory of equations; Professor Novak, college geometry, synthetic projective geometry; Professor Williams, theory of functions of a complex variable.

University of Virginia. June 27 to August 20: Professor Botts, advanced calculus, applied mathematics; Professor Floyd, advanced analysis; Professor Hoyle, differential equations, applied mathematics; Professor Whyburn, transformation theory.

University of Wisconsin. June 27 to August 19: Professor Sokolnikoff, higher mathematics for engineers; Professor Bing, higher mathematics for engineers, projective geometry; Professor Young, advanced calculus, partial differential equations; Professor MacDuffee, survey of the foundations of algebra, theory of numbers and Diophantine equations; Professor Colvin, mathematical applications; Professor Langer, harmonic analysis; Staff, determinants and matrices, Laplace transforms.

University of Wyoming. June 13 to July 15: vector analysis; advanced calcu-

lus; ordinary differential equations; projective geometry; history of mathematics; methods of teaching secondary mathematics. July 18 to August 19: theory of equations; partial differential equations; college geometry; fundamental concepts of mathematics; curve fitting; abstract algebra.

PERSONAL ITEMS

Professor C. O. Oakley of Haverford College was the delegate of the Mathematical Association to the Fifty-third Annual Meeting of the American Academy of Political Social Science which was held at Philadelphia on April 8–9, 1949.

Professor A. C. Schaeffer of Purdue University and Professor D. C. Spencer of Stanford University have been awarded jointly the Bôcher prize for analysis by the American Mathematical Society.

The French Academy of Sciences has announced the following awards in mathematics for 1948: Poncelet Prize to Georges Valiron of the University of Paris; Carrière Prize to P. J. Dubreil of the University of Paris; Dickson Prize to Julien Kravtchenko of the University of Grenoble; Grand Prize to Henri Millous of the University of Bordeaux; Albert I. de Monaco Prize to Jacques Hadamard of the College of France and Polytechnic School; Laplace Prize to Francois Morin of the Polytechnic School; Becquerel Foundation Prize to André Bloch of the Polytechnic School.

Oklahoma Agricultural and Mechanical College announces: Professor Arnold Dresden of Swarthmore College and Professor J. L. Walsh of Harvard University have been appointed Visiting Professors for the Summer Session, 1949. Professor J. H. Zant, assistant head of the Department of Mathematics, is serving as a member of the General Council of the American Society of Engineering Education as representative of the Mathematics Division, as Secretary of the Mathematics Division of A.S.E.E., and as a member of the Board of Directors of the National Council of Teachers of Mathematics.

At Purdue University, Miss Mary Robbins has been appointed to an instructorship and Assistant Professor W. P. Reid has resigned.

Stanford University reports that a Conference on Non-Linear Mechanics was held on November 8–12, 1948. Talks were given by Richard Bellman, R. Bishop, S. P. Diliberto, Alfred Horn, J. LaSalle, Solomon Lefschetz, N. Minorsky, Balth. van der Pol, H. Schaffner, D. C. Spencer, Eleanor Yost.

University of California at Berkeley announces the following appointments for Spring, 1949: Professor Lamberto Cesari of Bologna University as visiting professor, Wanda Szmielew and Ting-Kwan Pan as lecturers in mathematics.

The University of Virginia announces: Dr. E. E. Floyd, now Fine instructor at Princeton University, has been appointed to an assistant professorship; Mr. V. L. Klee, Jr., Atomic Energy Commission Fellow at the University of Virginia, has been appointed to an assistant professorship; Acting Assistant Professor Truman Botts has been appointed Assistant Professor; Assistant Professor R. H. Bing of the University of Wisconsin will be Acting Professor for the 1949–50 session; Professor E. J. McShane will be on leave for the 1949–50 session.

The University of Wisconsin makes the following announcements: Professor H. W. March is on leave of absence during the second semester of 1948–49; Professor R. E. Langer, chairman of the Department of Mathematics, has been elected President of the Mathematical Association of America.

Professor K. Ananda-Rau of Presidency College, Madras, India, has retired.

Mr. L. F. Boron of the University of Illinois has been appointed to an instructorship at the University of Maine.

Mr. R. W. Butcher, Queens View, Kingston, Ontario, has been appointed Lecturer in Actuarial Science at the University of Manitoba.

Mr. G. M. Dillon, formerly instructor at Long Island University, is now a member of the Pension Statistics Section, Treasury Department, E. I. duPont deNemours Company, Wilmington, Delaware.

Dr. Joseph Gillis of Sunderland, England, has been appointed senior assistant at the Weizmann Institute, Rehovot, Israel.

Assistant Professor Banesh Hoffmann of Queens College has been promoted to an associate professorship.

Associate Professor W. G. Hubert, chairman of the Department of Mathematics of City College of New York City, has been promoted to a professorship.

Mr. P. F. Hultquist, formerly assistant at the University of Wisconsin, has been appointed to an instructorship at College of Mines and Metallurgy, El Paso, Texas.

Dr. H. D. Huskey, formerly chief of the Machine Development Laboratory, National Applied Mathematics Laboratories, National Bureau of Standards, Washington, D. C., is now Chief of the Machine Development Unit, Institute for Numerical Analysis, National Bureau of Standards, Los Angeles, California.

Dr. Cornelius Lanczos of Boeing Aircraft Company has accepted an appointment as staff mathematician with the National Bureau of Standards, Washington, D. C.

Mr. Z. I. Mosesson has become a Fellow of the Society of Actuaries and has been promoted to the position of Senior Actuarial Assistant with the Prudential Insurance Company of America.

Mr. M. W. Oliphant has been appointed to an instructorship at Georgetown University.

Professor H. A. Pérsico of the University of LaPlata has been appointed to a professorship at the University of Cuyo, San Luis, Argentina.

Mr. A. V. C. Pleijel of Lund University has been appointed to a professorship at the Royal Institute of Technology, Stockholm, Sweden.

Associate Professor R. E. Byrne of the California Institute of Technology died September 17, 1948 at the age of thirty-seven years.

Mr. H. I. Treiber who was employed at Watson Laboratories, Red Bank, New Jersey, died on February 27, 1949.

Professor John Williamson of Queens College, New York, died on February 8, 1949.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE MAY MEETING OF THE KENTUCKY SECTION

The May meeting of the Kentucky Section of the Mathematical Association of America was held at Berea College, Berea, Kentucky, on Saturday, May 8, 1948. Professor D. W. Pugsley, Chairman of the Section, presided.

Fifty-eight persons attended the meeting, including the following nineteen members of the Association: M. C. Brown, A. E. Cook, H. H. Downing, W. L. Fields, Charles Hatfield, Aughtum S. Howard, W. R. Hutcherson, S. J. Jasper, W. L. Moore, Smith Park, Sallie E. Pence, Ruth E. Porter, D. W. Pugsley, S. L. Riggs, G. G. Roberts, W. J. Robinson, Florence V. Rohde, D. E. South, Guy Stevenson.

The following officers were elected for the coming year: Chairman, W. J. Robinson, Centre College; Secretary, Sallie E. Pence, University of Kentucky.

The following papers were presented:

1. *Remarks concerning some required mathematics at Berea College,* by Professor W. R. Hutcherson, Berea College.

As a part of Berea College's general education program, basic mathematics (arithmetic, elementary algebra, and plane geometry) is required of all students falling in the lower fifth of the freshman class. These students meet five days a week for two semesters, receiving no college credit for the work. Mathematics and astronomy constitute one-fourth of the material included in the physical science course. Every student of the college is required to take this five credit course. The eight lessons on mathematics are calculated to enrich the student's understanding of the place of mathematics in the modern world, rather than to emphasize drill work in elementary mathematics.

2. *Matrix algebra and linear networks,* by Professor W. L. Fields, Louisville Municipal College.

This paper was concerned with the derivation of the matrices associated with linear electrical networks. Matrices were derived for both series and shunt impedances. The solution of the system

$$dE = Iz\,dl, \qquad dI = Ey\,dl$$

for a transmission line was obtained by matrix methods.

3. *A theorem on homogeneous functions,* by Mr. S. J. Jasper, University of Kentucky.

The speaker established the following theorem on homogeneous functions: If $f(x, y)$ is continuous in a region R of the xy-plane, if it is homogeneous of order n, and has continuous partial derivatives in R of at least order $n+1$, then there exists a function $G(y)$ such that the $n+2$ partial derivatives of order $n+1$ can be expressed as products of G and powers of $(-x)$ and y.

4. *A converse theorem on homogeneous functions,* by Professor H. H. Downing, University of Kentucky.

In this paper it was shown that a function $f(x, y)$ which, together with its partial derivatives up to at least order $n+1$, satisfies certain continuity conditions, and (a) if certain equations involv-

ing the partial derivatives of $f(x, y)$ multiplied by products of powers of x and y hold; or (b) if there exists a $G(x, y)$ such that the partial derivatives of f of order $n+1$ can be expressed in terms of G (as in Mr. Jasper's paper), then $f(x, y)$ is homogeneous in x and y and of order n.

5. *An extension of a problem in the Monthly*, by Professor W. J. Robinson, Centre College.

The question of generalizing Problem 4272 of the November, 1947 issue of this MONTHLY was considered. It was shown that a generalization of the formula could be effected if the two parameters involved had certain values, but that such could not be done for certain other values.

6. *Visual aids in the teaching of mathematics*, by Professor W. L. Moore and Professor Guy Stevenson, University of Louisville.

Professor Moore reported on the use of models and slides in the mathematics department of the University of Louisville. He exhibited a model representing triple integration, and slides showing the steps in triple integration, as well as a slide projecting the coordinate system on the blackboard. The idea for the last slide appeared in a recent issue of this MONTHLY.

7. *Existence of a two-dimensional potential flow with finite wake past a strictly convex profile, symmetric with respect to the flow at infinity*, by Dr. G. L. Tiller, University of Kentucky.

In 1929 Weinstein proved the existence of a jet flow by approximating a smooth curve nozzle by a polynomial and letting the number of sides become infinite. In 1947 Pulliam used the same general method to prove the existence of a flow with wake extending to infinity. The procedure employed by Weinstein and Pulliam is used in this paper. The principal result obtained is the theorem: There exists a two dimensional potential flow with finite wake with $\omega'(D)=0$, (where the point D in the τ-plane corresponds to the point at infinity in the z-plane), past any strictly convex profile symmetric with respect to the flow at infinity and with continuously changing slope.

8. *Derivation of the formula for the Marchant square root table*, by Professor G. G. Roberts, Berea College.

The speaker gave a brief discussion of mechanical calculators, calling attention to the particular uses of several different machines. He then discussed the Marchant square root table, and illustrated by examples how the table made possible the extraction of square roots correct to five significant digits. In the process only a single division (or multiplication) is required.

9. *Some mathematical aspects of music*, by Miss Florence V. Rohde, University of Kentucky.

According to Joseph Schillinger, who developed the Schillinger system of musical composition, music may be projected into space by means of graphs. Mechanical trajectories are the inherent patterns of musical motion; thus music is capable of expressing everything which can be translated into a form of motion. Music may be composed by taking a system of number values, transforming them into geometric relations, and then into corresponding components of rhythm, melody, and harmony. Variation may be achieved through modification of the inherent geometrical relations. The natural harmonic series, arithmetic progressions, geometric progressions, involution series, logarithmic series, progressive additive series, prime number series, and others, all may be transformed into music. In addition to a presentation of some of Schillinger's ideas, the application of the harmonic series and certain elementary principles of physics to the construction and playing of the woodwind and brass instruments was shown by the speaker.

SALLIE E. PENCE, *Secretary*

THE NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The annual meeting of the Philadelphia Section of the Mathematical Association of America was held at the University of Pennsylvania, Philadelphia, Pennsylvania, on Saturday, November 27, 1948. Professor E. P. Starke of Rutgers University presided.

There were fifty-one present, including the following forty-two members of the Association: H. W. Brinkmann, Geo. Y. Cherlin, Bro. Damian Connelly, E. H. Cutler, James Elmer Davis, F. L. Dennis, Arnold Dresden, R. L. Erickson, A. B. Farnell, N. J. Fine, C. D. Firestone, W. H. Gottschalk, Theodore Hailperin, J. R. Holzinger, J. R. Kline, P. A. Knedler, C. E. Langenhop, V. V. Latshaw, Marguerite Lehr, Max LeLeiko, F. L. Manning, Clifford Marburger, D. L. McDonough, S. S. McNeary, A. E. Meder, Jr., W. R. Murray, A. B. Neale, C. A. Nelson, C. O. Oakley, J. C. Oxtoby, M. A. Rader, G. E. Raynor, I. J. Schoenberg, Francis A. C. Sevier, C. A. Shook, L. L. Smail, E. P. Starke, A. W. Tucker, R. M. Walter, Jean B. Walton, H. M. Zerbe, H. J. Zimmerberg.

At the business meeting the following officers were elected for the coming year: Chairman, G. E. Raynor, Lehigh University; Secretary, C. O. Oakley, Haverford College. The Program Committee for the next meeting will be: Arnold Dresden (Chairman), Swarthmore College; N. J. Fine, University of Pennsylvania; and J. W. Tukey, Princeton University. The next meeting of the section will be held at Haverford College, November 26, 1949.

The program consisted of the following papers:

1. *Recent advances in symbolic logic*, by Professor Theodore Hailperin, Lehigh University.

An introductory sketch of the subject is presented based on the three levels of: (1) the propositional calculus; (2) the theory of quantification; and (3) the theory of membership. The classical and pre-1936 status of each level is outlined, and some later results, of particular interest to mathematicians, are described.

On the propositional calculus level the paper of Rosser and Turquette on axiom-schemes for many-valued logics, and McKinsey's solution of the decision problem for strict implication, are referred to.

In the theory of quantification, mention is made of Quine's method for testing formulae of one variable for provability. On non-classical lines, the functional calculus of Bochvar, based on a. three valued logic, and Barcan's, based on strict implication, are of interest.

In the theory of membership the works of Bernays, Quine, and Godel are commented upon

Finally, the development of recursive arithmetic is mentioned, and the impossibility proofs of Church, Post, Markov, and Post and Linial, are mentioned.

2. *On a problem in the theory of differential equations*, by Professor W. R. Wasow, Swarthmore College.

When a physical phenomenon is described mathematically by a differential equation it is almost always necessary to simplify the problem by omitting terms whose coefficients are very small. The justification of such a simplification requires particular attention, if the terms omitted are of higher order of differentiation than those retained. For example, the simplified differential equation of lower order may possess a periodic solution, while the full equation does not admit a periodic solution corresponding to it unless certain conditions are satisfied.

Another example occurs in the theory of linear differential equations in the complex domain. There it may happen that the simplified equation has multi-valued solutions, whereas the solutions of the full equation are all single-valued. It turns out that there exist solutions of the full differential equation which are approximately equal to a given multi-valued solution of the simplified equation in some domain; but the two solutions will then differ radically in some other complex domain. This analysis helps to explain the so-called "inner friction layers" in the theory of hydrodynamic stability.

3. *A geometric approach to the theory of games*, by Professor A. W. Tucker, Princeton University.

Given a two-person zero-sum game in which player I wins (and player II loses) an amount a_{ij} if player I chooses his ith mode of play, and player II his jth mode ($i=1, 2, \cdots, m$; $j=1, 2, \cdots, n$), to find: (1) a probability distribution $p_1, p_2, \cdots, p_m$ for player I that maximizes the minimum of the inner products

$$(p\cdot a)_j = p_1a_{1j} + p_2a_{2j} + \cdots + p_ma_{mj};$$

and (2) a probability distribution $q_1, q_2, \cdots, q_n$ for player II that minimizes the maximum of the m inner products

$$(a\cdot q)_i = a_{i1}q_1 + a_{i2}q_2 + \cdots + a_{in}q_n.$$

A solution exists, and for these p's and q's min $(p\cdot a)_j =$ max $(a\cdot q)_i =$ "value" of the game.

An equivalent geometric problem is: Given n points $A_1, A_2, \cdots, A_n$ in cartesian m-space, to find: (1) non-negative direction numbers $p_1, p_2, \cdots, p_m$ with unit sum so that the halfspace $p_1x_1+p_2x_2+\cdots+p_mx_m \geqq u$ contains $A_1, A_2, \cdots, A_n$ and maximizes u; and (2) barycentric coordinates $q_1, q_2, \cdots, q_n$ (non-negative, with unit sum) so that the point $q_1A_1+q_2A_2+\cdots+q_nA_n$ belongs to the corner $C_v : \{x \mid x_i \leqq v, i=1, 2, \cdots, m\}$ and minimizes v. A solution exists, with $u=v$.

The theory was illustrated with several elementary examples.

C. O. OAKLEY, *Secretary*

THE NOVEMBER MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The 1948 fall meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at the University of Pittsburgh, Pittsburgh, Pennsylvania, on Saturday, November 6, 1948. Professor J. B. Rosenbach, Chairman of the Section, presided at the morning session, and Professor J. S. Taylor presided at the afternoon session.

The meeting was attended by approximately 160 persons, including the following 40 members of the Association: J. O. Blumberg, R. C. Briant, A. M. Bryson, Helen Calkins, J. G. Christiano, A. B. Cunningham, H. B. Curry, H. L. Dorwart, R. H. Downing, Esther S. Dunkelberger, L. T. Dunlap, E. T. Frankel, Orrin Frink, R. E. Settig, W. O. Gordon, W. J. Harrington, Evan Johnson, Roberta Johnson, J. C. Knipp, H. R. Leifer, Sister Marie MacNeil, Rev. P. M. Mino, L. T. Moston, B. H. Mount, F. D. Murnaghan, J. H. Neelley, E. G. Olds, M. O. Peach, J. B. Rosenbach, E. A. Saibel, I. M. Sheffer, R. E. Smith, F. H. Steen, J. S. Taylor, Margaret Taylor, C. H. Vehse, M. L. Vest, E. D. Wells, E. A. Whitman, V. A. Zora.

The spring meeting of the Section will be held at the University of West Virginia, Morgantown, West Virginia, Saturday, May 7, 1949.

The following papers were presented:

1. *A normal form for the equation of the straight line in space*, by Professor H. L. Dorwart, Washington and Jefferson College. This paper will appear in an early issue of *Scripta Mathematica.*

2. *Tests for convergence*, by Professor Orrin Frink, Pennsylvania State College.

Professor Frink discussed generalizations of the ordinary ratio test for convergence of series of positive terms, and in particular the theorem that $\sum a_n$ converges if $\lim (a_n/a_{n-k})^n < e^{-k}$. He also discussed various forms of Kummer's ratio test. It was shown that when Ermakoff's test is stated in the form "the series converges if $\lim (na_n < a_{[1n\, n]}) < 1$," it is unnecessary to assume that the terms are decreasing. As to method of proof, it was shown that it is usually sufficient to establish that there is a case where the test fails. Failure in one case insures success in the other cases.

3. *On the teaching of mathematics at the freshman and sophomore levels*, by Professor F. D. Murnaghan, Carnegie Institute of Technology.

Professor Murnaghan advocated the use of vector methods in teaching college algebra, trigonometry and analytic geometry. The central formula of trigonometry is the one which furnishes the cosine of the difference of two angles, and this is an immediate consequence of the fact that the magnitude of a vector is independent of the reference frame. The concept of the scalar product of two vectors yields without any manipulation the formula for the distance from a point to a line (in the plane) or to a plane (in space), and clarifies the concept (so confusing to a beginner) of the positive and negative sides of a directed line or oriented plane. He also advocated the use of the scalar product in defining determinants, and the use of matrices in the discussion of conics and quadric surfaces. In calculus he advocated the use of the upper and lower bound concepts (as applied to a bounded collection of numbers). This is a much simpler concept than that of a limit, and is much more natural than the latter when defining the definite integral. The speaker's ideas on the use of vectors in the teaching of analytic geometry are fully explained in his book *Analytic Geometry* (Prentice-Hall, 1946) and those on the possibility of presenting calculus rigorously without making it dull or forbidding are given in his book *Differential and Integral Calculus* (Remsen Press, 1947).

4. *An elementary presentation of some basic theorems on linear differential equations*, by Professor H. B. Curry, Pennsylvania State College.

The following three theorems are discussed: (1) if n linearly independent solutions of a linear differential equation are given, any solution is a linear combination of them; (2) the solutions given by the usual process for an equation with constant coefficients actually are linearly independent; (3) the formulation of the method of undetermined coefficients for the non-homogeneous equation with constant coefficients. The first two theorems are proved rigorously by mathematical induction without any appeal to existence theorems, and without any use of determinants. Thus it is not necessary to have gaps in the logic at these points, even for immature students. The third theorem is stated and proved more simply than usual.

5. *An unusual approach to maxima and minima*, by Professor F. H. Steen, Allegheny College.

Beginning with the elementary fact that of all rectangles with a given perimeter the square has the largest area, the speaker developed methods for solving rapidly a large number of maximum and minimum problems involving one or more independent variables, and derived the usual formula for the slope of a polynomial curve, all without use of the limit concept.

6. *What is statistical quality control?*, by Professor E. G. Olds, Carnegie Institute of Technology.

Statistical quality control is the application of statistical methods to the improvement of the manufacturing operation. A narrower definition, which has gained support because of certain wartime developments, is that statistical quality control is the use of control charts and sampling tables for quality assurance. This is too restrictive, but the philosophy and principles which form the basis for process control and acceptance sampling are fundamental to statistical quality control in the large.

As an amplification of the author's conception of statistical quality control, a description of the two-semester course now offered in the College of Engineering and Science, Carnegie Institute of Technology is presented.

Then, after sketching the history of statistical quality control, the paper briefly describes the nature of process control and acceptance sampling, giving simple examples for each.

7. *A problem in linear recurrence relations*, by I. M. Sheffer, Pennsylvania State College.

A solution of the recurrence relation

$$u_{n+2} + au_{n+1} + bu_n = 0 \tag{1}$$

is uniquely determined by the initial values u_0 and u_1. One may ask if there exist infinitely many pairs of positive integers (n_j, k_j) and infinitely many numbers λ_i such that $\sum_{r=0}^{n_j} u_r = \lambda_j u_{k_j}$ $(j=1, 2, \cdots)$ for arbitrary choice of u_0, u_1. (An example is the case considered in this MONTHLY, vol. 54, 1947, problem 4272). Certain asymptotic aspects of this problem are considered.

8. *A method of calculating the number of primes less than a certain integer*, by Mr. R. E. Gettig, University of Pittsburgh.

Enumeration of the composites divisible only by the primes from P_k to $P_m \leqq [n/P_k] < P_{m+1}$, where k is arbitrary and n the assigned upper bound, leads to an easier and more direct solution for many values of n than does the use of Meissel's reduction formula. In particular, if n be chosen as the product of small powers of the first $(k-1)$ primes, the Euler ϕ-functions may be used and the auxiliary table of Meissel's method dispensed with completely. The author calculated $\pi(n)$ by both methods for $n = 30030 = 2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13$ and found the proposed method to yield a decided advantage.

9. *Construction of harmonic functions of two variables*, by Mr. M. O. Peach, Carnegie Institute of Technology.

It is shown that the problems of (a) determining whether a given function (of two variables) is harmonic, (b) finding the conjugate harmonic function, and (c) finding the function of a complex variable of which the given harmonic function is the real part, can be solved by a simple algebraic procedure, not involving differentiation or integration. A function is harmonic if it can be expressed in the form $f(z)+g(\bar{z})$ where $z=x+iy$ and $\bar{z}=x-iy$. The conjugate function is $(-i)f(z)+if(\bar{z})$. The function of a complex variable of which it is the real part $2f(z)$.

B. H. MOUNT, JR., *Acting Secretary*

THE DECEMBER MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at The Johns Hopkins University, Baltimore, Maryland, on Saturday, December 4, 1948. Mr. Michael Goldberg, Chairman of the Section, presided.

There were seventy-three persons attending the meeting including the following forty-seven members of the Association: R. P. Bailey, N. H. Ball, W. E. Bleick, S. G. Bourne, R. S. Burington, H. H. Campaigne, C. R. Clark, G. R. Clements, G. F. Cramer, C. H. Denbow, J. A. Duerksen, Anselm Fisher, J. H. Giese, Michael Goldberg, R. A. Good, E. C. Gras, J. R. Hammond, E. K. Haviland, M. A. Hyman, S. B. Jackson, Walter Jennings, Sidney Kaplan, L. M. Kells, D. C. Lewis, Carol V. McCamman, Florence M. Mears, Emanuel Mehr, Joseph Milkman, A. K. Mitchell, T. W. Moore, W. K. Morrill, R. W. Rector, Irwin Roman, R. E. Root, W. G. Rouleau, E. D. Schell, Erwin Schmid, Veryl G. Schult, W. F. Shenton, A. D. Sollins, C. A. Spicer, C. F. Stephens, O. M. Thomas, P. D. Thomas, J. A. Tierney, C. C. Torrance, Beryl W. Williams.

The spring meeting of the Section will be held at the University of Virginia.

The following papers were presented at the morning session:

1. *The logarithmic function is unique*, by Professor Joseph Milkman, United States Naval Academy.

The operation of the slide rule for multiplication and division depends on the fact that $\log a+\log b=\log ab$. A slide rule with different scales could be made if there were three continuous functions f, g and h such that $f(x)+g(y)=h(xy)$. It was proved that: (1) if f, g and h are continuous functions satisfying this equation, then $f(x)=k\ln c_1x$, $g(y)=k\ln c_2y$ and $h(xy)=k\ln c_1c_2xy$; (2) if $f(x)+f(y)=f(xy)$ for all positive real numbers x and y, and $f(x)$ is bounded in some closed interval $a\leqq x\leqq b$, then $f(x)$ is continuous for all positive x. The method was extended to prove that the only continuous functions satisfying the equation $\sum_{i=1}^n f_i(x_i)=f(x_1x_2\cdots x_n)$ for all $x>0$ are $f_i(x_i)=k\ln c_ix_i$, $i=1, 2, \cdots, n$.

2. *On the differential equation $y'=f(y)$*, by Dr. Sylvan Wallach, The Johns Hopkins University, introduced by Professor Morrill.

Necessary and sufficient conditions for the existence of nontrivial solutions of the differential equation $y'=f(y)$ were given. This was followed by a description of the possible solutions and of the method of obtaining them. No assumption was made as to the zeros of $f(y)$.

3. *Evaluation of roots of a polynomial by successive square roots*, by Professor John Tyler, United States Naval Academy, introduced by the Secretary.

Expansions of various functions by successive square roots were given, and limiting values of arithmetical and geometrical means were obtained. Examples were given to illustrate the solution of the equation $f(x)=0$. The equation was written in the form $Q_r(x)=q(x)$, with $Q_r(x)$ an iterated quadratic form, and $q(x)$ a polynomial of degree less than the degree of $Q_r(x)$, or a fraction. By solving $Q_r(x)=0$ for x, $f(x)=0$ was written in radical form, and approximations for the roots were obtained by the use of a finite difference equation.

4. *Modifications of authalic projections*, by Mr. P. D. Thomas, U. S. Coast and Geodetic Survey.

In the usual authalic projection of the sphere or oblate spheroid upon a plane, the projection is bounded by curves which pass through the poles of the projection. For the various types that have been devised, the distortion for values of the longitude beyond 90° is so great that they are considered unsatisfactory for more than a hemisphere. By replacing the poles of the projection by lines of given length (a multiple of the map equatorial length) parallel to the equator, this extreme distortion is avoided except in the region of the poles and near the bounding meridian of the projection. Mr. Thomas derived equations for a modification which replaces the poles of an authalic

projection by straight lines parallel to the map equator, which preserves the equal area property, and which may be applied to existing authalic projections whose parallels are straight lines.

5. *A unified theory of special functions*, by Dr. C. A. Truesdell, Naval Research Laboratory, introduced by the Secretary.

The objective of the paper was to provide a theory to discover and coordinate formal relationships satisfied by special functions. To this end, a class of linear partial differential-difference equations with variable coefficients was reduced to a single equation

$$\frac{\partial}{\partial z} F(z, \alpha) = F(z, \alpha + 1)$$

whose very simple properties were then developed. Among the numerous special functions to which the results apply are Bessel, Legendre, Laguerre, Hermite, and hypergeometric functions. A great many power series expansions, contour integral representations, generating series, definite integral formulae, and other connections among these functions may be deduced as special cases of about a dozen general formulae.

FLORENCE M. MEARS, *Secretary*

CALENDAR OF FUTURE MEETINGS

Joint Meeting with American Society for Engineering Education, Troy, New York, June 20–21, 1949.

Thirty-first Summer Meeting, Boulder, Colorado, August 29–30, 1949.

Thirty-third Annual Meeting, New York City, December 30, 1949.

The following is a list of the Sections of the Association with dates of future meetings in so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, West Virginia University, Morgantown, May 7, 1949.

ILLINOIS, Bradley University, Peoria, May 13–14, 1949

INDIANA, University of Notre Dame, May 7, 1949

IOWA

KANSAS

KENTUCKY, Centre College, Danville, May 14, 1949

LOUISIANA-MISSISSIPPI, Centenary College, Shreveport, Louisiana, Spring, 1950

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, University of Virginia, Charlottesville, May 14, 1949

METROPOLITAN NEW YORK, Spring, 1950

MICHIGAN

MINNESOTA, Gustavus Adolphus College, St. Peter, May 7, 1949

MISSOURI, Spring, 1950

NEBRASKA, Lincoln, May, 1949

NORTHERN CALIFORNIA, Berkeley, January 28, 1950

OHIO, April, 1950

OKLAHOMA, Oklahoma City, October 14, 1949

PACIFIC NORTHWEST, University of Washington, Seattle, Spring, 1950

PHILADELPHIA, Haverford College, November 26, 1949

ROCKY MOUNTAIN

SOUTHEASTERN, University of Florida, Gainesville, March, 1950

SOUTHERN CALIFORNIA, Immaculate Heart College, Hollywood, March 11, 1950

SOUTHWESTERN

TEXAS, Abilene, Spring, 1950

UPPER NEW YORK STATE, Syracuse University, Spring, 1950

WISCONSIN, Lawrence College, Appleton, May 14, 1949

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LAPLACE

SIR EDMUND WHITTAKER, University of Edinburgh

The editor of the MONTHLY has done me the honor to invite me to contribute an article in connection with the bicentenary of Laplace, which falls this year. As Laplace's collected works, which were published under the auspices of the French Academy of Sciences in 1878–1912, occupy fourteen large volumes, it will obviously not be possible to review his discoveries in detail: a mere catalogue of them would more than exhaust the space at my disposal. I do not propose, therefore, to do much more than to give an account of the dramatic experience, very early in his career, when, by solving a problem which had baffled both Euler and Lagrange, he came to be recognized as the greatest living mathematician. I wish to take the opportunity also of correcting some errors with regard to his life and character which have crept into well-known popular histories of mathematics.

When Laplace began his active career as a mathematician, more than eighty years had elapsed since the publication of Newton's *Principia*. For long after its first appearance, that greatest of all works of science had met with considerable opposition. The most eminent mathematicians of the end of the seventeenth century—Huygens, Leibnitz, John Bernoulli, Cassini,—declared against the Newtonian theory of gravitation; even in Cambridge, Newton's own University, natural philosophy continued for long to be studied in the text-book of Rohault, a Cartesian work. It was not until 1745 that the theory of the motions of the heavenly bodies began to be carried beyond the stage it had reached in the *Principia*; and for forty years after that, attention was focused almost exclusively on certain phenomena, which were well attested by astronomical observation, and which seemed to be irreconcilable with the Newtonian theory.

The most striking of these was what was called the great inequality of Jupiter and Saturn. From a comparison of ancient and current observations, it was found that for many centuries the mean motion, or average angular velocity round the sun, of Jupiter, had been continually increasing, while that of Saturn had been continually decreasing. Hence it could be inferred, by Kepler's third law, that the mean distance of Jupiter from the sun was always decreasing, and that of Saturn increasing; so the ultimate fate of Jupiter would be to fall into the sun, while Saturn would wander into space and be lost altogether to the solar system. This of course assumed that the phenomenon was truly secular, that is to say, that it proceeded always in the same direction, with a cumulative effect; but the observational evidence gave no support to any alternative view. Attempts to bring the great inequality of Jupiter and Saturn within the compass of the Newtonian theory of gravitation were made by Euler in 1748 and 1752, and by Lagrange in 1763, but without success. In 1773 Laplace, then aged twenty-four, took the matter up. Carrying the work to a higher degree of approximation than his predecessors, he was surprised to find that in the expression

for the disturbing action of Jupiter upon the mean motion of Saturn, the secular terms cancelled each other out; and following up the clue thus provided, he was able to establish a general theorem, to the effect that the mean motions of the planets cannot have any secular accelerations whatever as a result of their mutual attractions. It seemed as if he, like Euler and Lagrange, had failed to account for the observed fact, and not only so, but had even shown the Newtonian theory to be definitely incompatible with it. However, instead of accepting this consequence, he inferred that the inequality of Jupiter and Saturn was not truly secular, but was periodic, of very long period; and with this new idea, he set to work again, examining the way in which long-period inequalities might arise. Any long-period inequality must arise from a term of the same long period in the perturbing function. Now there were no long-period terms with appreciable coefficients in the perturbing function of one planet or another, so the coefficient of such a term must be very small, and the term might be represented as $p \sin qt$, where p is small, and where q must also be small since the period is long. But Laplace saw that, by the double integration which was involved in the passage from the perturbing function to the expression for the inequality, this term would become $(p/q^2) \sin qt$; and if q were small enough, the coefficient (p/q^2) might be quite large, even though p were very small. This was the true solution of the problem, and all that remained to be done was to identify the small term $p \sin qt$ in the perturbing function which produced the result. Laplace now recalled that five times the mean motion of Saturn is very nearly equal to twice the mean motion of Jupiter; so if n, n' are the mean motions, then $5n-2n'$ is very small, and the term in $\sin [(5n-2n')t]$ in the perturbing function would be of very long period; moreover, its coefficient would involve, as factors, comparatively high powers of the eccentricities and inclinations of the orbits, and so would be very small. Thus it would satisfy all the conditions required. So he arrived finally at a complete explanation of the great inequality; it was not secular, but was periodic, of very long period; and it was due, ultimately, to the fact that the mean motions of the two planets were nearly commensurable. The comparison of his theory with observation left no doubt of its truth.

From the point of view of the solution of the differential equations of motion of the planets, the ratio of the mean motions is classified as one of the arbitrary constants of the solution; and thus Laplace's discovery showed that the nature of the solution of differential equations may be profoundly influenced by the circumstance that one of the arbitrary constants of the solution is a rational or an irrational number. This is the root cause of great difficulties with regard to the convergence of the series of Celestial Mechanics, a subject which calls urgently for further research.*

Laplace's explanation of the great inequality of Jupiter and Saturn was the first of a long series of triumphs, which are recorded in the 2000 pages of his *Traité de Mécanique Céleste*, and which achieved the complete justification of the

* References for the literature may be found in the last chapter of my book *Analytical Dynamics.*

Newtonian theory. He made also many important discoveries in theoretical physics, and indeed he was interested in everything that helped to interpret Nature; pure mathematics for its own sake, however, did not greatly appeal to him, and his contributions to pure mathematics were mostly thrown off as mere by-products of his great works in natural philosophy. Yet there are several cases where a part of one of Laplace's papers, after undergoing a straightforward development at the hands of others, has come to be regarded as an important branch of pure mathematics, to which a substantial volume may fitly be devoted. For instance, E. W. Hobson's great work *The Theory of Spherical and Ellipsoidal Harmonics* (Cambridge, 1931) represents the natural flowering of Laplace's memoir of 1782 on the attraction of spheroids. G. Doetsch's book, *Theorie und Anwendung der Laplace-Transformation* (Berlin, 1937) describes the inevitable evolution of the method of "generating functions" introduced by Laplace in the *Théorie analytique des probabilités*; and any book on the Theory of Determinants consists chiefly of developments of the expansion-theorem which Laplace discovered when he was twenty-three (namely, that any determinant is equal to the sum of all the minors that can be formed from any selected set of its rows, each minor being multiplied by its algebraic complement); for most theorems on determinants can be proved by equating two different Laplace expansions of the same determinant to each other.

When in the course of his researches he comes to a situation where a heavy piece of pure mathematical working is needed, he often says, "Il est facile de voir," and gives the result without saying how he got it. His power of solving problems in pure mathematics has perhaps never been equalled, but he seems to have thought nothing of it, and to have assumed that it was possessed by all the readers of his works.

Coming lastly to personal matters, it is surprising that such learned, and usually well-informed and careful writers as D. E. Smith and Florian Cajori should have fallen into serious error regarding Laplace's origin and early history. The former says, "He was born in poverty, and owed his early education to the interest which his promise excited in men of intellectual power. Of these days of struggle he never spoke. Almost the first reliable records that we have of his life show him studying and afterwards teaching mathematics in the military school at Beaumont"; while Cajori's account is: "Very little is known of his early life. When at the height of his fame, he was loath to speak of his boyhood, spent in poverty. Some rich neighbors who recognized the boy's talent assisted him in securing an education. As an extern he attended the military school at Beaumont, where at an early age he became teacher of Mathematics."

For these statements there is no foundation. Laplace's father, Pierre de Laplace, was Syndic of Beaumont, proprietor of the small estate of Mérisier in the neighborhood, and well-connected, being the nephew of Maitre Olivier de Laplace, Chirurgien Royal. There is no reason to suppose that he was unable or unwilling to pay for the education of his only son Pierre-Simon; and our disbelief becomes complete when we learn that the "military school at Beaumont"

had no existence at that time. The school in which the boy was actually educated between the ages of 7 and 16 was attached to a Benedictine priory at Beaumont, and most of the teachers were Benedictine monks; they employed other teachers however, one of whom seems to have been Pierre-Simon's uncle, Father Louis de Laplace, a secular priest. At the age of 16, Pierre-Simon proceeded to the University of Caen, where he seems to have remained for five years, and where his first mathematical paper was written (it was published in the *Miscell. Taurin.*, then edited by Lagrange).*

Cajori adds "The political career of this eminent scientist was stained by servility and suppleness." It is difficult to see what justification there is for this statement. Laplace placed his scientific knowledge and ability at the disposal of whatever Government was in power; he was President of the Bureau des longitudes, President of the Commission de réorganisation de l'École polytechnique, and so forth; and was highly esteemed both by Napoleon, who made him a Count, and by the restored Bourbons, who further promoted him to be a Marquis. It is unnecessary to postulate "servility and suppleness" in order to account for the fact that he was *persona grata* to the successive rulers of France; his eminence as the greatest living man of science, and his value as an organizer of scientific education and research, made them glad to enlist his services, and to reward him with dignities never bestowed before nor since on a mathematician.

EQUIVALENCE RELATIONS IN ALGEBRAIC SYSTEMS

R. R. STOLL, Lehigh University

1. Introduction. This note is addressed primarily to an interested reader of modern algebra and its purpose is to present an elementary exposition of equivalence relations including a variety of illustrations, along with applications to group theory.

Although the concept of an equivalence relation is a prerequisite for a careful phrasing as well as a full understanding of a great variety of statements in mathematics, frequently the notion as such is met for the first time in an introductory text on modern algebra [see, for example, references **2**, **3**]. Its inclusion in such a text is a certainty due to the usefulness of the notion in clarifying the intent of many definitions and in developing efficiently fundamental results for groups and other algebraic systems. However, since its applicability is not restricted to algebra, it seems worth while to examine the concept independently.

* The facts regarding Laplace's family, school, and University, are fully discussed in an article by Professor Karl Pearson in Biometrika 21 (1929), 202.

2. Equivalence relations. A binary relation $\mathcal{R}$ over a set $S=\{a, b, \cdots\}$ is any function $\mathcal{R}(a, b)$, where a, b range independently over S, with values in the set consisting of two elements: true, false. In the event $\mathcal{R}(a, b)=$true, we write

$$a\mathcal{R}b,$$

and in the contrary case a solidus is superimposed upon the $\mathcal{R}$.

A binary relation $\mathcal{R}$ over S is called an *equivalence relation* if it satisfies the axioms,

(1) Reflexitivity: $a\mathcal{R}a$ for every a in S.
(2) Symmetry: $a\mathcal{R}b$ implies $b\mathcal{R}a$.
(3) Transitivity: $a\mathcal{R}b$, $b\mathcal{R}c$ imply $a\mathcal{R}c$.

For a characterization of equivalence relations another definition is needed. A partition $P=P(C_\alpha)$ of S into classes C_α is a collection of non-empty subsets C_α such that each element of S belongs to one and only one subset.

THEOREM 1: *An equivalence relation $\mathcal{R}$ over the set S defines a partition $P(C_\alpha)$ of S where the class C_a containing a consists of all x in S such that $x\mathcal{R}a$. Conversely, a partition of S defines an equivalence $\mathcal{R}$ where $a\mathcal{R}b$ if and only if a and b belong to the same class.*

Proof: By (1), a is a member of C_a, so that every element is in at least one class. But if a is in both C_b and C_d, so that $a\mathcal{R}b$ and $a\mathcal{R}d$, (2) implies $b\mathcal{R}a$ and then using (3), $b\mathcal{R}d$. Hence $C_b=C_d$ and a is in only one class.

For the converse, it is clear that (1), (2), (3) are satisfied.

In the set of integers the familiar notions of $a|b$ (a divides b), $a<b$, $(a, b)=1$ (a is relatively prime to b), and $|a-b|<1$ are examples of binary relations satisfying one or more of (1), (2), (3). We list next several relations that are actually equivalences.

E_1. The relation $a\equiv b \pmod{n}$ in the set of integers. This example is responsible for, in the general case, the notation $a\equiv b(\mathcal{R})$ in place of $a\mathcal{R}b$, and the phrase "a residue class mod $\mathcal{R}$" to denote a class C_a.

E_2. The relation $f(x)=g(x)$, for almost all $x\in I$, in the set of all real, single-valued functions $f(x)$ defined over an interval I.

E_3. The relation A similar to B in the set of all n by n matrices with entries from a field.

E_4. The relation T_α similar to T_β in the set of all plane triangles.

E_5. The relation A is obtainable from B by a rotation and translation, in the set of all proper conic sections.

E_6. This example, along with several definitions arising from special cases of it, is included for future reference. Let G and G' denote two sets and σ a single-valued correspondence from G onto G', *i.e.*, σ is a rule associating with each element $g\in G$, a unique element $g'=\sigma(g)\in G'$ such that $\sigma(x)=g'$ has a solution $x\in G$ for every $g'\in G$. Then $g_1\equiv g_2$, if and only if $\sigma(g_1)=\sigma(g_2)$, is an equivalence relation in G.

In the event $\sigma(x)=g'$ always has a unique solution for all $g'\in G'$, σ is a one-one correspondence between G and G'. If G and G' are multiplicative systems, *e.g.*, groups, and σ has the property $\sigma(ab)=\sigma(a)\sigma(b)$, then σ is called a homomorphism and G' a homomorphic image of G. If, moreover, σ is one-one, the homomorphism is called an isomorphism.

E_7. Our final example, the equality relation, is in a sense the most fundamental since in its analysis we arrive at the postulates assumed for an equivalence relation. Indeed, if one reflects on the everyday usage of this relation as a synonym for either identity or a qualified degree of likeness, *e.g.*, when we say two triangles are "equal," we may have in mind only the agreement of the areas, one soon comes to the conclusion that equality satisfies axioms (1) through (3). On the other hand, Theorem 1 demonstrates that these are sufficient to assure the desired separation into classes, and hence that they characterize equality. As such, any equivalence relation may be called an "equals" relation.

3. A fundamental application. The last example above leads immediately to one fundamental application of equivalence relations. On one hand we have seen that the equivalence axioms characterize equality. On the other hand, when dealing with a concrete mathematical system, we invariably have as an inherent part of the system a rule for deciding whether two elements are one and the same, or whether they are distinct, *i.e.*, we have given an equality. Thus when one realizes the duplication in effort that frequently arises in the analysis of various concrete systems and decides to study an abstract system whose properties are those underlying various concrete systems, one such assumed property will be the existence of an equivalence relation. Usually this basic relation is denoted by $=$, to distinguish it from further equivalences that may be defined.

Invariably at some stage of an investigation of a mathematical system S, one wishes to identify elements which although unequal, exhibit a certain likeness which we symbolize by $\mathcal{L}$. Any suitable criterion for indiscernibility must lead to a partition of S; hence $\mathcal{L}$ must be an equivalence relation. That is, $\mathcal{L}$ must measure up to the equivalence axioms to serve the purpose intended for it. Each of the examples E_1 through E_6 exhibits a likeness $\mathcal{L}$ in the set at hand. An equivalence class mod $\mathcal{L}$ consists of all elements of S indiscernible with respect to $\mathcal{L}$. With the way cleared for emphasis upon equivalence classes, it is sometimes in order to regard these classes as elements of a new set Σ and assign them names. For example, recall the definition of a cardinal number. In transfinite arithmetic one calls two sets equivalent if and only if they can be put in one-one correspondence, and then defines the cardinal number of a set S as the set of all classes which are equivalent with S. Accompanying the transition from S to Σ is the replacement of equivalence mod $\mathcal{L}$ by equality (identity) of elements in Σ. The reverse step is frequently used in computations: calculations with elements of Σ are performed mod $\mathcal{L}$ with elements of S chosen from the Σ-elements in question (*i.e.* so-called representatives of the equivalence classes). Calculations with rational numbers serve as an example of this.

4. A restricted type of equivalence relation. The minimum requirements for an algebraic system certainly include the existence of a set S with an equality relation for which there is defined a binary law of composition, *i.e.* a single-valued function $a\cdot b$ (or simply ab) of pairs a, b such that $a\cdot b$ is in S for a, b in S. Adopting this as our starting point, we superimpose an equivalence $\mathcal{R}$ on S in order to point out how one is led to an important restricted type of equivalence relation. Namely, denoting by Σ the set of equivalence classes C_a mod $\mathcal{R}$, we raise this question: Can an operation $\odot$ be defined in Σ based upon the operation $\cdot$ in S? We proceed along the lines of what might be one's first attempt* to investigate this question by tentatively defining

$$C_a \odot C_b = C_{a\cdot b}, \tag{4}$$

thus apparently making the product dependent upon the choice of class representatives. This difficulty is overcome if

$$C_{a'} = C_a \quad \text{and} \quad C_{b'} = C_b \quad \text{imply} \quad C_{a'\cdot b'} = C_{a\cdot b}$$

or, what amounts to the same thing,

$$a'\mathcal{R}a \quad \text{and} \quad b'\mathcal{R}b \quad \text{imply} \quad a'b'\mathcal{R}ab. \tag{5}$$

An equivalent form of (2) is

$$a'\mathcal{R}a \quad \text{implies} \quad a'x\mathcal{R}ax \quad \text{and} \quad xa'\mathcal{R}xa \quad \text{for all } x. \tag{6}$$

This sufficient condition is also a necessary one upon $\mathcal{R}$ in order that (4) be a well-defined operation. Equivalences satisfying (6) will be called regular.

Then if $\mathcal{R}$ is regular we observe that the correspondence ϕ of S onto Σ defined by

$$\phi(x) = C_a \quad \text{if and only if} \quad x \in C_a, \tag{7}$$

is a homomorphism (see E_6). For if along with $\phi(x) = C_a$, $\phi(y) = C_b$ then $\phi(xy) = C_{xy} = C_x \odot C_y$ using (4). But since $C_x = C_a$ and $C_y = C_b$ we have finally $\phi(xy) = C_a \odot C_b$. The homomorphism (7) is usually referred to as the "homomorphism of inclusion" or the "natural homomorphism."

In passing, we notice that if $\mathcal{R}$ is the equality relation in S the above has the following significance. We take as the elements of our algebraic system the equivalence classes mod $\mathcal{R}$ and assume that R is regular with repect to $\cdot$. Then (4) is a well-defined operation. Thus, a careful definition of a group begins with the existence of a set S, a binary operation $(\cdot)$, an equivalence $(=)$, regular with respect to $\cdot$. The group elements are then the equivalence classes mod $=$.

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* As an alternative "stimulus" for (4) we might take the following as our criterion for multiplication in Σ: Σ should be a homomorphic image of S under a correspondence mapping all elements of S belonging to an equivalence class onto an element of Σ. But the existence of such a homomorphism immediately implies the existence of one mapping the class containing a upon C_a and the homomorphism property then requires that (4) hold.

To illustrate this point, let S denote the set of all plane rotations ρ of a given equilateral triangle into itself. If $\mathfrak{R}$ is defined by $\rho\mathfrak{R}\rho'$ if and only if the number of degrees in both agree, we obtain an infinite cyclic group. Whereas if $\mathfrak{R}$ is defined by $\rho\mathfrak{R}\rho'$ if and only if they produce the same permutation of the vertices, a cyclic group of order three results.

5. Further applications to groups. Returning to the central idea of the preceding section, let us specialize our system to a group G with a regular equivalence $\mathfrak{R}$. Then, defining an operation in $\Gamma = \{C_a, C_b, \cdots\}$ by (4) gives a homomorphic image of G. But it is easily seen that a homomorphic image of a group is necessarily a group. Hence Γ is a group; its unit element is C_e where e is the unit in G. Thus, shrinking a group with the aid of a regular equivalence produces a homomorphic image.

Conversely, given a homomorphic image Γ of G, a partition, and therefore an equivalence $\mathfrak{R}$, is defined in G (see E_6) and the homomorphism property immediately demonstates that $\mathfrak{R}$ is regular. Hence the problem of finding all homomorphic images of G amounts to that of finding all regular equivalences over G.

Concerning this problem we first mention the coset decomposition of a group G with respect to a subgroup H. Define

$$a\mathfrak{R}b \quad \text{if and only if} \quad a = hb,\ h \in H.$$

Since $a = ae$, $a\mathfrak{R}a$; if $a = hb$ then $b = h^{-1}a$, so that $\mathfrak{R}$ is symmetric. Finally $a = hb$, $b = h'c$ implies $a = (hh')c$, so that $\mathfrak{R}$ is transitive and hence an equivalence. The equivalence class C_a is Ha, a so-called right coset of H, and

$$G = \Sigma Ha.$$

Since $a\mathfrak{R}b$ implies $ax\mathfrak{R}bx$, $\mathfrak{R}$ is right regular, *i.e.*, satisfies the first part of (6).

But conversely, starting with a right regular equivalence $\mathfrak{R}$ in G we find that C_e is a subgroup and $C_a = C_e a$, since $b\mathfrak{R}a$ implies $b_a{}^{-1}\mathfrak{R}e$; hence $ba^{-1} \in C_e$, $b \in C_e a$ and conversely. Hence the various right regular equivalences in G are determined by the right coset decompositions of G with respect to its subgroups. It is of interest to note that once the elements equivalent to the unit e are known in a right regular equivalence, the entire decomposition of G is known.

Similar results are obtained for left regular equivalences and we can now analyze the regular ones. If $\mathfrak{R}$ is regular, then on one hand it defines the left coset decomposition with respect to the subgroup H of all elements x such that $x\mathfrak{R}e$. On the other hand it defines a right coset decomposition with respect to the same subgroup. Hence $\mathfrak{R}$ stems from a subgroup for which the left cosets are identical with its right cosets. Such a subgroup N we call normal, and have

$$xN = Nx \quad \text{for all} \quad x \in G.$$

Clearly this property characterizes the normal subgroups. Conversely, a normal subgroup defines a regular equivalence.

If N is a normal subgroup of G we now know that its cosets xN form a group with multiplication rule (4):

$$aN \cdot bN = abN$$

and unit N. The resulting (factor) group, G/N, is homomorphic to G and conversely every homomorphic image of G can be duplicated by (*i.e.* is isomorphic to) such a factor group.

The analysis of a ring along the same lines as above yields the corresponding homomorphism theorem just as efficiently.

6. Conclusion. In conclusion we mention that the set E of all possible equivalence relations definable over a given set S has been made an object of study [1, 4]. If operations are defined appropriately, E becomes an example of a lattice.

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AN APPLICATION OF NEWTON'S POWER-SUM FORMULAS

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1. Introduction. If $f(x)$ and $g(x)$ are real polynomials, then we may ask under what conditions

$$f(g(x)) = g(f(x)) \tag{1}$$

identically in x. Without further restrictions, there are clearly an infinite number of $g(x)$'s which satisfy (1) for a fixed $f(x)$, as for example all the polynomials of type $f(\cdots(f(x))\cdots)$, where the operator f is iterated n times. The main purpose of the present note is to prove in an elementary way that for a given $f(x)$ the only $g(x)$'s having the same degree as $f(x)$ and satisfying (1) are: $g(x) = f(x)$ if the degree is even, and one other possibility if the degree is odd and greater than one (see Theorem I). An application of this result is made to obtain certain uniqueness properties of families of mutually commutative polynomial operators. For the general properties of polynomial substitution one may consult the papers listed in the references.

2. The main theorem. Let f and g refer to the polynomial operators defined by

$$f(x) = \sum_{i=0}^{n} a_i x^{n-i} \quad \text{and} \quad g(x) = \sum_{i=0}^{n} b_i x^{n-i},$$

where the coefficients are real. Also, let fg mean the operator defined by the substitution $f(g(x))$. Then we can prove

THEOREM I: *If f and g are real polynomial operators of equal degree $n>1$, then $fg=gf$ holds if and only if $f=g$, unless $f(x)$ is of the form $a_0(x-c)^{2k+1}+b_1(x-c)^{2k-1}+\cdots+b_{2k}(x-c)+c$, in which case $g(x)$ may also be $-f(x)+2c$, where $c=-a_1/na_0$, $n=2k+1$.*

Geometrically this means that if $y=f(x)$ is symmetric to a point on $y=x$, then and only then does it have a non-trivial commuting polynomial of the same degree. If $n=1$, each point on $y=x$ defines an infinite bundle of commuting linear functions.

Proof. Let r_i and s_i, $i=1,\cdots,n$ be the zeros of $f(x)$ and $g(x)$ respectively. Then $fg=gf$ implies, as a necessary condition,

$$a_0\prod_{i=1}^{n}(g(x)-r_i)=b_0\prod_{i=1}^{n}(f(x)-s_i).$$

Thus the zeros of $f(g(x))$ are distributed in two ways, once among the factors $\{g(x)-r_i\}$ and again among the factors $\{f(x)-s_i\}$. This distribution can be effectively utilized by application of Newton's power-sum formulas. Let σ_i and S_i, $i=1,\cdots,n$ denote the usual elementary symmetric functions and the power sums of the roots, respectively. Then Newton's formulas can be expressed as follows:

$$\begin{pmatrix} 1 & & & & \\ -\sigma_1 & 1 & & & \\ \sigma_2 & -\sigma_1 & 1 & & \\ \vdots & & & \ddots & \\ (-1)^{n-1}\sigma_{n-1} & \cdots & \cdots & \cdots & 1 \end{pmatrix}\begin{pmatrix} S_1\\ S_2\\ S_3\\ \vdots\\ S_n\end{pmatrix}=\begin{pmatrix}\sigma_1\\ -2\sigma_2\\ 3\sigma_3\\ \vdots\\ (-1)^{n-1}n\sigma_n\end{pmatrix}.$$

These may be applied to each factor and added over the factors. Two expressions for each S_i of $f(g(x))$ will thus be generated, and by equating them the desired relations among the a_i and b_i appear. The first two steps and the critical last step (for $n>3$) are as follows:

$$S_1=\sum_{i=1}^{n}(-b_1/b_0)=\sum_{i=1}^{n}(-a_1/a_0),\quad\text{implying}\quad a_1/a_0=b_1/b_0.$$

$$S_2=\sum_{i=1}^{n}[(-b_1/b_0)^2-2b_2/b_0]=\sum_{i=1}^{n}[(-a_1/a_0)^2-2a_2/a_0];\ \text{hence}\ a_2/a_0=b_2/b_0.$$

. . . .

$$S_n=\sum_{i=1}^{n}[\cdots+(-1)^{n-1}n(b_n-r_i)/b_0]=\sum_{i=1}^{n}[\cdots+(-1)^{n-1}n(a_n-s_i)/a_0];$$

hence $(1/b_0)\sum (b_n - r_i) = (1/a_0)\sum (a_n - s_i)$; and finally $n(a_nb_0 - b_na_0) = a_1 - b_1$.

Now, an examination of the condition $fg=gf$ gives immediately that $a_0b_0^n = a_0^nb_0$. So if n is even, then $a_0=b_0$; otherwise $a_0=\pm b_0$. Hence

(a). For $a_0=b_0$, we get $a_i=b_i$ for $i=1, \cdots, n-1$. Furthermore, $na_0(a_n-b_n) = 0$, or $a_n=b_n$. These are obviously sufficient conditions.

(b). For $a_0=-b_0$, we get $a_i=-b_i$ for $i=1, \cdots, n-1$, and $na_0(a_n+b_n) = -2a_1$, or $b_n=-a_n-2a_1/na_0$. That is, $g(x)$ differs from $-f(x)$ by a constant. The implication of this necessary condition is clear if one writes $g(x)=-f(x)+k$ and substitutes in $fg=gf$. For then $f(-f(x)+k)=-f(f(x))+k$, or $f(k-y) = -f(y)+k$. Setting $y=z-k/2$ shows that $f(x)$ must be symmetric to $(k/2, k/2)$. Such a function will be of the form given in the statement of the theorem, and a direct check shows that it commutes with $-f(x)+2c$.

3. **Application.** As an application of Theorem I, we may show

THEOREM II. *Any family F of real mutually commutative polynomial operators, with at least one operator of each degree,* has exactly one of each degree. The prime degree operators determine the whole family.*

This is a corollary of the

LEMMA. *If F_0 is any family of real mutually commutative polynomial operators containing at least one operator of even degree, then F_0 contains only one operator of each degree represented.*

Proof. Let e be the degree of the even operator f_e which is in F_0 by hypothesis. Then, by Theorem I, f_e is the only operator of degree e in F_0, this being true for any even degree. Let f_a and g_a be of degree a and in F_0. Then f_af_e and g_af_e commute with one another, and being of even degree must be equal. Hence $f_a=g_a$. This proves the Lemma.

In particular, the Lemma implies that in F of Theorem II there is just one operator of each degree. If f_2 in F is given, then f_2^n is in F for all n, where f^n means f iterated n times. In fact, when the prime degree operators f_{p_i} are given, then for $a=\prod p_i^{a_i}$, the composite $\prod f_{p_i}^{a_i}$ is in F. Hence $f_a=\prod f_{p_i}^{a_i}$. This proves Theorem II.

Perhaps the best known family of type F are the Tschebycheff polynomials.

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* A study of such families has been made by H. Thielman and H. Block.

THE COMPOSITION OF QUADRATIC BINARY FORMS*

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1. Introduction. Though the general basis for the composition of forms lies in the ideal theory, the composition of binary quadratic forms was first developed independent of such theory; in fact, it antedated the existence of ideal theory. Dedekind† laid the basis for the relationship between the composition of ideals and that of forms and Weber‡ developed it. But neither used it to prove Gauss's theorem on duplication nor for other important results in the theory of binary forms. The literature in general seems somewhat hazy on the precise details of the subject. It seems worthwhile therefore to give a self-contained elementary development of the theory assuming neither knowledge of ideal theory nor of quadratic forms. It will be seen that the use of ideal theory adds elegance to the treatment and gives promise of further application.

We first define and develop some of the properties of ideals in a quadratic field. Let $\mathfrak{F}$ be a field obtained by adjoining $\sqrt{\Delta}$ to the field of rational numbers where Δ is a non-square integer congruent to 1 or 0 (mod 4); that is, $\mathfrak{F}$ consists of all numbers of the form $a+b\sqrt{\Delta}$ where a and b are rational numbers. Let $\sigma=\frac{1}{2}(\sqrt{\Delta}+1)$ or $\frac{1}{2}\sqrt{\Delta}$ according as $\Delta\equiv 1$ or 0 (mod 4) and call quadratic integers those numbers expressible in the form $x+y\sigma$ where x and y are rational integers. (We reserve the name "integer" for rational integers.) The set of all quadratic integers in $\mathfrak{F}$ we denote by $J(\Delta)$ or J. It is easy to see that J is closed under addition, subtraction and multiplication; that is, the sum, difference and product of any two quadratic integers are quadratic integers. If α is an element of J we call the number obtained from α by replacing $\sqrt{\Delta}$ by $-\sqrt{\Delta}$ its conjugate and denote the conjugate of α by α^c. Call $\alpha\alpha^c=N(\alpha)$, the norm of α, and see that $N(\alpha)N(\beta)=N(\alpha\beta)$. If α is a quadratic integer, $N(\alpha)$ is an integer.

If now $\alpha_1, \alpha_2, \cdots, \alpha_n$ is a set of numbers of J, we call all numbers $\tau_1\alpha_1+\tau_2\alpha_2+\cdots+\tau_n\alpha_n$ where the τ_i range over all numbers of J, the ideal $\mathfrak{J}=(\alpha_1, \alpha_2, \cdots, \alpha_n)$. If the α_i are such that every number in $\mathfrak{J}$ is expressible uniquely in the form $x_1\alpha_1+x_2\alpha_2+\cdots+x_n\alpha_n$ where the x_i are integers, we call $\alpha_1, \alpha_2, \cdots, \alpha_n$ a basis of $\mathfrak{J}$ and write $\mathfrak{J}=[\alpha_1, \alpha_2, \cdots, \alpha_n]$ with brackets instead of parentheses. Two ideals are said to be equal if they contain the same numbers. An ideal consisting of all numbers $\alpha\rho$, where α is a fixed number of J and ρ ranges over all numbers of J is called a principal ideal and is written (α). If $\mathfrak{J}=[\alpha_1, \alpha_2]$ we call $\mathfrak{J}^c$ the ideal $[\alpha_1^c, \alpha_2^c]$ and $u\mathfrak{J}$ the ideal $[u\alpha_1, u\alpha_2]$.

2. Some properties of quadratic ideals. We first need some classical results on ideals.

* The author wrote this paper while on Sabbatic leave from Cornell University and with the aid of a grant from The Research Corporation.

† Dirichlet-Dedekind, Zahlentheorie, ed. 2 (1871), Suppl. XI pp. 488–497.

‡ Heinrich Weber, Lehrbuch der Algebra, ed. 2 (1908), pp. 368–397.

THEOREM 1. *Every ideal* $\mathfrak{J}$ *of* J *has a basis* $[r, s+u\sigma]$ *where* r, s, u *are integers,* r *and* u *both being positive. Furthermore,* r *is the g.c.d. of the integers in* $\mathfrak{J}$ *and* u *is the g.c.d. of the coefficients of* σ *in the quadratic integers of* $\mathfrak{J}$. *(Hence* r *and* u *are uniquely determined by* $\mathfrak{J}$.*)*

To prove this, first notice that $\mathfrak{J}$ contains an integer since if α is a quadratic integer in $\mathfrak{J}$, then $N(\alpha)$ is also in $\mathfrak{J}$. If then we let r be the least positive integer in $\mathfrak{J}$ we see that it must divide all integers in $\mathfrak{J}$ since if it does not divide an integer b in $\mathfrak{J}$, then $b=qr+r'$, where r' is a positive integer less than r; b and qr being in $\mathfrak{J}$ implies that r' is in $\mathfrak{J}$ contrary to the supposition that r has least positive value. By the same reasoning we can show that u has the properties desired.

It remains to show that every number of $\mathfrak{J}$ is a unique linear combination of r and $s+u\sigma$ with integer coefficients. To this end let $a+b\sigma$ be a number of $\mathfrak{J}$. Then $b=ux$ determines an integer x and $a+b\sigma-x(s+u\sigma)=a-xs$ is an integer in $\mathfrak{J}$ and hence is equal to ry for some integer y; thus $a+b\sigma=yr+x(s+u\sigma)$. Furthermore $yr+x(s+u\sigma)=y'r+x'(s+u\sigma)$ implies that $x=x'$, $y=y'$ and the representation in terms of the basis is unique. This completes the proof of the theorem.

There are certain restrictions on r, s and u of the theorem above if r, $s+u\sigma$ are to form a basis for an ideal. For instance, 3 and $1+2\sqrt{3}$ do not form a basis of an ideal since the equation $(1-2\sqrt{3})(1+2\sqrt{3})=-11$ shows that -11 is in the ideal $(3, 1+2\sqrt{3})$ while -11 is not expressible in the form $3x+(1+2\sqrt{3})y$ for integers x and y. We now find necessary and sufficient conditions on integers r, s and u with $ru\neq 0$ that $(r, s+u\sigma)$ shall form a basis for an ideal. Suppose first that $(r, s+u\sigma)$ is such a basis. Then all numbers of the ideal are of the form

$$t = r(x_1 + y_1\sigma) + (s + u\sigma)(x_2 + y_2\sigma^c)$$

for integers x_1, x_2, y_1, y_2. The coefficient of σ in t is $ry_1+ux_2-sy_2$ and since, from the theory of numbers, y_1, x_2 and y_2 can be chosen so that this coefficient is the g.c.d. of r, u and s, we see that u, being by Theorem 1 a divisor of all coefficients of σ in numbers of $\mathfrak{J}$, must divide the g.c.d. of r, u and s; hence the equations $r=au$, $s=eu$ determine integers a and e. If, on the other hand, we choose y_1, x_2 and y_2 so that $ry_1+ux_2-sy_2=0$, that is, $ay_1+x_2-ey_2=0$, and let $\sigma+\sigma^c=\epsilon$ where ϵ is 1 or 0 according as $\Delta\equiv 1$ or 0 (mod 4) we have

$$(1)\qquad \begin{aligned} t &= rx_1 - (rs/u)y_1 + y_2k/u \\ k &= s^2 + su\epsilon + u^2\sigma\sigma^c = N(s + u\sigma). \end{aligned}$$

Since t is an integer in $\mathfrak{J}$ it must, by Theorem 1, be divisible by r for all values of x_1, y_1 and y_2; hence $k\equiv 0 \pmod{ru}$. Thus we have shown that if $(r, s+u\sigma)$ is a basis under the restrictions of Theorem 1, then r and s are divisible by u and $N(s+u\sigma)$ is divisible by ru. Conversely if these conditions hold we see by reference to the above that u divides all coefficients of σ for numbers of $\mathfrak{J}$ and that,

since all integers in $\mathfrak{J}$ are expressible in the form of t in (1), r divides all integers in $\mathfrak{J}$. Hence we have proved

THEOREM 2. *$\mathfrak{J}=[r, s+u\sigma]$ for r, s and u integral and $ru\neq 0$ if and only if $r=ua$, $s=ue$ determine integers a and e and $N(s+u\sigma)\equiv 0 \pmod{ru}$. That is, we can write $\mathfrak{J}=[au, eu+u\sigma]=u[a, e+\sigma]$ with $N(e+\sigma)\equiv 0 \pmod{a}$.*

We call a basis $[r, s+u\sigma]$ of an ideal a *reduced basis* if r, s and u are integers, r and u both positive. Theorem 2 gives an alternative way of writing a reduced basis.

Suppose $[\alpha_1, \alpha_2]$ and $[\beta_1, \beta_2]$ are two equal or distinct ideals for which the following equations hold:

$$\alpha_1 = \beta_1 t_{11} + \beta_2 t_{21}$$
$$\alpha_2 = \beta_1 t_{12} + \beta_2 t_{22}. \tag{2}$$

Then we may say that the basis $[\alpha_1, \alpha_2]$ is taken into the basis $[\beta_1, \beta_2]$ by the linear transformation T whose matrix is

$$T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}.$$

If the elements of T are integers and its determinant ± 1 ,we call the transformation unimodular; if its determinant is $+1$ we call it properly unimodular. Considering (α_1, α_2) and (β_1, β_2) as two-row matrices the equations (2) may be written in matric notation as follows:

$$(\alpha_1, \alpha_2) = (\beta_1, \beta_2)T.$$

Next we prove

THEOREM 3. *Two ideals $[\alpha_1, \alpha_2]$ and $[\beta_1, \beta_2]$ are equal if and only if there is a unimodular transformation taking one basis into the other.*

If there is a unimodular transformation taking one into the other the numbers in the ideals are the same, since (2) and T unimodular implies that $x\alpha_1+y\alpha_2 =(xt_{11}+yt_{12})\beta_1+(xt_{21}+yt_{22})\beta_2$ and if the coefficients of α_1 and α_2 are integers, so are the coefficients of β_1 and β_2. Suppose on the other hand, that the ideals are equal. Then there are matrices T and T' with integer elements such that the matric equations $(\alpha_1, \alpha_2)=(\beta_1, \beta_2)T$ and $(\beta_1, \beta_2)=(\alpha_1, \alpha_2)T'$ hold. Then $(\alpha_1, \alpha_2) =(\alpha_1, \alpha_2)T'T$ and since the representation of α_1 and α_2 by α_1 and α_2 is unique we have $T'T=I$, the identity matrix, which shows that $|T'||T|=1$ and hence $|T|=\pm 1$. Thus the theorem is proved.

If $\mathfrak{J}$ has the reduced basis $[ua, ue+u\sigma]$ we define the norm of $\mathfrak{J}$, denoted by $N(\mathfrak{J})$, to be ru and see that $N(\mathfrak{J})=u^2a$. If $\mathfrak{J}_1$ and $\mathfrak{J}_2$ are two ideals in J, their product $\mathfrak{J}_1\mathfrak{J}_2=\mathfrak{J}_3$ is defined to be that ideal consisting of all numbers in J expressible in the form $\sum\rho_i\alpha_i\beta_i$ where the α_i are in $\mathfrak{J}_1$, the β_i are in $\mathfrak{J}_2$ and the ρ_i are quadratic integers. Since the set of numbers expressible in the form

$\sum \rho_i \alpha_i^c \beta_i^c$ coincides with the conjugates of the set of numbers expressible in the form $\sum \rho_i \alpha_i \beta_i$ we see that $\mathfrak{J}_1\mathfrak{J}_2 = \mathfrak{J}_3$ implies $\mathfrak{J}_1^c\mathfrak{J}_2^c = \mathfrak{J}_3^c$. Multiplication of ideals is associative and commutative since quadratic integers have these properties. We call an ideal $\mathfrak{J} = [ua, ue+u\sigma]$ primitive if 1 is the g.c.d. of a, $2e+\sigma+\sigma^c$, and c where $c = N(e+\sigma)/a$. The following important result holds even if the ideals concerned are neither primitive nor principal but the general proof is much more difficult and the more restricted result is sufficient for our purposes here.

THEOREM 4. *If each of the ideals $\mathfrak{J}_1$, $\mathfrak{J}_2$ is either primitive in J or principal, then* $N(\mathfrak{J}_1\mathfrak{J}_2) = N(\mathfrak{J}_1)N(\mathfrak{J}_2)$.

We prove this theorem by showing that it follows from Lemma 1 below. This consequence is direct since, using the lemma, $(N(\mathfrak{J}_1))(N(\mathfrak{J}_2)) = \mathfrak{J}_1\mathfrak{J}_1^c\mathfrak{J}_2\mathfrak{J}_2^c = \mathfrak{J}_3\mathfrak{J}_3^c = (N(\mathfrak{J}_3))$, where $\mathfrak{J}_3 = \mathfrak{J}_1\mathfrak{J}_2$, and hence $N(\mathfrak{J}_1)N(\mathfrak{J}_2) = N(\mathfrak{J}_3)$. It remains to show

LEMMA 1. *If the ideal $\mathfrak{J}$ is primitive or principal, then $\mathfrak{J}\cdot\mathfrak{J}^c$ is the principal ideal consisting of all multiples of $N(\mathfrak{J})$ by numbers of J.*

To prove this first for $\mathfrak{J}$ primitive, write $\mathfrak{J} = [ua, ue+u\sigma]$ and see that all elements of $\mathfrak{J}\cdot\mathfrak{J}^c$ are linear combinations of u^2a^2, $u^2a(e+\sigma)$, $u^2a(e+\sigma^c)$, $u^2aN(e+\sigma)/a = u^2ac$, with integer coefficients. Thus all numbers of $\mathfrak{J}\cdot\mathfrak{J}^c$ are divisible by u^2a. Furthermore, addition of the second and third numbers above, shows that $\mathfrak{J}\cdot\mathfrak{J}^c$ contains u^2ag where g is the g.c.d. of a, $2e+\sigma+\sigma^c$ and c which, by the primitiveness of $\mathfrak{J}$, is equal to 1. This shows that $\mathfrak{J}\cdot\mathfrak{J}^c$ is the principal ideal (u^2a).

If, on the other hand, $\mathfrak{J}$ is principal it may be written in the form $u(v+w\sigma)$ where v and w are relatively prime. Then $\mathfrak{J}\cdot\mathfrak{J}^c = (u^2N(v+w\sigma))$ and it remains to show that $N(\mathfrak{J}) = u^2N(v+w\sigma)$. We do this by finding a reduced basis for $\mathfrak{J}_1 = (v+w\sigma)$. Now every number of $\mathfrak{J}_1$ is expressible in the form $L = (v+w\sigma)(x+y\sigma^c)$ for integers x and y. But $L = vx+wyN(\sigma)+(\sigma+\sigma^c)yv+(wx-yv)\sigma$ shows that x and y may be chosen so that the coefficient of σ is 1. All values of x and y which make $wx-yv=0$ are integral multiples of $x=v$ and $y=w$. Hence all integers in $\mathfrak{J}_1$ are integral multiples of $v^2+w^2N(\sigma)+(\sigma+\sigma^c)vw = N(v+w\sigma)$ which is therefore the norm of $\mathfrak{J}_1$.

It is convenient to call $d(\mathfrak{J})$, the determinant of an ideal $[\alpha_1, \alpha_2]$, the expression $-(\alpha_1^c\alpha_2-\alpha_2^c\alpha_1)^2/4$. To justify this definition by showing that its value is independent of the particular basis chosen let $(\alpha_1', \alpha_2') = (\alpha_1, \alpha_2)T$ where T is unimodular. Then

$$\begin{vmatrix} \alpha_1' & \alpha_2' \\ \alpha_1'^c & \alpha_2'^c \end{vmatrix} = \begin{vmatrix} \alpha_1 & \alpha_2 \\ \alpha_1^c & \alpha_2^c \end{vmatrix} |T|$$

which implies that the two determinants are equal except perhaps for sign, and hence that the change of basis does not alter $d(\mathfrak{J})$. Furthermore if $\mathfrak{J} = [r, s+u\sigma]$,

where r, s and u are the properties imposed in Theorem 1, we have $d(\mathfrak{J}) = -r^2u^2(\sigma-\sigma^c)^2/4 = -N^2(\mathfrak{J})(\sigma-\sigma^c)^2/4 = -N^2(\mathfrak{J})\Delta/4$.

3. A correspondence between ideal classes and classes of quadratic forms. We first need some of the terminology of quadratic forms. Let the binary form be denoted by $f=ax^2+bxy+cy^2$, where a, b and c are integers. The matrix

$$F = \begin{bmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{bmatrix}$$

we call the matrix of f and the determinant of F, $ac-\frac{1}{4}b^2$, is called the determinant of f and is denoted by $d(f)$. Using matric multiplication we have

$$f = [x, y]\begin{bmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

or $f=(x, y)F(x, y)^T$.* We say that two forms f_1 and f_2 are *equivalent* if there is a unimodular transformation taking one into the other, that is, if f becomes $f'=a'x'^2+b'x'y'+c'y'^2$ in virtue of the transformation $(x, y)=(x', y')T$, T being unimodular. Two equivalent forms f_1 and f_2 are said to be in the same class and we write $f_1 \cong f_2$. If there is a unimodular transformation of determinant $+1$ taking f_1 into f_2 we say that the forms are properly equivalent and that they are in the same proper class; if there is a unimodular transformation of determinant -1 taking one form into the other the forms are called improperly equivalent. Two form f_1 and f_2 of the same determinant are said to be in the same genus if one can be taken into the other by a linear transformation with rational elements whose denominators are prime to twice the determinant of the forms; we then write $f_1 \vee f_2$.

Suppose $\mathfrak{J}=[ua, ue+u\sigma]$. Then $N(uax+u(e+\sigma)y)=u^2N(ax+(e+\sigma)y) = u^2a(ax^2+bxy+cy^2)$ where

$$(3) \qquad b = 2e + \sigma + \sigma^c, \qquad ac = N(e + \sigma).$$

Furthermore $ac-\frac{1}{4}b^2 = -\frac{1}{4}(\sigma-\sigma^c)^2 = -\frac{1}{4}\Delta$. Moreover, we have seen that any basis of $\mathfrak{J}$ may be obtained from its reduced basis by a unimodular transformation. This transformation will leave unaltered the norm and determinant of the ideal and takes f into an equivalent form. Thus we have proved

THEOREM 5. *If* $\mathfrak{J}=[\alpha_1, \alpha_2]$ *is in* $J(\Delta)$, *then*

$$(4) \qquad N(\alpha_1x + \alpha_2y) = N(\mathfrak{J})f(x, y)$$

where $f(x, y)=ax^2+bxy+cy^2$ *and* $d(f) = -\frac{1}{4}\Delta$, *that is,* $d(\mathfrak{J})=N^2(\mathfrak{J})d(f)$. *If the basis is reduced, the relationship between the basis and the coefficients of f is given by equations* (3).

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* The superscript T denotes "transpose" and I denotes "inverse."

The following theorem establishes the correspondence in the other direction.

THEOREM 6. *With a class of forms there is associated by* (5) *below an ideal* $\mathfrak{J}$ *in* $J(\Delta)$ *where* $b^2-4ac=\Delta$ *and for some basis of* $\mathfrak{J}$ *and some form* f *of the class, equation* (4) *holds with* $f=ax^2+bxy+cy^2$.

To prove this suppose that m is an integer least in absolute value represented by the class of forms. Then there must be a solution in integers x_0, y_0 of the equation $f=m$. Now x_0 and y_0 must be relatively prime since the square of any common factor would divide m denying the supposition that m is least. We may then determine integers v and w such that $x_0v-y_0w=1$ and the transformation

$$\begin{bmatrix} x_0 & w \\ y_0 & v \end{bmatrix}$$

is unimodular and will take f into a form whose leading coefficient is m. Hence we write f in the form in Theorem 6 with a a non-zero integer least in absolute value of all the numbers represented by f. Then with f we associate the ideal

$$I = \left(a, \frac{b+\sqrt{b^2-4ac}}{2}\right) = (\alpha_1, \alpha_2) \tag{5}$$

in $J(\Delta)$ where $\Delta=b^2-4ac$, and see that $N(\alpha_1x+\alpha_2y)=af$. It remains to show that $a=N(\mathfrak{J})$ and hence (α_1, α_2) is a basis. If $\Delta\equiv 1 \pmod 4$ we may take $r=a$, $s=(b-1)/2$, $u=1$ and, since $\sigma=\frac{1}{2}(1+\sqrt{b^2-4ac})=\frac{1}{2}(1+\sqrt{\Delta})$, we have $N(s+u\sigma)=ac\equiv 0 \pmod a$ and the conditions of Theorem 2 are satisfied. If $\Delta\not\equiv 1 \pmod 4$, b is even and we may take $s=\frac{1}{2}b$, $u=1$, $\sigma=\frac{1}{2}\sqrt{\Delta}$ and again $N(s+u\sigma)=ac\equiv 0 \pmod a$. Furthermore the correspondence between coefficients and basis is the same as in (3).

Notice that the form which we have made correspond to $\mathfrak{J}$ depends on the particular basis but that the class of that form is independent of the basis. In both cases the ideal $\mathfrak{J}=[\alpha_1, \alpha_2]$ and the form f satisfy condition (4). However, though all forms associated with any ideal by the above means are properly or improperly equivalent to one another, there may be several ideals associated with one form; for instance $\mathfrak{J}$ and $\rho\mathfrak{J}$, with ρ a quadratic integer, are associated with the same class of forms. To obtain uniqueness of correspondence we say that two ideals $\mathfrak{J}_1$ and $\mathfrak{J}_2$ are in the same class (or ideal class) if there exist quadratic integers ρ_1 and ρ_2 such that $\rho_1\mathfrak{J}_1=\rho_2\mathfrak{J}_2$. Let Σ denote an ideal class (that is, the set of all ideals in a given class) and Γ the set of all forms (properly or improperly) equivalent to a given form. As above, Σ^c denotes the class of ideals obtained from Σ by replacing each number by its conjugate. We prove

THEOREM 7. *Two primitive ideals* $\mathfrak{J}_1$ *and* $\mathfrak{J}_2$ *in* $J(\Delta)$ *are associated by* (4) *with the same class* Γ *of forms if and only if the ideals are in the same class* Σ *or conjugate classes* Σ *and* Σ^c.

To prove this let $\mathfrak{J}_i=[\alpha_i, \beta_i]$ and $N(\alpha_ix_i+\beta_iy_i)=N(\mathfrak{J}_i)f_i(x_i, y_i)$ for $i=1, 2$

and $f_1(x_1, y_1) \cong f_2(x_2, y_2)$. If T is a unimodular transformation taking f_1 into f_2 we have the matric equation $(x_1, y_1) = (x_2, y_2)T$ and $f_1(x_1, y_1) = f_2(x_2 y_2)$ identically in x_2 and y_2. Now $(\alpha_1, \beta_1)^T = T^I(\alpha_1', \beta_1')^T$ defines a new basis α_1', β_1' of $\mathfrak{J}_1$ and $(x_1, y_1)(\alpha_1, \beta_1)^T = (x_2, y_2)(\alpha_1', \beta_1')^T$. Thus $N(\alpha_1 x_1 + \beta_1 y_1) = N(\mathfrak{J}_1) f_2(x_2, y_2)$ implies $N(\alpha_1' x_2 + \beta_1' y_2) = N(\mathfrak{J}_1) f_2(x_2, y_2)$ and replacing α_1', β_1' by α_1, β_1 we have

$$N(\alpha_i x_2 + \beta_i y_2) = N(\mathfrak{J}_i) f_2(x_2, y_2), \qquad i = 1, 2 \tag{6}$$

identically in x_2 and y_2. If a and c are the coefficients of x_2^2 and y_2^2 (they must be different from zero) respectively in $f_2(x_2, y_2)$ we have, taking the pairs of values (1, 0) and (0, 1)

$$N(\alpha_i) = N(\mathfrak{J}_i)a, \qquad N(\beta_i) = N(\mathfrak{J}_i)c, \tag{7}$$

and hence

$$N(\alpha_1)N(\mathfrak{J}_2) = N(\alpha_2)N(\mathfrak{J}_1), \qquad N(\beta_1)N(\mathfrak{J}_2) = N(\beta_2)N(\mathfrak{J}_1). \tag{8}$$

Then the ideals $\mathfrak{J}_4 = \alpha_1 \mathfrak{J}_2$ and $\mathfrak{J}_3 = \alpha_2 \mathfrak{J}_1$ have equal norms by (8) and Theorem 4 and are associated with the same class Γ of forms. Furthermore, $\mathfrak{J}_4$ and $\mathfrak{J}_3$ are in the same ideal class as $\mathfrak{J}_2$ and $\mathfrak{J}_1$ respectively. Thus, multiplying (6) by $N(\alpha_2)$ and $N(\alpha_1)$ respectively we have, equating coefficients of x_2^2, $x_2 y_2$, y_2^2 on the left side,

$$\alpha_4 \overset{c}{\alpha}_4 = \alpha_3 \overset{c}{\alpha}_3, \qquad \alpha_4 \overset{c}{\beta}_4 + \overset{c}{\alpha}_4 \beta_4 = \alpha_3 \overset{c}{\beta}_3 + \overset{c}{\alpha}_3 \beta_3, \qquad \beta_4 \overset{c}{\beta}_4 = \beta_3 \overset{c}{\beta}_3 \tag{9}$$

where $\mathfrak{J}_4 = [\alpha_4, \beta_4]$ and $\mathfrak{J}_3 = [\alpha_3, \beta_3]$. But the determinants of $\mathfrak{J}_4$ and $\mathfrak{J}_3$ are equal since they depend only on Δ and the norms. Hence

$$\alpha_4 \overset{c}{\beta}_4 - \overset{c}{\alpha}_4 \beta_4 = \pm (\alpha_3 \overset{c}{\beta}_3 - \overset{c}{\alpha}_3 \beta_3) \tag{10}$$

If the positive sign holds we have, adding (10) to the second equation of (9) $\alpha_4 \beta_4^c = \alpha_3 \beta_3^c$ which, with $\beta_4 \beta_4^c = \beta_3 \beta_3^c$, implies $\alpha_4/\beta_4 = \alpha_3/\beta_3$ and hence $\mathfrak{J}_4$ and $\mathfrak{J}_3$ are in the same ideal class. If the negative sign holds, $\alpha_4 \beta_4^c = \alpha_3^c \beta_3$ which, with $\beta_4 \beta_4^c = \beta_3 \beta_3^c$, gives $\alpha_4/\beta_4 = \alpha_3^c/\beta_3^c$ and $\mathfrak{J}_4$ and $\mathfrak{J}_3^c$ are in the same class. This completes the proof that $f_1 \cong f_2$ implies that the corresponding ideal classes are equal or conjugate.

On the other hand, if $\Sigma_1 = \Sigma_2^c$ we may choose $[\alpha_1, \alpha_2]$ and $[\alpha_1^c, \alpha_2^c]$ as bases and $N(\alpha_1 x + \alpha_2 y) = = N(\alpha_1^c x + \alpha y_2^c)$ shows that the forms are equal.

Notice that if we define a form to be primitive when 1 is the g.c.d. of its coefficients, the definition of a primitive ideal and equations (3) show that a form is primitive if and only if the corresponding ideal class is primitive. Then our correspondence will be completed by the following result.

THEOREM 8. *The primitive ideal classes Σ and Σ^c are equal if and only if each corresponding class Γ is improperly equivalent to itself.*

To prove this we may take $[a, e+\sigma]$ as the representative of Σ and $[a, e+\sigma^c]$

as its conjugate. Suppose they are in the same class, that is, there exist quadratic integers α and β such that $\alpha\mathfrak{J}=\beta\mathfrak{J}^c$. Then $N(\alpha)\mathfrak{J}=\alpha^c\beta\mathfrak{J}^c$ and, since the coefficient of σ in every number of the ideal on the left is divisible by $N(\alpha)$ we see that $N(\alpha)$ divides $\alpha^c\beta$ and that $\mathfrak{J}=\gamma\mathfrak{J}^c$ for some quadratic integer γ. Taking the norm of both sides we see that $N(\gamma)$ is 1 and hence that the ideal (γ) is the ideal (1). Thus $\mathfrak{J}=\mathfrak{J}^c$ which implies that there exist integers t and v such that $ta+v(e+\sigma)=e+\sigma^c$. Equating coefficients of the rational and irrational parts we have $v=-1$ and $ta=2e+\sigma+\sigma^c$ which must be solvable for an integer t. Thus from (3) the form corresponding to the ideal is $ax^2+bxy+cy^2=f$ in which $b\equiv 0 \pmod{a}$. Then f may be taken into itself by the product of two transformations: first

$$H=\begin{bmatrix}1 & h\\ 0 & 1\end{bmatrix}$$

where h is chosen so that $2ah+b$ is 0 or a, and second

$$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}H^I \quad\text{or}\quad \begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}H^I$$

in the respective cases. Thus f is improperly equivalent to itself and the same will be true of every form in the class of f.

Conversely, if f may be taken into itself by the transformation T of determinant -1, we have $T^TFT=F$ where F is the matrix of f. Then for S any properly unimodular transformation, S^ITS takes S^TFS into itself where S^TFS is in the same proper class as F. We wish to choose S so that S^ITS has one of two simple forms. First we show that $T^TFT=F$ where T is the matrix with elements t_{ij} implies that $t_{11}+t_{22}=0$, that is, $\lambda_1+\lambda_2=0$ where λ_1 and λ_2 are the roots of $|T-\lambda I|=0$, the vertical lines denoting the determinant of the symbol enclosed. To this end notice that $\lambda_1\lambda_2=|T|=-1$ and let

$$R=\begin{bmatrix}-t_{12} & -t_{12}\\ t_{11}-\lambda_1 & t_{11}-\lambda_2\end{bmatrix}.$$

Then either $t_{12}=0$ in which case $|T|=-1$ implies $t_{11}t_{22}=-1$ and hence $t_{11}+t_{22}=0$ or else R is nonsingular since $\lambda_1\lambda_2=-1$ and $\lambda_1+\lambda_2$ real implies $\lambda_1\neq\lambda_2$. Then R^ITR has the form

$$\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$$

in view of the fact that $\lambda_i^2-(t_{11}+t_{22})\lambda_i-1=0$, and takes R^TFR into itself. Thus if $g=ax^2+bxy+cy^2$ is the form whose matrix is R^TFR (a, b, c need not be integral here) R^ITR takes it into $\lambda_1^2ax^2+b\lambda_1\lambda_2xy+c\lambda_2^2y_2^2$. Since $\lambda_1\lambda_2=-1$ we have $b=0$ and since the determinant of g is not zero we have $\lambda_1^2=\lambda_2^2=1$. Hence λ_1 and λ_2 are 1 and -1 in some order and their sum is zero. Take $\lambda_1=1$, $\lambda_2=-1$.

Next let u be the g.c.d. of t_{12} and $t_{11}-1$ and let $-t_{12}=us_{11}$ and $t_{11}-1=us_{21}$ define integers s_{11} and s_{21}. Since the latter are relatively prime we can determine s_{12} and s_{22} so that

$$S_1 = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}$$

has determinant 1. Then, since the first column of S_1 is proportional to that of R we have

$$TS_1 = S_1 \begin{bmatrix} 1 & r \\ 0 & s \end{bmatrix}$$

for integers r and s. Furthermore $|T| = -1$, $|S_1| = 1$ implies $s=-1$.

If we define the matrices

$$S_2 = \begin{bmatrix} 1 & (w-r)/2 \\ 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & w \\ 0 & -1 \end{bmatrix}$$

where $w=1$ or 0 according as r is odd or even, we see that S_2 is properly unimodular and

$$\begin{bmatrix} 1 & r \\ 0 & -1 \end{bmatrix} S_2 = S_2 D.$$

Then, letting $S=S_1S_2$ we have $TS=SD$ or $S^I TS=D$. Hence $F_1=S^T FS$ defines a matrix F_1 for which $T^T FT=F$ implies $D^T F_1 D=F_1$.

Thus we have shown that if f can be taken into itself by a transformation of determinant -1, there is in the same proper class as f a form f_1 which is taken into itself by the transformation D above with $w=1$ or 0. Write $f_1=ax^2+bxy+cy^2$ and see that D takes f_1 into $a(x+wy)^2+b(x+wy)(-y)+cy^2=ax^2+xy(2aw-b)+(aw^2-bw+c)y^2$. Thus $b=2aw-b$ and $b\equiv 0 \pmod{a}$. Then, retracing our argument in the first part of this theorem we see that the corresponding ideal $\mathfrak{J}$ has the property that $\mathfrak{J}^c=\mathfrak{J}$. This completes the proof and permits us to make the definition: an ideal class Σ is called ambiguous if it is equal to its conjugate Σ^c. A class of quadratic forms is ambiguous if any (and hence every) form in the class may be taken into itself by a transformation of determinant -1. Hence we have just shown that there is a 1-1 correspondence between ambiguous classes of primitive ideals and ambiguous classes of primitive forms while the correspondence for non-ambiguous classes is 2-2. We therefore understand that in any correspondence between Σ and Γ classes, if Σ corresponds to Γ then Σ^c corresponds to Γ^c. It is this correspondence which enables us to prove expeditiously several important properties of binary quadratic forms.

4. Composition of ideal classes and classes of forms. If Γ_1 and Γ_2 are classes of forms associated with the ideal classes Σ_1 and Σ_2 in $J(\Delta)$ we define the class as-

sociated with $\Sigma_1\Sigma_2$ in $J(\Delta)$ to be the class $\Gamma_1\Gamma_2$ and this class is said to be derived from Γ_1 and Γ_2 by composition. Since Δ determines the determinant of the forms, their determinants are all equal. Furthermore, composition is commutative and associative since the product of ideals has these properties. If $\mathfrak{J}_i=[\alpha_i, \beta_i]$ we have, from equations (7) that $N(\alpha_i)=N(\mathfrak{J}_i)a_i$ where a_i is the leading coefficient of the form f_i associated with $\mathfrak{J}_i$. If the ideals $\mathfrak{J}_i$ are primitive, $N(\mathfrak{J}_1)N(\mathfrak{J}_2)=N(\mathfrak{J}_1\mathfrak{J}_2)$ implies $N(\alpha_1, \alpha_2)=N(\mathfrak{J}_1\mathfrak{J}_2)a_1a_2$. Thus a_1a_2 will be the leading coefficient of some form in $\Gamma_1\Gamma_2$. Hence we have proved the following important theorem:

THEOREM 9. *If a_1 and a_2 are represented by primitive forms of classes Γ_1 and Γ_2 then a_1a_2 is represented by forms of the class $\Gamma_1\Gamma_2$.*

We shall need the following two results on binary quadratic forms.

THEOREM 10. *A form f is primitive if and only if it represents a number prime to $8d(f)$.*

To prove this let p_i be any prime factor of $8d(f)$ where $f=ax^2+bxy+cy^2$. If p_i is prime to a, let $x_i=1$, $y_i=0$; if p_i is a divisor of a and not of c let $x_i=0$, $y_i=1$; if p_i divides both a and c, it does not divide b and we let $x_i=y_i=1$; for these cases the value of f will be prime to p_i. By the Chinese remainder theorem we can choose $x\equiv x_i \pmod{p_i}$, $y\equiv y_i \pmod{p_i}$ for all prime divisors p_i of $8d(f)$ and, for such an x and y, f will be prime to $8d(f)$. On the other hand, if g is the g.c.d. of the coefficients of f, it is a divisor of $8d(f)$ and divides all numbers represented by f.

THEOREM 11. *Two primitive binary quadratic forms f and g with integral coefficients are in the same genus if and only if their determinants are equal and there are integers a and w prime to $8d(f)$ such that $f=a$ and $g=w^2a$ are solvable in integers.*

We know by Theorem 10 that f represents a number a prime to $8d(f)$ and hence we may take f to be $ax^2+bxy+cy^2$. If f and g are of the same genus we know from the definition of genus that there is a transformation

$$\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} w^{-1}$$

taking g into f where the t_{ij} are integers and w is an integer prime to $8d(f)$. Then $(t_{11}/w, t_{21}/w)$ is a solution of $g=a$ and hence $g=aw^2$ is solvable in integers.

Conversely suppose $g=w^2a$ has a solution t_{11}, t_{21} in integers. Then there are integers u and v such that $t_{11}v-t_{21}u=q$ where q is the g.c.d. of t_{11} and t_{21} and hence q^2 divides w^2a. Then the transformation

$$\begin{bmatrix} t_{11}/w & uw/q \\ t_{21}/w & vw/q \end{bmatrix} = T$$

has determinant 1 and takes g into $g'=ax^2+b'xy+c'y^2$ where b' and c' need not be integers while $qb'\equiv b \pmod 2$. Since q^2 divides w^2a, the denominators of the elements of T and of b' and c' are prime to $8d(f)$. Then the transformation

$$S = \begin{bmatrix} 1 & (b-b')/2a \\ 0 & 1 \end{bmatrix}$$

takes g' into $g''=ax^2+bxy+c''y^2$. Since $d(g'')=d(g)=d(f)$ we have $c''=c$ and $g''=f$. Hence TS takes g into f and TS is a transformation with rational elements whose denominators are prime to $8d(f)$. This shows that f and g are in the same genus.

Since the genus of a form is determined by any odd integer prime to the determinant and represented by the form we have

THEOREM 12. *If* $\Gamma_1 \mathrm{v} \Gamma_1'$ *and* $\Gamma_2 \mathrm{v} \Gamma_2'$, *then* $\Gamma_1\Gamma_2 \mathrm{v} \Gamma_1'\Gamma_2'$.

If a form represents 1 we say that it is in the principal class Γ_0 and call the corresponding ideal class the principal ideal class Σ_0. The genus containing the principal class is called the principal genus. Notice that the principal ideal class contains the ideal $(1)=J$. We now prove

THEOREM 13. *The primitive ideal classes of a given* $J(\Delta)$ *and hence the primitive classes of forms, form a multiplicative group. Also* $\Sigma\Sigma^c=\Sigma_0$ *and* $\Gamma\Gamma^c=\Gamma_0$ *for primitive classes* Σ *and* Γ *and their conjugates.*

First we see that Γ_1 and Γ_2 primitive imply that $\Gamma_1\Gamma_2$ is primitive since Theorem 10 shows that forms f_1 and f_2 of Γ_1 and Γ_2, respectively, represent numbers a_1 and a_2 prime to $8d(f)$, and hence, by Theorem 9 forms of $\Gamma_1\Gamma_2$ represent a number a_1a_2 prime to $8d(f)$ which shows by Theorem 10 that $\Gamma_1\Gamma_2$ is primitive.

Now Lemma 1 shows that if $\mathfrak{J}$ is a primitive ideal. $\mathfrak{J}\cdot\mathfrak{J}^c$ is the principal ideal $(N(\mathfrak{J}))=N(\mathfrak{J})\cdot(1)$ and hence $\mathfrak{J}\cdot\mathfrak{J}^c$ is in the principal ideal class.

The principal ideal class is the unit element of the group. We have shown the existence of the inverse and the closure property. Furthermore the associative property holds from the same property for quadratic integers. Hence we have shown that the primitive ideal classes form a group and, if we choose our correspondence so that $\Gamma_0\Gamma=\Gamma$ for every Γ we have the group properties for the classes of primitive forms.

5. Consequences of the composition of forms. We first prove the rather startling result embodied in

THEOREM 14. *If h is the number of proper classes of forms in the principal genus for a given determinant, then the number of proper classes in each genus of primitive forms is h.*

To prove this let $\Gamma_0, \Gamma_2, \cdots, \Gamma_h$ be the classes in the principal genus and let Γ be a representative of any other genus. Then $\Gamma\Gamma_0, \cdots, \Gamma\Gamma_h$ are all in the

genus of Γ by Theorem 12. No two are in the same class since $\Gamma\Gamma_i = \Gamma\Gamma_j$ would imply, by multiplication by Γ^c, that $\Gamma_i = \Gamma_j$.

Next we have Gauss's celebrated theorem on duplication.

THEOREM 15. *If Γ is any class in the principal genus of primitive forms there is a class Γ_1 such that $\Gamma_1^2 = \Gamma$.*

Let $\mathfrak{J} = [a, e'+\sigma]$ be an ideal corresponding to Γ. Then $N\{ax+(e'+\sigma)y\} = N(\mathfrak{J})f(x, y) = af(x, y)$ where $f(x, y) = ax^2+bxy+cy^2$. If g is a form in the principal class it represents 1 and $f \vee g$ implies the existence of a transformation

$$\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} w^{-1}$$

of determinant 1 taking f into g with w prime to $8d(f)$ and t_{ij} integers. That is, $f=1$ has a solution $(t_{11}/w, t_{12}/w)$. Hence $f = w^2$ has a solution t_{11}, t_{21} in integers. Since any common factor of t_{11} and t_{21} divides w and may be removed, we may consider t_{11} and t_{21} to be relatively prime. Then t'_{12} and t'_{22} may be chosen so that

$$\begin{bmatrix} t_{11} & t'_{12} \\ t_{21} & t'_{22} \end{bmatrix}$$

is unimodular and takes f into $w^2x^2+b'xy+c'y^2$ where b' and c' are integers. Hence we may assume w^2 to be the leading coefficient of f and an ideal associated with f is $\mathfrak{J} = [w^2, e+\sigma]$.

Our proof will be complete if we can show that $I_1^2 = \mathfrak{J}$ where $\mathfrak{J}_1 = [w, e+\sigma]$. Since w is prime to 2Δ, it is also prime to $b' = 2e+\sigma+\sigma^c$ and hence there are integers x and y such that $wx+(2e+\sigma+\sigma^c)y = 1$. Now $\mathfrak{J}_1^2 = (w^2, w(e+\sigma), (e+\sigma)^2)$ and hence $\mathfrak{J}_1^2$ contains $xw(e+\sigma)+y(e^2+2e\sigma+\sigma^2) = e+\sigma-yN(e+\sigma)$. But $N(e+\sigma) \equiv 0 \pmod{w^2}$ from Theorem 2 and hence $e+\sigma$ is in $\mathfrak{J}_1^2$ which shows that $\mathfrak{J}_1^2 = [w^2, e+\sigma]$ and completes the proof.

Finally we prove

THEOREM 16. *The number of primitive ambiguous classes of given determinant is equal to the number of genera of primitive forms.*

Since a class Γ is ambiguous if and only if $\Gamma^2 = \Gamma_0$ we see that the ambiguous classes form a subgroup of the group of classes of all primitive forms of the given determinant. Also $\Gamma_1^2 = \Gamma_2^2$ implies $\Gamma_1 = \Gamma_2\Gamma_a$ where Γ_a is ambiguous. Let q be the number of ambiguous classes, g the number of genera and h the number of classes in each genus. Then gh/q is the number of distinct squares of classes. This, from Theorem 15, is equal to h and hence $g = q$.

Notice that if there are q' ambiguous classes in the principal genus, each genus has q' or 0 ambiguous classes.

MATHEMATICAL NOTES

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AN OPERATOR APPROACH TO MATRIX THEOREMS

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1. Introduction. A known theorem on matrices is the following:[1]

If a matrix has two of the three properties in one of the following sets, then it has all three properties in that set:

(a) *Real, orthogonal, unitary;*
(b) *Symmetric, orthogonal, involutory;*
(c) *Hermitian, unitary, involutory.*

A generalization of the above theorem will be given whose proof employs linear operators rather than the matrices themselves. Finally, a finite projective geometry will be exhibited to schematize the algebraic results. This geometric representation unifies certain matrix concepts which heretofore have been treated only separately, if at all.

2. Notation. Matrix A will be called regular provided it is complex and nonsingular. Hereafter all matrices considered will be regular, and no mention of their elements will be made.

Define eA, kA, nA, tA, vA to be the matrices A, $\bar{A}$, $-A$, A', A^{-1}, respectively, of our customary notation. Let (A, B) denote a matrix couple, i.e., an ordered pair of matrices. Define $p(A, B)$, $q(A, B)$, and $r(A, B)$ to be the matrix AB, the matrix couple (nA, B), and the matrix couple (B, A), respectively.

Let the domain of an operator be the set of all elements for which the operator is defined, and let the range of an operator be the set of all image elements. Denote the operators e, k, n, t, v by x_i, and the operators e, q, r by z_i. Thus the domain of operator x_i is a set of matrices, while the domains of operators p, q, and r are sets of matrix couples. We now extend the domain of x_i to matrix couples by defining:

$$x_i(A, B) =_{Df} (x_iA, x_iB).$$

Note, however, that the range of operator p is a set of matrices, while the ranges of operators q and r are sets of matrix couples.

Equality of two operators is designated by $\doteq$. The product of two operators is designated by ordered juxtaposition beginning at the right.

3. Operator Theorems. Using the preceding definitions and notation we state several operator theorems without proof. These operator theorems can be

[1] See C. C. MacDuffee, The Theory of Matrices, p. 25, Theorem 18.1.

translated into their corresponding matrix theorems simply by inspection.

INVOLUTION THEOREM: $(\forall x_i)[x_i^2 \doteq e]$.
COMMUTATION THEOREM: $(\forall x_i)(\forall x_j)[x_i x_j \doteq x_j x_i]$.
INVOLUTION THEOREM: $(\forall z_i)[z_i^2 \doteq e]$.
ANTI-COMMUTATION THEOREM: $rq \doteq qrn$.
COMMUTATION THEOREM: $(\forall x_i)(\forall z_j)[x_i z_j \doteq z_j x_i]$.
EXISTENCE THEOREM: $(\forall x_i)(\exists z_i)[x_i p \doteq p x_i z_i]$.

4. Standard Subsets and Properties. We now define 14 proper subsets of the group of regular matrices. We call them standard subsets, and they in turn define 14 matrix properties, called standard properties, each of which corresponds to its particular subset as follows:

Standard Property	*Customary Notation*	*Operator Notation*	*Operator Equation*	*Abbreviation of Standard Property*
1. A is real	$A=\overline{A}$	$eA=kA$	$e\doteq k$	R
2. A is pure imaginary	$A=-\overline{A}$	$eA=nkA$	$e\doteq nk$	
3. A is symmetric	$A=A'$	$eA=tA$	$e\doteq t$	S
4. A is skew symmetric	$A=-A'$	$eA=ntA$	$e\doteq nt$	
5. A is involutory	$A=A^{-1}$	$eA=vA$	$e\doteq v$	I
6. A is skew involutory	$A=-A^{-1}$	$eA=nvA$	$e\doteq nv$	
7. A is orthogonal	$A=A'^{-1}$	$eA=vtA$	$e\doteq vt$	O
8. A is skew orthogonal	$A=-A'^{-1}$	$eA=nvtA$	$e\doteq nvt$	
9. A is Hermitian	$A=\overline{A}'$	$eA=ktA$	$e\doteq kt$	H
10. A is skew Hermitian	$A=-\overline{A}'$	$eA=nktA$	$e\doteq nkt$	
11. A is nominal[2]	$A=\overline{A}^{-1}$	$eA=vkA$	$e\doteq vk$	N
12. A is skew nominal	$A=-\overline{A}^{-1}$	$eA=nvkA$	$e\doteq nvk$	
13. A is unitary	$A=\overline{A}'^{-1}$	$eA=vktA$	$e\doteq vkt$	U
14. A is skew unitary	$A=-\overline{A}'^{-1}$	$eA=nvktA$	$e\doteq nvkt$	

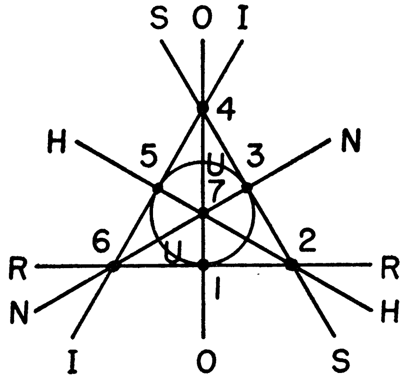
5. Theorem. Let a set closed, associative, and commutative under a binary operation be called an abelian semigroup. We wish to find those sets of standard properties which form semigroups under the operation of property conjunction. Let $x_i \doteq e, k, t, v$. Since $x_1 \doteq e$ and $x_2 \doteq e$ imply $x_1 x_2 \doteq e$, examination of the next to the last column in the above table reveals seven closed sets of standard properties. Further, since the operation of property conjunction is associative and commutative, we have established the following theorem:

THEOREM: *Each of the sets* 1: $\{R, U, O\}$, 2: $\{R, S, H\}$, 3: $\{S, N, U\}$, 4: $\{S, O, I\}$, 5: $\{H, I, U\}$, 6: $\{R, N, I\}$, 7: $\{O, N, H\}$ *is an abelian semigroup under property conjunction.*

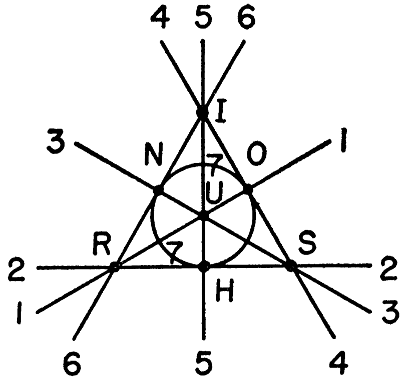
[2] This term is introduced by the author for an un-named type mentioned briefly by Turnbull and Aitken, An Introduction to the Theory of Canonical Matrices, p. 33.

6. Projective Geometric Representation. We present the finite projective geometry $PG(2, 2)$ and its dual to geometrize the algebraic results. In the original geometry a line denotes a matrix property, a point denotes a semigroup, the incidence of a point on a line denotes that a semigroup contains a property, and the intersection of two lines denotes the conjunction of two properties. Here a necessary and sufficient condition that three properties form a semigroup is that they lie on a point. In the dual a necessary and sufficient condition that three properties form a semigroup is that they lie on a line.

Projective Geometry $PG(2, 2)$



Dual



In conclusion it should be mentioned that the Involution, Commutation, and Existence Theorems for our operators permit a novel proof of the proposition that of all standard subsets only the real, orthogonal, and unitary ones form groups under matrix multiplication. This proof, though direct, is too long to be given here.

AN ELEMENTARY PROOF OF STÄCKEL'S THEOREM

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The system associated in the classical dynamics with the name of Stäckel is a holonomic system with n degrees of freedom, for which the kinetic energy function T and the potential energy function V have the particular forms

$$T = \frac{1}{2}\sum_r \frac{1}{v_{1r}}\dot{q}_r^2, \qquad V = \sum_r v_{1r}w_r,$$

where the summations run from 1 to n, and the functions v_{1r} and w_r which occur in the formulae have the following properties:

i. There exists a matrix (u_{rs}) in which the elements u_{rs} of the rth row are functions only of the rth Lagrangian coordinate q_r; the motion takes place in a domain D of the q-space in which the determinant U of (u_{rs}) does not vanish; (v_{rs}) is the matrix inverse to (u_{rs}). (In particular the elements v_{1r} which occur in

the formula for T are the minors of the elements in the first *column* of the matrix (u_{rs}) divided by U.)

ii. The coefficient w_r is a function only of q_r.

iii. The element u_{rs} has a continuous derivative with respect to q_r, and the coefficient w_r has a continuous derivative with respect to q_r, in D. (It follows that the elements v_{rs} have continuous first derivatives with respect to all the q's.)

Stäckel's theorem asserts that there are n first integrals of the equations of motion, namely

$$\sum_r v_{kr}\left(\frac{1}{2}\frac{\dot{q}_r^2}{v_{1r}^2} + w_r\right) = \alpha_k, \qquad k = 1, 2, \cdots, n,$$

where the α_k are constants. (The special case $k=1$ is of course the familiar integral of energy.)

The natural context for the theorem is the general dynamical theory of Hamilton and Jacobi, and the proof usually given (for example in Stäckel's papers [1] and the standard text-books [2]) presents the result as a corollary of the Hamilton-Jacobi theorem. It is not difficult however to derive the result directly from the Lagrangian equations of motion.

We prove first the following Lemma: *For all values of s, k, and r we have*

$$v_{1s}\frac{\partial v_{kr}}{\partial q_s} - v_{ks}\frac{\partial v_{1r}}{\partial q_s} = 0.$$

The proof of this Lemma is simple. We have, for all values of k and m,

$$\sum_r v_{kr}u_{rm} = \delta_{km},$$

whence, differentiating with respect to q_s, and remembering that u_{rm} is a function of q_r only, we obtain

$$\sum_r \frac{\partial v_{kr}}{\partial q_s} u_{rm} + v_{ks}\frac{du_{sm}}{dq_s} = 0.$$

If we write the same equation with 1 in place of k, and eliminate du_{sm}/dq_s, we find

$$\sum_r u_{rm}\left(v_{1s}\frac{\partial v_{kr}}{\partial q_s} - v_{ks}\frac{\partial v_{1r}}{\partial q_s}\right) = 0.$$

If we denote the expression in the parentheses by θ_r (k and s being fixed) we have n equations

$$\sum_r u_{rm}\theta_r = 0,$$

one for each value of m. The determinant of the coefficients is U, and does not

vanish, so each $\theta_r = 0$. This proves the Lemma.

To deduce Stäckel's theorem we have

$$
\begin{aligned}
&\frac{d}{dt}\sum_r v_{kr}\left(\frac{1}{2}\frac{\dot{q}_r^2}{v_{1r}^2} + w_r\right) \\
&\qquad = \sum_r \left(\frac{1}{2}\frac{\dot{q}_r^2}{v_{1r}^2} + w_r\right)\frac{d}{dt}v_{kr} + \sum_r v_{kr}\left(\frac{\dot{q}_r}{v_{1r}}\right)\frac{d}{dt}\left(\frac{\dot{q}_r}{v_{1r}}\right) + \sum_r v_{kr}\frac{dw_r}{dt}.
\end{aligned}
$$

Substituting for $d/dt(\dot{q}_r/v_{1r})$ from Lagrange's equations of motion, we obtain

$$
\begin{aligned}
&\sum_r \left(\frac{1}{2}\frac{\dot{q}_r^2}{v_{1r}^2} + w_r\right)\sum_s \frac{\partial v_{kr}}{\partial q_s}\dot{q}_s + \sum_r v_{kr}\frac{\dot{q}_r}{v_{1r}}\left\{-\frac{1}{2}\sum_s \frac{\dot{q}_s^2}{v_{1s}^2}\frac{\partial v_{1s}}{\partial q_r}\right. \\
&\qquad \left. - \frac{\partial}{\partial q_r}\left(\sum_s v_{1s}w_s\right)\right\} + \sum_r v_{kr}\frac{dw_r}{dq_r}\dot{q}_r \\
&\qquad = \sum_{r,s}\frac{1}{2}\frac{\dot{q}_r^2\dot{q}_s}{v_{1r}^2}\frac{\partial v_{kr}}{\partial q_s} + \sum_{r,s} w_r\frac{\partial v_{kr}}{\partial q_s}\dot{q}_s - \sum_{r,s}\frac{1}{2}v_{kr}\frac{\dot{q}_r}{v_{1r}}\frac{\dot{q}_s^2}{v_{1s}^2}\frac{\partial v_{1s}}{\partial q_r} \\
&\qquad - \sum_{r,s} v_{kr}\frac{\dot{q}_r}{v_{1r}}\frac{\partial v_{1s}}{\partial q_r}w_s - \sum_r v_{kr}\frac{\dot{q}_r}{v_{1r}}v_{1r}\frac{dw_r}{dq_r} + \sum_r v_{kr}\frac{dw_r}{dq_r}\dot{q}_r.
\end{aligned}
$$

The last two sums disappear, and, interchanging the dummy suffixes r and s in the third and fourth sums, we obtain

$$
\begin{aligned}
&\sum_{r,s}\frac{1}{2}\frac{\dot{q}_r^2\dot{q}_s}{v_{1r}^2}\frac{\partial v_{kr}}{\partial q_s} - \sum_{r,s}\frac{1}{2}\frac{\dot{q}_r^2\dot{q}_s}{v_{1r}^2}\frac{v_{ks}}{v_{1s}}\frac{\partial v_{1r}}{\partial q_s} + \sum_{r,s} w_r\frac{\partial v_{kr}}{\partial q_s}\dot{q}_s - \sum_{r,s} w_r\frac{v_{ks}}{v_{1s}}\frac{\partial v_{1r}}{\partial q_s}\dot{q}_s \\
&\qquad\qquad = \sum_{r,s}\left(\frac{1}{2}\frac{\dot{q}_r^2}{v_{1r}^2} + w_r\right)\dot{q}_s\left(\frac{\partial v_{kr}}{\partial q_s} - \frac{v_{ks}}{v_{1s}}\frac{\partial v_{1r}}{\partial q_s}\right),
\end{aligned}
$$

which vanishes in virtue of the Lemma. The theorem follows.

The result has been known for more than half a century, so from the point of view of dynamical theory the present exposition is unimportant. From the point of view of the teaching of dynamics, however, it may be important; by means of it the theorem can be introduced and used in an elementary course which contains Lagrange's equations but does not embrace the more recondite theory of Hamilton and Jacobi.

References

1 Stäckel. Math. Ann., vol. 42, 1893, p. 537; Comptes Rendus, Paris, vol. 116, 1893, p. 485 and p. 1284, vol. 121, 1895, p. 489.

2. See for example Appel, *Mécanique Rationelle* (1923), vol. II, p. 433; Levi-Civitá e Amaldi, *Meccanica Razionale* (1927), vol. II, part 2, p. 420.

A THEOREM ON POSITIVE DEFINITE MATRICES

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Let us write

$$|A| = |A|_+ - |A|_-$$

for the complete expansion of the determinant of a matrix A, so that the first member on the right-hand side of the equation is the sum of those terms which have positive signs in the determinant expansion. Then we have the following result.

THEOREM: *If A is a positive definite Hermitian matrix then* $|A|_- \geqq 0$.

Proof. Let A be a positive definite Hermitian matrix. By a well known theorem* there exists an $n\times n$ triangular matrix (t_{ij}) with $t_{ij}=0$ for $j>i$ such that

$$A = (t_{ij})(\bar{t}_{ij})'. \tag{1}$$

By definition,

$$|A|_- = \sum a_{1i_1} \cdots a_{ni_n}, \tag{2}$$

the sum taken over all odd permutations of $1, \cdots, n$.

Using the decomposition (1) we have

$$a_{ik} = \sum_{\nu=1}^{n} t_{i\nu}\bar{t}_{k\nu}$$

and so

$$a_{1i_1} \cdots a_{ni_n} = \sum_{\nu_1=1}^{n} \cdots \sum_{\nu_n=1}^{n} t_{i_1\nu_1} \cdots t_{i_n\nu_n} \cdot \bar{t}_{1\nu_1} \cdots \bar{t}_{n\nu_n}.$$

Hence by (2)

$$|A|_- = \sum_{\nu_1=1}^{n} \cdots \sum_{\nu_n=1}^{n} |t_{i\nu_j}|_- \cdot \bar{t}_{1\nu_1} \cdots \bar{t}_{n\nu_n}. \tag{3}$$

The multiple sum in (3) has a value independent of the order of summation and this value is therefore the average of the sum corresponding to any set of permutations of the orders of summation. Take this set to be that of the R even permutations and observe that in each of them the first (determinantal) factor has the same value. The residual factors add up to $|\bar{t}_{i\nu_j}|_+$, and so we have

$$|A|_- = \sum_{\nu_1=1}^{n} \cdots \sum_{\nu_n=1}^{n} \frac{1}{R} |t_{i\nu_j}|_- \cdot |\bar{t}_{i\nu_j}|_+. \tag{4}$$

If $\nu_i=\nu_j$ for some $i\neq j$, then $|t_{i\nu_j}|=0$, and hence $|t_{i\nu_j}|_+ = |t_{i\nu_j}|_-$. Therefore

* See e.g. F. D. Murnaghan, The theory of group representations, Baltimore, 1938, pp. 20–22.

the two determinantal factors in (4) are conjugate and their product is positive. If no such i and j exist it will be shown that the product vanishes. For, in this case $|t_{i\nu_j}|$ has as its columns a permutation of the columns of $|t_{ij}|$ and the latter has in its formal expansion only one term different from 0. Hence either $|t_{i\nu_j}|_+$ or $|t_{i\nu_j}|_-$ will vanish. This establishes the theorem.

CLASSROOM NOTES

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CERTAIN BASIC THEOREMS ON LINEAR DIFFERENTIAL EQUATIONS*

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1. Introduction. The theorems to be discussed are three, viz.: 1) the theorem that if n linearly independent solutions of a linear differential equation of the nth order are known, any other solution is a linear combination of them; 2) the theorem that the different solutions usually given for an equation with constant coefficients actually are linearly independent; and 3) the theorem stating the method of finding a particular integral of the nonhomogeneous equation with constant coefficients. These theorems can hardly be unfamiliar to you. Nevertheless, a survey of otherwise suitable textbooks (for an elementary course at The Pennsylvania State College) shows that in practically all of them the treatment of the first two theorems is logically inadequate, while that of the third theorem is unnecessarily complicated. Since the objective of this Association is the improvement of college teaching, it is appropriate to emphasize the presentation of these theorems here, even though that involves some reiteration of what has been known for a long time.

In regard to the first of these theorems, what is usually done is to show, quite correctly, that any linear combination of solutions (of the homogeneous equation) is a solution, and then to say that a linear combination, with arbitrary constant coefficients, of n solutions is "the general solution" because it contains n arbitrary constants. The only proof offered of the principle implicit here—viz., that a solution of an nth order equation embraces every solution if it depends on n arbitrary constants—is a heuristic argument in the front of the book. All this argument shows is that a general solution of the form stated is what one would normally expect; and the existence of singular solutions for $n=1$ shows that there are exceptions. An exact formulation and proof of the principle would require consideration of existence and uniqueness theorems for differential

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* An address presented to the Allegheny Mountain Section of the Association at Pittsburgh, Pa., on November 6, 1948.

systems which are altogether too difficult for the kind of course here considered. Now it happens that the student solves equations of the first order by deriving necessary consequences of the equation, as in elementary algebra; and accordingly, if he has learned to reason correctly, he proves, in each particular case, that the solution or solutions which he finds are exhaustive. The principle mentioned is, therefore, not logically necessary for his argument; and so a heuristic and vague treatment of it is quite suitable. But in the study of linear equations one of the methods we wish to use gives sufficient, rather than necessary, conditions. In that case the first of our three theorems becomes an essential part of the logic. If we want to teach our students to reason correctly, an adequate proof of the theorem should be given. Somewhat similar remarks apply, of course, to the second theorem.

The proofs here given of these theorems are, in one sense, elementary in character. Of course, they are presented to you in a more abstract and general form than may be suitable for a class. The first two proofs, which depend on a general n, are by mathematical induction. However the proofs are intrinsically no more difficult than those of other theorems depending on general n with which the student is familiar, and they can be clarified by the usual devices of good teaching. No originality is claimed for the proofs. Parts of them will even be found in some existing textbooks, e.g. that of Agnew (cited below); but I know of no textbook in English which contains all of them.

2. The completeness theorem. Let $y_1, y_2, \cdots, y_n$ be n linearly independent solutions of the linear differential equation of the nth order

$$L(y) = 0. \tag{1}$$

Let Y be any other solution. Then it is to be shown that there exist constants $C_1, C_2, \cdots, C_n$ such that

$$Y = C_1y_1 + C_2y_2 + \cdots + C_ny_n. \tag{2}$$

This theorem is clear if $n=1$; it is, in fact, shown in the discussion of the linear equation of the first order. In order to establish it generally we employ mathematical induction. Assume the theorem true for equations of order $n-1$; we then show it is true for those of order n.

To this end we use the standard technique for reducing the order when one solution is known (and thus obtain that theorem as a by-product). If we substitute $Y=Zu$ in (1) we have an identity of the form

$$L(Zu) = ZL(u) + Z'L_1(u) + Z''L_2(u) + \cdots + Z^{(n)}L_n(u),$$

where the L_i are linear differential operators whose nature does not concern us. If u is a particular solution of (1), the term in Z vanishes and we have

$$L(uZ) = M(Z'), \tag{3}$$

where M is a linear differential operator of order $n-1$.

Now suppose $y_1, y_2, \cdots, y_n$ are linearly independent solutions of (1), and let Y be any other solution. Let

$$z_1 = \frac{y_1}{y_n}, z_2 = \frac{y_2}{y_n}, \cdots, z_{n-1} = \frac{y_{n-1}}{y_n}, Z = \frac{Y}{y_n}. \tag{4}$$

Then $z_1', z_2', \cdots, z_{n-1}', Z'$ are solutions v of

$$M(v) = 0,$$

as we see from (3) with $u = y_n$. Moreover $z_1', \cdots, z'_{n-1}$ are linearly independent. For if

$$C_1 z_1' + C_2 z_2' + \cdots + C_{n-1} z_{n-1}' = 0,$$

then, integrating, we have, for same constant C_n,

$$C_1 z_1 + C_2 z_2 + \cdots + C_{n-1} z_{n-1} + C_n = 0;$$

thence, multiplying by y_n and using (4), we have

$$C_1 y_1 + C_2 y_2 + \cdots + C_{n-1} y_{n-1} + C_n y_n = 0$$

and so all the C's are 0. By the hypothesis of the induction there must exist $C_1, C_2, \cdots, C_{n-1}$ such that

$$Z' = C_1 z_1' + C_2 z_2' + \cdots + C_{n-1} z_{n-1}'.$$

Hence, for some C_n,

$$Z = C_1 z_1 + C_2 z_2 + \cdots + C_{n-1} z_{n-1} + C_n.$$

Multiplying by y_n we have (2).

It will be noted that this argument involves nothing more difficult than the theorem on the reduction of the order for a linear differential equation—a theorem which is given in most textbooks.

3. Linear independence of the basis. The second theorem, on the linear independence of the basic solutions of an equation with constant coefficients is evidently equivalent to the following: If $p_1, p_2, \cdots, p_n$ are all distinct numbers, and $P_1, P_2, \cdots, P_n$ are polynomials, then

$$P_1 e^{p_1 x} + P_2 e^{p_2 x} + \cdots + P_n e^{p_n x} \tag{5}$$

cannot vanish identically unless all the P_i vanish identically.

This theorem is clear when $n = 1$. Suppose it true for a given n; we prove it is true for $n+1$.

Let $p_1, p_2, \cdots, p_n, p_{n+1}$ be distinct numbers and let $P_1, P_2, \cdots, P_{n+1}$ be polynomials such that

$$P_1 e^{p_1 x} + P_2 e^{p_2 x} + \cdots + P_n e^{p_n x} + P_{n+1} e^{p_{n+1} x} = 0. \tag{6}$$

If $P_{n+1} \equiv 0$, then all the $P_i = 0$ by the hypothesis of the induction. If not we can divide (6) by $P_{n+1}e^{p_{n+1}x}$, obtaining

$$R_1e^{r_1x} + R_2e^{r_2x} + \cdots + R_ne^{r_nx} + 1 = 0$$

where the R_i are rational, $r_i = p_i - p_{n+1}$. Differentiating, we have

$$(R_1' + r_1R_1)e^{r_1x} + \cdots + (R_n' + r_nR_n)e^{r_nx} = 0$$

If we clear of fractions, the left side becomes an expression of form (5) which vanishes identically. Since the common denominator does not vanish identically, the hypothesis of the induction shows that for all i

$$R_i' + r_iR_i = 0,$$

$$R_i = C_ie^{-r_ix}.$$

This contradicts the fact that the R_i are rational. Hence all P_i in (6) are identically 0.

4. Method of undetermined coefficients. As already stated the difficulty with the third theorem is that the general case is long and complicated—one author takes a full half page merely to state the theorem; furthermore the formulations usually given do not show why the method applies.

Suppose, then, we have to solve an equation

$$\phi(D)[y] = f(x). \tag{7}$$

It is supposed that we know how to find the complementary function, and particular solution of (7) is sought. The cases where the method of undetermined coefficients applies are those in which there exists an operator $\psi(D)$ such that

$$\psi(D)[f(x)] = 0. \tag{8}$$

$$\psi(D)\phi(D)[y] = 0. \tag{9}$$

In other words every solution y of (7) will be a solution of (9), and hence must be a specialization of the general solution of (9). Thus to get the general solution of (7) we can substitute the general solution of (9) in (7) and see what relations must hold among the arbitrary constants in order that (7) hold.

Furthermore, given any solution of (9) which is also a solution of (7), if we delete the terms which are part of the complementary function of (7), the remaining terms must also constitute a solution of (7). Hence a particular integral of (7) can be obtained by the following rule. Find $\psi(D)$ such that (8) holds; write the general solution of (9) and strike out terms in the complementary function of (7); then use the method of undetermined coefficients to specialize the constants so that (7) holds.

This rule entails the ones usually stated. Suppose, for instance,

$$f(x) = x^pe^{kx}.$$

Then

$$\psi(D) = (D - k)^{p+1}.$$

Let $\phi(D)=(D-k)^q\omega(D)$, where $\omega(k)\neq 0$. Then the general solution of (9) consists of

$$(C_0 + C_1x + C_2x^2 + \cdots + C_{p+q}x^{p+q})e^{kx} + \cdots$$

where the dots on the right indicate terms also present in the *c.f.* of (7). The first q terms indicated are also part of the *c.f.* of (7). Hence the particular integral is of the form

$$y = x^q(C_q + C_{q+1}x + \cdots + C_{p+q}x^p)e^{kx},$$

where the constants are to be determined by method of undetermined coefficients.†

PYTHAGOREAN TRIPLES

W. R. TALBOT, Jefferson City, Missouri

Pythagorean triples are such popular topics with undergraduates, a brief and simple method for their determination as special cases of the solution of the more general equation $x^2+y^2=z^n$ may be of interest.

The sum of two real squares as a^2+b^2 may be written as the square of the absolute value of the complex number $a+ib=\rho(\cos\theta+i\sin\theta)$. By DeMoivre's Theorem $(a+ib)^n=\rho^n(\cos n\theta+i\sin n\theta)$. Then $|(a+ib)^n|=\rho^n=|a+ib|^n$. Let $(a+ib)^n$ be expanded to give $A+iB$ where A and B are real. Then $|(a+ib)^n|=(A^2+B^2)^{1/2}=\rho^n=|a+ib|^n=(a^2+b^2)^{n/2}$. Then $A^2+B^2=(a^2+b^2)^n$. The solution of the equation is $x=A$, $y=B$, and $z=a^2+b^2$ where $(a+ib)^n=A+iB$.
If $n=2$, $x=a^2-b^2$, $y=2ab$, and $z=a^2+b^2$.
If $n=3$, $x=a^3-3ab^2$, $y=3a^2b-b^3$, and $z=a^2+b^2$, etc.

The solutions are easily written from the binomial expansion as the terms are alternately real and imaginary and the signs alternate by pairs of successive terms. When n is odd, the expansion has an even number of terms and x will have one term without any b and y will have one term without any a, and when n is even both of these terms are in x; hence, primitive solutions are derived when a and b are relatively prime and of different parity.

SOLUTIONS OF A TRIGONOMETRIC EQUATION

W. R. RANSOM, Tufts College

The solution given for

$$a\sin x + b\cos x = c$$

† This special case is discussed by this method in Agnew, R. P., Differential Equations, New York, McGraw Hill Book Company, 1942.

which appeared in Mattheson's note (*Rational Solutions of a Certain Trigonometric Equations*, this MONTHLY, vol. 55, p. 574 (1948)) avoids the extraneous answers that appear when either sine or cosine is expressed in terms of the other and the radical then removed by squaring. The introduction of an auxiliary angle is another means of avoiding extraneous roots; it is often used in numerical cases, and it may be of interest to observe this method of solution in general form.

Take h as the positive square root of a^2+b^2, and k as one square root of h^2-c^2. We can select a particular angle, M, for which $\sin M=b/h$ and $\cos M=a/h$. Upon dividing the given equation by h we obtain:

$$\cos M \sin x + \sin M \cos x = c/h$$

whence $\sin (M+x)=c/h$, and $\cos (M+x)=\pm k/h$. We can then determine $\sin x$ and $\cos x$ as follows:

$$\sin x = \sin \left[(M+x)-M\right] = \frac{c}{h}\cdot\frac{a}{h} - \frac{\pm k}{h}\,\frac{b}{h} = \frac{ac \mp bk}{a^2+b^2}$$

$$\cos x = \cos \left[(M+x)-M\right] = \frac{\pm k}{h}\,\frac{a}{h} + \frac{c}{h}\,\frac{b}{h} = \frac{bc \pm ak}{a^2+b^2}\,.$$

These values check in the equation, and there are no missing or extraneous roots. They also confirm the conclusion that the functions $\sin x$ and $\cos x$ are rational when k is rational, that is when $a^2+b^2=c^2+k^2$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 871. *Proposed by W. R. Ransom, Tufts College*

Is there any value of k other than $k=1$ for which $k \tan x \cot kx=1$?

E 872. *Proposed by Leo Moser, University of Manitoba*

An enthusiastic problemist proposes at least one problem every day. In order not to overwork problem editors, however, he does not propose more than 730 problems a year. Given any positive integer n, show that he proposes exactly n problems in some set of consecutive whole days. (Dedicated to Victor Thébault.)

E 873. *Proposed by I. M. Hostetter, Oregon State College*

Let a vector be interpreted as a directed line segment and let $A * B$ be a symbolic product of two arbitrary vectors A and B such that (1) $A * B$ is a scalar, (2) $A * B = B * A$, (3) $A * (B+C) = A * B + A * C$. Is there an interpretation of $A * B$ other than the familiar inner, or scalar, product: $|A||B| \cos \theta$, where $|A|$, $|B|$ are the (positive) lengths of A and B, and θ is the angle between A and B?

E 874. *Proposed by C. S. Ogilvy, Trinity College*

If $y_1 = x^{1/2}$, $y_2 = (x+y_1)^{1/2}, \cdots, y_m = (x+y_{m-1})^{1/2}$, prove that $\lim_{m\to\infty} y_m$ is integral if and only if x is of the form $n(n-1)$, $n = 2, 3, \cdots$, in which case the limit is n.

E875. *Proposed by Victor Thébault, Tennie, Sarthe, France*

A life-buoy is formed by two right circular cones joined by their common base. (1) Show that the centroid of the solid coincides with the centroid of the vertices of the quadrangle obtained by cutting the surface of the buoy by an axial plane. (2) Determine the ratio of the altitudes of the two cones in order that the buoy may be in equilibrium when it rests, on a horizontal plane, on one of the two surfaces, or indifferently on one or the other of the two surfaces, of the two cones.

SOLUTIONS

Perimeters of Pythagorean Triangles

E 828 [1948, 427]. *Proposed by Fritz Herzog, Michigan State College*

For positive integral x, let $P(x)$ denote the number of distinct primitive Pythagorean triangles of perimeter x. Show that $P(x)$ is unbounded; in fact, for any given non-negative integer r, the equation $P(x) = r$ has an infinitude of solutions. (Compare Problem E 812 [1948, 248].)

I. *Solution by the Proposer.* Let the sides of any primitive Pythagorean triangle be written in the form $2st$, s^2-t^2, s^2+t^2, where s, t are integers with $s>t>0$, $(s, t)=1$, and $s \not\equiv t \pmod 2$. The correspondence between the primitive Pythagorean triangles and the pairs s, t is bi-unique. (See Hardy and Wright, *An Introduction to the Theory of Numbers*, Theorem 225, p. 189.) The perimeter of the triangle is then equal to $2su$, where $u = s+t$. The above conditions on s and t read, in terms of s and u,

$$(1) \qquad s < u < 2s, \qquad (s, u) = 1, \qquad u \equiv 1 \pmod 2.$$

Thus we obtain $P(x) = 0$ for odd x and assume from now on that x is even ($x = 2y$). According to the above, $P(2y)$ equals the number of those divisors u of y which satisfy (1), s being equal to y/u. If, in particular, y is odd and square-free then the second and third conditions of (1) will be satisfied for all divisors u of y so that $P(2y)$ equals the number of those divisors u of y for which, with $s = y/u$,

$$1 < u/s < 2. \tag{2}$$

We shall now exhibit an infinitude of odd squarefree y with $P(2y)=r$, for any given $r \geqq 0$. We denote by q_n the nth odd prime, and we make use of the relation

$$\lim_{n\to\infty} (q_{n+1}/q_n) = 1. \tag{3}$$

(See Hardy and Wright, *loc. cit.*, Theorem 8, p. 10.)

It is obvious that $P(2q_n)=0$ for $n \geqq 1$. Also $P(2q_nq_{n+1})=1$ if $q_{n+1}/q_n<2$, which by (3) is true for sufficiently large n. To solve $P(2y)=2$ we put $y=3\cdot 5\ q_nq_{n+1}$, where $q_{n+1}/q_n<6/5$, which by (3) holds for sufficiently large n. The inequalities $3\cdot 5q_n/q_{n+1}>3\cdot 5^2/6>2$, $q_nq_{n+1}/3\cdot 5>5\cdot 7/3\cdot 5>2$, and $1<5^2/3\cdot 6<5q_n/3q_{n+1}<5q_{n+1}/3q_n<5\cdot 6/3\cdot 5=2$ show that among the divisors u of y only $u=5q_n$ and $u=5q_{n+1}$ satisfy (2). Hence $P(2\cdot 3\cdot 5\ q_nq_{n+1})=2$. Finally, in order to solve $P(2y)=r$, where r is a given integer $\geqq 3$, we put $y=q_nq_{n+1}\cdots q_{n+r-1}Q$, where Q is a prime such that

$$\frac{q_{n+1}q_{n+2}\cdots q_{n+r-1}}{q_n} < Q < \frac{2q_nq_{n+1}\cdots q_{n+r-2}}{q_{n+r-1}}\,. \tag{4}$$

Such a prime Q exists for sufficiently large n because, as $n\to\infty$, the outer members of (4) approach ∞ (since $r\geqq 3$) and their ratio, *viz.* $2(q_n/q_{n+r-1})^2$, approaches 2 by repeated application of (3); from that the existence of Q for sufficiently large n follows by again applying (3). From (4) we conclude first that $Q>q_{n+r-1}$ (since $r\geqq 3$) so that y is odd and squarefree. Secondly, (4) implies the following inequalities:

$$\frac{q_nq_{n+1}\cdots q_{n+r-1}}{Q} > \frac{q_{n+r-1}^2}{2} > 2,$$

$$\frac{q_nq_{n+1}Q}{q_{n+2}q_{n+3}\cdots q_{n+r-1}} > q_{n+1}^2 > 2,$$

$$1 < \frac{q_nQ}{q_{n+1}q_{n+2}\cdots q_{n+r-1}} < \frac{q_{n+r-1}Q}{q_nq_{n+1}\cdots q_{n+r-2}} < 2.$$

These inequalities show that the only divisors u of y satisfying (2) are $u=q_{n+j}Q$, with $j=0, 1, \cdots, r-1$. This shows that $P(2q_nq_{n+1}\cdots q_{n+r-1}Q)=r$ and completes the proof.

II *Solution by Leo Moser, University of Manitoba.* $P(x)$ is equal to the number of different factorizations of $x/2$ into two factors m and n, $(m, n)=1$, $m<n<2m$. Such a factorization will be called "favorable."

Let p_i denote the ith prime. An elementary consequence of the prime number theorem is that for any $e>0$ there exists a j such that for $i>j$, $p_i\div p_{i+1}>1-e$.

Now take $r>2$ and consider $x=p_i^{r-2}p_{i+1}p_{i+2}\cdots p_{i+r}$. It is clear that for e sufficiently small the only "favorable" factorizations of x are

$$x/2 = (p_i^{r-2}p_{i+k})(p_{i+1}p_{i+2}\cdots p_{i+r}/p_{i+k}), \qquad k = 1, 2, \cdots, r.$$

Hence $P(x)=r$, and since i is restricted only by the condition $i>j$, there exist infinitely many solutions of $P(x)=r$, $r>2$.

To handle the case $r=2$ we again invoke the prime number theorem, which guarantees the existence of infinitely many sets of four primes which are "almost" proportional to 2, 3, 4, 5. With $x/2$ the product of four such primes $P(x)=2$.

Also solved by L. J. Burton.

Sufficient Condition for Continuity in an Interval

E 829 [1948, 427] *Proposed by S. H. Gould, Purdue University*

Let $f(x)$ be defined in a closed interval $[a, b]$ and have the property of assuming, in any subinterval $[c, d]$, every value between $f(c)$ and $f(d)$ Prove that $f(x)$ is continuous if no rational value is infinitely often assumed.

Solution by the Proposer. Suppose $f(x)$ to be discontinuous at x_0, say from above on the left. Then, for some $\epsilon>0$, we may choose $x_1<x_2<\cdots<x_n<\cdots\to x_0$ such that $f(x_n)>f(x_0)+\epsilon$ for all x_n. Let m be rational between $f(x_0)$ and $f(x_0)+\epsilon$. Take y_1 in (x_1, x_0) with $f(y_1)=m$, then $x_{i_1}>y_1$ and y_2 in (x_{i_1}, x_0) with $f(y_2)=m$, and similarly for $y_3, y_4, \cdots$. Then $f(x)$ assumes the rational value m infinitely often in (x_1, x_0), which is contrary to hypothesis.

Also solved by L. J. Burton, A. S. Day, Leo Moser and Alex Tytun.

Rider to I-47

E 831 [1948, 498]. *Proposed by K. W. Crain, Purdue, University*

If squares be constructed on the legs of a right triangle, the lines (which do not lie along the sides of the triangle) drawn from each end of the hypotenuse to a vertex of the opposite square intersect on the altitude which passes through the vertex of the right angle.

I. *Solution by Ragnar Dybvik, Levanger, Norway*. Let the right triangle, placed on a rectangular coordinate system, have its vertices at the points $A(0, b)$, $B(a, 0)$, $C(0, 0)$. The concerned vertices of the squares are the points $D(b, -b)$ and $E(-a, a)$. Let CH be the altitude on the hypotenuse. The equations of the lines BD, EA, CH are found to be

$$\begin{aligned} (a+b)x + ay - ab &= 0, \\ -bx - (a+b)y + ab &= 0, \\ -ax + by &= 0. \end{aligned}$$

Since the sum of the left members of these equations is identically zero it follows that the three lines are concurrent.

II. *Solution by W. B. Clarke, San Jose, California.* Using the notation o solution I and designating the intersections of CA and DB, CB and EA by M, N it is easily shown that $CM/MA = a/b$, $BN/NC = a/b$, $AH/HB = b^2/a^2$. Therefore $(BN)(CM)(AH) = (NC)(MA)(HB)$ and AN, BM, CH are concurrent by Ceva's theorem.

Also solved by Louis Berkofsky, Sheldon Best, W. E. Buker, L. J. Burton W. J. Cherry, R. L. Clayton, A. C. Cohen, Jr., William Douglas, Gertrude Ehrlich, B. K. Gold, Frank Harary, Frank Herlihy, Banesh Hoffmann, R. E Horton, S. J. Jasper, L. M. Kelly, Roger Lessard, Marie Madden, Donald Marsh, Eugene McLachlan, R. D. McWilliams, Leo Moser, C. S. Ogilvy, W. O Pennell, O. M. Rasmussen, C. C. Richtmeyer, P. D. Thomas, W. I. Thompson C. W. Trigg (three ways), E. H. Vance, Margaret Willerding, Maud Willey J. E. Winter, the proposer, and R. V. Andree's freshman engineering class at University of Wisconsin.

Editorial Note. Trigg pointed out that this problem has occurred three times in *School Science and Mathematics*: as problem 563, Mar. 1919; problem 841, Dec. 1924; problem 1651, May 1940. In addition to proofs similar to those given above there appear some using only high school geometry. One proof shows that the three lines in question are the altitudes of triangle ABT, where T is the intersection of the remote sides of the squares prolonged.

No one observed that the theorem is true for *any* triangle. In fact we have: *Given any triangle ABC with squares ADFC, BEGC described either both externally or both internally on the sides CA and CB, then AE, BD intersect on the altitude through C and AG, BF intersect on DE.* For the first part of the theorem we may replace the squares by two similar rectangles.

Rational Points on a Circle

E 832 [1948, 498]. *Proposed by V. E. Dietrich, Purdue University*

If a circle has a center with at least one irrational coordinate, then there are at most two points on the circle with rational coordinates.

Solution by C. S. Ogilvy, Trinity College. Suppose there are three points on the circle, each with rational coordinates. Then the equation of the circle can be found by substituting the coordinates of the three points, one pair after another into the equation

$$x^2 + y^2 + Ax + By + C = 0.$$

The resulting three simultaneous linear equations have rational coefficients, and hence yield rational values for A, B, C. This contradicts the hypothesis, since $-A/2$ and $-B/2$ are the coordinates of the center of the circle.

Also solved by F. Bagemihl, Murray Barbour, L. J. Burton, B. B. Dressler, N. J. Fine, L. B. Hedge, L. M. Kelly, Roger Lessard, Norman Miller, Leo Moser, S. T. Packer, C. F. Pinzka, C. W. Trigg, Alex Tytun, and the proposer.

Miller gave the following examples. The circle $(x-\sqrt{2})^2+y^2=3$ has on it the rational points (0, 1) and (0, -1); $(x-\sqrt{2})^2+y^2=3-2\sqrt{2}$ has the single rational point (1,0); $(x-\sqrt{2})^2+y^2=\sqrt{3}$ has no rational point. The circle $(x-\sqrt{2})^2+(y-\sqrt{2})^2=6$ has on it the rational points (1, -1) and (-1, 1); $(x-\sqrt{3})^2+(y-\sqrt{2})^2=5$ has the single rational point (0, 0); $(x-\sqrt{3})^2+(y-\sqrt{2})^2=4$ has no rational point.

R. W. Hamming pointed out that the words "irrational" and "rational" can be replaced by "transcendental" and "algebraic" respectively.

A City Park

E 833 [1948, 498]. *Proposed by P. D. Thomas, Washington, D. C.*

A surveyor in laying out a square park area in a city found it was necessary because of obstructions to shorten two opposite sides by one foot, but both the length and width were in integral feet. To check his work he ran a diagonal of the resulting rectangle. Imagine his surprise to find that the semi-perimeter (diagonal+length+width) was in integral rods! What were the dimensions of the field?

Solution by C. W. Trigg, Los Angeles City College. If the sides and diagonal of the park are x, $x-1$, y feet, respectively, then

$$2(y+2x-1)=33s,$$

where s is an integer. Also $y^2=x^2+(x-1)^2$, which may be written as

$$2y^2=(2x-1)^2+1.$$

Hence we seek a solution of the Pellian equation $w^2-2y^2=-1$ such that $w+y$ is a multiple of 33. Two consecutive solutions of this equation are $(w_0, y_0)=(-1, 1)$ and $(w_1, y_1)=(1, 1)$. All other solutions are given by the recurrence formulas

$$w_{n+2}=6w_{n+1}-w_n, \qquad y_{n+2}=6y_{n+1}-y_n.$$

The two smallest solutions such that $w+y\equiv 0 \pmod{33}$ are

$$(8119, 5741) \text{ and } (318281039, 225058681).$$

Since the latter solution corresponds to a park of side 159140520 feet, it represents an impossible situation. Therefore the smaller pair provides the unique solution in which the sides and diagonal of the park are 4060, 4049, 5741 feet, respectively, the semi-perimeter being 840 rods.

Also solved by Murray Barbour, E. M. Berry, L. J. Burton, John Cromelin, Monte Dernham, William Douglas, G. B. Huff, Roger Lessard, Leo Moser, Leola Odland, C. S. Ogilvy, S. T. Parker, Alex Rosenberg, C. M. Sandwick, W. R. Talbot, John Walker, and the proposer.

Moser said that it can be shown, from the recurrence formulas, that the semi-perimeter will be divisible by 33 if and only if n is divisible by 6.

The problem of finding Pythagorean triangles whose legs are consecutive integers is considered by Kraitchik, *Mathematical Recreations*, p. 100.

Fibonacci Numbers as Determinants

E 834 [1948, 498]. *Proposed by Don Walter, Pomona College*

Show that

$$F_n = \begin{vmatrix} 1 & -1 & 1 & -1 & 1 & -1\cdots \\ 1 & 1 & 0 & 1 & 0 & 1\cdots \\ 0 & 1 & 1 & 0 & 1 & 0\cdots \\ 0 & 0 & 1 & 1 & 0 & 1\cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\cdot\cdot\cdot \end{vmatrix},$$

where F_n is the nth term of the Fibonacci sequence 1, 1, 2, 3, 5, $\cdots$, x, y, $x+y$, $\cdots$ and the determinant is of order $n-1$.

Solution by Alex Tytun, New York, N. Y. Denoting the above determinant by D_n, it is seen that $D_2=1$, $D_3=2$. It remains to show that $D_n=D_{n-1}+D_{n-2}$, $n\geqq 4$. In D_n subtract the $(n-3)$th column from the $(n-1)$th, the $(n-4)$th from the $(n-2)$th, $\cdots$, the first from the third, obtaining

$$D_n = \begin{vmatrix} 1 & -1 & 0 & 0 & 0 & 0\cdots \\ 1 & 1 & -1 & 0 & 0 & 0\cdots \\ 0 & 1 & 1 & -1 & 0 & 0\cdots \\ 0 & 0 & 1 & 1 & -1 & 0\cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\cdot\cdot\cdot \end{vmatrix}.$$

By expanding this determinant with reference to the first row, there results the desired relation.

Also solved by L. J. Burton, R. G. Buschman, Gertrude Ehrlich, Roger Lessard, Donald Marsh, Leo Moser, S. T. Parker, C. F. Pinzka, C. W. Trigg, and the proposer.

Recurrence Relations Defining Null Sequence

E 835 [1948, 498]. *Proposed by Kenneth May, Carleton College*

If x_1, y_1, and a are real numbers, and for all integral $n\geqq 1$ we have $x_{n+1}=a(x_n^2-y_n^2)$ and $y_{n+1}=2ax_ny_n$, for what values of x_1 and y_1 will $\lim_{n\to\infty} x_n=\lim_{n\to\infty} y_n=0$?

Solution by Norman Miller, Queen's University, Ontario. Represent the point (x_1, y_1) by the polar coordinates (r, θ) and suppose $a\neq 0$. By applying the given formulas we find that the point (x_{n+1}, y_{n+1}) is, in polar coordinates, $(a^{2^n-1}r^{2^n}, 2^n\theta)$. Hence, if $(x_1^2+y_1^2)^{1/2}=r<|a|^{-1}$, the points (x_n, y_n) approach the origin as a limit as n increases; if $r=|a|^{-1}$, the points remain on the circle

$r=|a|^{-1}$ and, in general, approach no limit; if $r>|a|^{-1}$, the points recede indefinitely from the origin.

Also solved by L. J. Burton, B. B. Dressler, N. J. Fine, W. Fulks, J. F. Heyda, Roger Lessard, Donald Marsh, S. T. Parker, C. F. Pinzka, and the proposer.

Chains of Circles

E 840 [1948, 576]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Inscribe three equal circles (A), (B), (C) in the corresponding interior angles of a triangle ABC such that we may insert between (B) and (C) a chain of tangent circles equal to (B) and (C) and all touching side BC, and similar chains between (C) and (A) and between (A) and (B). What is the condition for possibility of solution, and how many solutions are there for a given triangle ABC?

Solution by C. M. Sandwick, High School, Easton Pa. Suppose the problem solved and that A', B', C' are the centers of circles (A), (B), (C). Clearly, the sides of triangle $A'B'C'$ are commensurable, and therefore, since triangles $A'B'C'$ and ABC are similar, the sides of triangle ABC must be commensurable.

On the other hand, suppose the sides a, b, c of triangle ABC are commensurable, m being a common unit of measure. Then there are positive integers, p, q, r such that $a=pm$, $b=qm$, $c=rm$. Let I be the incenter of ABC and draw $BD=m/2$ perpendicular to BC on the opposite side of BC from A. Let DI cut BC in E, and let the perpendicular to BC at E cut BI in B'. Draw triangle $A'B'C'$ with sides parallel to those of ABC, A', B', C' falling on AI, BI, CI, respectively. Let the sides of $A'B'C'$ be denoted by a', b', c' and let $B'E$ be denoted by t. Then, from similar triangles, it is easy to show that $a'/2t=a/m=p$. Similarly, $b'/2t=q$ and $c'/2t=r$. Therefore there exists a solution to the problem such that there are $p+1$ circles in the chain along side BC and having centers on $B'C'$, $q+1$ circles in the chain along CA and having centers on $C'A'$ and $r+1$ circles in the chain along AB and having centers along $A'B'$.

If one solution exists, then, of course, there are infinitely many solutions, obtained by taking $m=g/k$, where g is the greatest common measure of a, b, c and k is any positive integer.

Also solved by P. R. Beesack, Richard Courter, C. F. Pinzka, and the proposer.

Folding an Envelope into Tetrahedra

E 841 [1948, 640]. *Proposed by C. W. Trigg, Los Angeles City College*

(a) Can any sealed rectangular envelope, after a single straight cut, be folded into two congruent tetrahedra? Will the position of the cut affect the size of the tetrahedra?

(b) How should the cut be made to make the total number of folds and unfolds a minimum?

(c) What should be the relative dimensions of the envelope in order that the tetrahedra be regular?

Solution by the Proposer. (a) Since the areas of congruent tetrahedra are equal, the envelope must be cut so that the two pieces will have equal areas. Hence the cut must pass through the center of the rectangle.

Let A, B, C, D be the corners of the envelope, with AD the shorter side, K the center of the front, K' the center of the back, EKF parallel to AD and GKH parallel to BA. The corresponding lines on the back together with the diagonals divide the surface of the envelope into sixteen congruent triangles, such that the adjacent angles at the corners and the center are alternately equal and complementary, the adjacent angles at the center of the edges are right angles, and the adjacent triangles are symmetric with reference to those common sides which are edges or are parallel to edges. (To facilitate future folding, fold the envelope along the diagonals and along EF before cutting.)

The cut MKN may be made in four essentially different ways:

1. Cut along line EKF. Unfold AE, AD, DF. Fold out along AK, AK', DK, DK'. Then KEK', and KFK' fold out into straight lines forming triangles AKK' and DKK' which are congruent to ADK and ADK'. Thus an isosceles tetrahedron is formed in three unfolds and four folds. The other half of the envelope can be folded similarly to give a tetrahedron congruent to the first one.

2. Cut through K at an angle to EF so that M falls on AB and N on DC. Then triangles MEK, MEK', NFK, NFK' are congruent. Hence, when the half-envelope is unfolded and folded as in way 1, an additional fold along KEK' will bring M into coincidence with N completing the fourth face DKK' of a tetrahedron congruent to those in way 1. (Three unfolds, five folds.)

3. Cut along BDK. Unfold AD and AB. Fold out AK and AK' and fold in along KEK'. Thus with two unfolds and three folds a tetrahedron congruent to those in way 1 is secured.

4. Cut through K so that M falls on BC and N on AD. Then triangles MGK, MGK', NHK, NHK' are congruent. Unfold AN, AB, BM. Fold out along AK, AK', BK, BK' until KEK' is a straight line. When the figure is folded in along KK', triangles ANK and BMK join and triangles ANK' and BMK' join to form faces congruent to AKK' and BKK' and to ADK. Hence this tetrahedron is congruent to those in way 1. (Three unfolds, five folds.)

Therefore the size of the tetrahedra is independent of the direction of the cut.

A necessary condition that these tetrahedra be formed is that the angles formed at a corner of the envelope by a diagonal with the edges be unequal. Thus the *single exception* is that of a square envelope for which angles EAK, EAK', HAK, HAK' are equal to 45°. When the cutting, unfolding, and folding of a square envelope is done in any one of the four ways, instead of a tetrahedron, two superposed squares are secured. That is, a square envelope after a single cut may be folded into two square envelopes.

(b) The cut should be made along a diagonal (way 3) to minimize the unfolding and folding.

(c) If the tetrahedron is to be regular, each face will be an equilateral tri-

angle, whence $AD/KH = 2/\sqrt{3}$. Thus the required relative dimensions of the envelope are given by $AD/GH = AD/AB = 1/\sqrt{3}$.

It may be observed further that if the envelope be slit along the edge, and then folded along the lines joining the midpoints of the slit edge to the other corners, a single isosceles tetrahedron is obtained in three unfolds and four folds.

Also solved by D. W. Matlack.

Cubics with Rational Zeros and Bend Points

E 842 [1948, 640]. *Proposed by H. L. Lee, University of Tennessee*

Determine cubic functions $f(x)$ for which $f(x) = 0$ and $f'(x) = 0$ have rational roots.

Solution by Alan Wayne, Flushing, N. Y. Let $f(x) = (x-a)(x-b)(x-c)$, where a, b, c are rational. If any two of the zeros, say a and b, are equal, then $f(x)$ and $f'(x)$ both have rational zeros for all rational a and c. Suppose, then, no two of a, b, c are equal. The substitution $y = (x-a)/(b-a)$ transforms $f(x) = 0$ and $f'(x) = 0$ into $F(y) = y(y-1)(y-q) = 0$ and $F'(y) = 3y^2 - 2(q+1)y + q = 0$, where $q = (c-a)/(b-a)$. Now $f'(x)$ will have rational zeros if and only if $F'(y)$ has rational zeros, that is, if and only if

$$q^2 - q + 1 = r^2, \tag{1}$$

where r is rational. But r is rational if and only if there exists a rational number $v \neq -1$ such that $r = 1/(v+1)$. Then there is a rational number $t \neq 0$ such that $q = (v/t+1)/(v+1)$. We also have $v \neq 0$ and $t \neq 1$, for otherwise $q = 1$ and $b = c$. From (1), since $v \neq 0$, we find $v = -(1+t)/(t^2-t+1)$, and it further follows that $t \neq -1$, 2. If, in addition, $t \neq 1/2$, then $q = (t^2-2t)/(1-2t)$. Thus finally, since $c = (b-a)q+a$, we find that a necessary and sufficient condition for both $f(x)$ and $f'(x)$ to have *distinct* rational zeros is that

$$c = a(1 - t^2)/(1 - 2t) + b(t^2 - 2t)/(1 - 2t), \tag{2}$$

where a and b are any two distinct rational numbers and t is any rational number different from -1, 0, 1/2, 1, 2. Thus any two distinct rational numbers may be chosen for a and b and a corresponding c determined by (2). The cubic polynomial is then easily constructed.

Also solved by Joshua Barlaz, L. J. Burton, Roger Lessard, C. M. Sandwick, C. W. Trigg, and the proposer.

A related problem is E 328 [1939, 170].

Spheres Related to an Oblate Ellipsoid

E 843 [1948, 640]. *Proposed by P. D. Thomas, Washington, D. C.*

Show that the difference in the radii of the two spheres, one equivalent in area, the other equivalent in volume, to an oblate ellipsoid of revolution is of the fourth order in the eccentricity of the generating ellipse.

Solution by the Proposer. Let a, b, e where $b^2=a^2(1-e^2)$, be the usual parameters of the generating ellipse, and let r_a and r_v be the radii of the equivalent-area and equivalent-volume spheres, respectively. These two radii are then defined by

$$(1) \qquad 2r_a^2 = a^2 + b^2(\tanh^{-1} e)/e = a^2[1 + (1 - e^2)(\tanh^{-1} e)/e],$$

$$(2) \qquad r_v^3 = a^2 b = a^3(1 - e^2)^{1/2}.$$

Now

$$\tanh^{-1} e = (1/2) \ln [(1 + e)/(1 - e)]$$
$$= e + e^3/3 + e^5/5 + e^7/7 + \cdots,$$

whence, from (1),

$$r_a = a[1 - e^2(1/3 + e^2/15 + e^4/35 + \cdots)]^{1/2}$$
$$= a(1 - e^2/6 - 17e^4/360 - 67e^6/3024 - \cdots).$$

From (2) we have

$$r_v = a(1 - e^2)^{1/6} = a(1 - e^2/6 - 5e^4/72 - 55e^6/1296 - \cdots)$$

It is thus seen that $r_a - r_v = a(e^4/45 + \cdots)$.

Also solved by L. J. Burton, Roger Lessard, Hyman Orlin, and C. W. Trigg.

Trigg showed that if we denote by R_a and R_v the radii of the spheres equivalent in area and volume, respectively, to the *prolate* ellipsoid, then $R_a - R_v$ is also of the fourth order in e.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results found in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4350. *Proposed by V. L. Klee, Jr., University of Virginia*

Money deposited in Postal Savings draws interest at the rate of $\frac{1}{2}\%$ quarterly, beginning with the month following that in which it is deposited. Interest is not compounded, and is not paid until the deposit is withdrawn. Under the simplifying assumption that deposits and withdrawals are permissible in any

amount whatever, find the optimum pattern of withdrawal and re-deposit for an initial deposit A, principal and interest to be permanently withdrawn at the end of n months.

4351. *Proposed by Albert Wilansky, Lehigh University, Bethlehem, Pa.*

Let $f(x, y)$ be continuous for all (x, y). On each circle with center at the origin f assumes a minimum at certain points. Is the set of all such points throughout the plane connected?

4352. *Proposed by Paul Erdös, Syracuse University*

Denote by $f(n; a_1, a_2, \cdots, a_k)$ the number of positive integers $m \leqq n$ which are either divisors or multiples of one of the a's $(a_i > 1)$. Prove that

$$f(n; a_1, a_2, \cdots, a_k) \leqq f(n; 2, 3, \cdots, p_k),$$

$2, 3, \cdots, p_k$ being the first k primes.

4353. *Proposed by H. F. Sandham, Trinity College, Ireland*

Prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left[\frac{\log n}{\log 2}\right] = \gamma,$$

where $[x]$ denotes the integral part of x, and γ is Euler's constant.

4354. *Proposed by Victor Thébault, Tennie, Sarthe, France*

A necessary and sufficient condition for a tetrahedron to be isosceles is that each of two bialtitudes of the tetrahedron divide the opposite edges proportionally.

SOLUTIONS

A Functional Equation

3649 [1933, 610]. *Proposed by F. T. O'Doubler, Springfield, Mo.*

Solve the functional equation

$$f(xy) = [f(x)]^{y^\beta}[f(y)]^{x^\beta},$$

where β is a real constant and $f(x)$ is a real continuous and single-valued function of the real variable x.

Solution by N. A. Court, University of Oklahoma. With $g(x) \equiv \log f(x)$, we have

$$g(xy) = y^\beta g(x) + x^\beta g(y). \tag{1}$$

For $y = x$, (1) becomes

$$g(x^2) = 2x^\beta g(x),$$

and by induction it is readily shown that the equality

$$(N) \qquad g(x^n) = nx^{(n-1)\beta}g(x)$$

is valid for all positive integral values of n.

For $y=1$ and $y=x^{-1}$, (1) yields respectively

$$g(1) = 0, \qquad g(x^{-1}) = -x^{-2\beta}g(x).$$

Thus formula (N) is valid for $n=0, -1$, and an easy induction completes the proof that it is valid for all integral values of n.

If in (1) we replace x and y by $x^{1/u}$ and $x^{(u-1)/u}$, respectively, where u is an integer, positive or negative, we have

$$(2) \qquad g(x) = x^{(u-1)\beta/u}g(x^{1/u}) + x^{\beta/u}g(x^{(u-1)/u}),$$

but, by virtue of (N),

$$g(x^{(u-1)/u}) = g[(x^{1/u})^{(u-1)}] = (u-1)x^{(u-2)\beta/u}g(x^{1/u}),$$

with which (2) is easily reduced to

$$(3) \qquad g(x^{1/u}) = u^{-1}x^{(1-u)\beta/u}g(x),$$

so that formula (N) is valid when n is a rational fraction with unit numerator.

Let v/u be a rational fraction. Since, from (N),

$$g(x^{v/u}) = g[(x^{1/u})^v] = vx^{(v-1)\beta/u}g(x^{1/u}),$$

we have, making use of (3),

$$g(x^{v/u}) = vx^{(v-1)\beta/u}\cdot(1/u)\cdot x^{(1-u)\beta/u}g(x) = (v/u)x^{(v-u)\beta/u}g(x).$$

Thus (N) is valid for all rational values of n.

An irrational number may be considered as the limit of an infinite sequence of rational numbers. Since the formula (N) is valid for all these rational numbers, it will remain valid for the limit, since the function f, and therefore also g, is continuous by hypothesis. Thus (N) is valid for all real values of n.

By differentiating both sides of (N) with respect to the parameter n we have

$$\frac{dg(x^n)}{d(x^n)}\cdot x^n \log x = x^{-\beta}g(x)(x^{\beta n} + n\beta x^{\beta n}\log x).$$

This equation is valid for all values of n, and in particular for $n=1$. Hence

$$\frac{dg(x)}{g(x)} = \frac{dx}{x\log x} + \frac{\beta dx}{x}\cdot$$

By integration we obtain

$$\log g(x) = \log\log x + \beta\log x + \log c,$$

where c is an arbitrary constant. From this equation upon replacing $g(x)$ by $\log f(x)$, we derive the desired result,

$$f(x) = x^{(cx^\beta)}.$$

Tetrahedron, Hyperboloid, and Paraboloid

4197 [1946, 161]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a tetrahedron $T \equiv ABCD$ the perpendiculars from an arbitrary point P to the planes of the faces meet again the pedal sphere of P in A', B', C', D'. (1) The intersections of corresponding faces of T and $A'B'C'D'$ belong to a hyperboloid. Generalize. (2) If P is on a sphere with center O and if the perpendiculars to the faces meet it in A'', B'', C'', D'', the orthologic center of T and $A''B''C''D''$, other than P, is the focus of the inscribed paraboloid with the axis parallel to OP.

Solution by the Proposer. (1) If A_1, B_1, C_1, D_1 are the projections of P on the planes of the faces BCD, CDA, DAB, ABC of T, we have, taking into account both magnitudes and directions,

$$(PA_1)(PA') = (PB_1)(PB') = (PC_1)(PC') = (PD_1)(PD') = k^2, \quad \text{say,}$$

whence T and $A'B'C'D'$ are polar reciprocals with respect to the sphere with center P and radius k (k real or imaginary). Therefore the intersections of corresponding faces of T and $A'B'C'D'$ belong to a hyperboloid.

(2) The planes p_a, p_b, p_c, p_d drawn through A'', B'', C'', D'' perpendicular to PA'', PB'', PC'', PD'' are concurrent at the point P' on the sphere (O) diametrically opposite to P.

By a translation equal and parallel to $P'A$, the planes p_b, p_c, p_d may be made to coincide with the planes CDA, DAB, ABC. If P goes into P_1 and if B_1'', C_1'', D_1'' are the projections of P_1 on CDA, DAB, ABC, then B_1'', C_1'', D_1'' correspond to B'', C'', D'' under the translation. Therefore the perpendicular through A to the plane $B''C''D''$ is also perpendicular to the plane $B_1''C_1''D_1''$, and we see that this perpendicular is the isogonal of line AP_1 for the trihedral $A-BCD$.

The same reasoning applied to vertices B, C, D shows that the orthologic center Q obtained by drawing perpendiculars through A, B, C, D to the planes $B''C''D''$, $C''D''A''$, $D''A''B''$, $A''B''C''$ is the isogonal conjugate, with respect to T, of the point at infinity in the direction OP. The projections of Q on the faces of T are therefore coplanar and Q is thus the focus of the paraboloid of revolution inscribed in T and having its axis parallel to OP.

Also solved by M. R. Bouvaist, who used analytical methods for part (2).

Editorial Note. For authorities for certain parts of the above proof see exercise 35(b), p. 213, and articles 741 and 748 in N. A. Court's *Modern Pure Solid Geometry.*

The analogous theorems for the plane are also true, a line of collinearity corresponding to the hyperbolic group of lines in part (1).

Orthocentric Tetrahedron

4219 [1946, 471]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In an orthocentric tetrahedron $ABCD$ with the altitudes AA', BB', CC', DD', let H' be the inverse of the orthocenter H with respect to the circum-

sphere, which the lines $H'A$, $H'B$, $H'C$, $H'D$ meet again in A_1, B_1, C_1, D_1. Show that the tetrahedrons $A_1B_1C_1D_1$ and $A'B'C'D'$ are similar and that the volume of the first is 27 times that of the second.

*Solution by the Proposer.** This problem is a particular case of the following general proposition (which may be new):

With the vertices of a tetrahedron $T\equiv ABCD$ as centers we construct arbitrary spheres (A, m), (B, n), (C, p), (D, q). The six spheres of similitude of these four spheres taken in pairs have in common two points V and W such that $OV\cdot OW=R^2$, R being the radius of the circumsphere (O) of T.† The lines AV, BV, CV, DV intersect (O) again in A_1, B_1, C_1, D_1 and the lines AW, BW, CW, DW meet (O) again in A_2, B_2, C_2, D_2. The tetrahedrons $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ are equal, since V and W are inverse with respect to (O).

In the present problem six spheres of similitude passing through H and H' may be constructed since the Newtonian plane of the spheres $(A, m), \cdots,$ is the perpendicular bisector of the segment HH'. By Neuberg's Theorem the squares of the radii of $(A, m), \cdots,$ are determined to within a factor of proportionality.‡ The corresponding tetrahedrons $A_2B_2C_2D_2$ and $A_1B_1C_1D_1$ are equal. The original result then follows since $A_2B_2C_2D_2$ is the homothetic transform $(H, 3)$ of the tetrahedron $A'B'C'D'$.§

A Divisibility Problem

4267 [1947, 479]. *Proposed by C. F. Pinzka, Student, Rutgers University*

Let p be a prime greater than 3, and let r/ps be the sum of the harmonic series, $1+1/2+1/3+\cdots$ to p terms. Prove that p^3 divides $r-s$.

I. *Solution by Ernst Trost, Technikum Winterthur, Zürich, Switzerland.* From Fermat's theorem follows

$$\begin{aligned} x^{p-1}-1 &\equiv (x-1)(x-2)\cdots(x-p+1) \\ &= x^{p-1}-s_1x^{p-2}+\cdots-s_{p-2}x+s_{p-1} \pmod{p}, \end{aligned} \tag{1}$$

and hence we have

$$s_{p-1}=(p-1)!\equiv -1 \pmod{p} \qquad \text{(Wilson's theorem)}$$
$$s_{p-2}\equiv[1+1/2+\cdots+1/(p-1)](p-1)!\equiv 0 \pmod{p}$$
$$s_{p-3}\equiv 0 \pmod{p}.$$

From (1) with $p>3$, we have

$$(p-1)!\equiv s_{p-3}p^2-s_{p-2}p+s_{p-1} \pmod{p^3},$$

and thus

$$s_{p-2}\equiv ps_{p-3}\equiv 0 \pmod{p^2}.$$

* Translated and references supplied by W. E. Byrne, Virginia Military Institute.

† Court, Modern Pure Solid Geometry, p. 204.

‡ Court, loc. cit., p. 203.

§ Court, loc. cit., p. 265.

Now, if

$$1+\frac{1}{2}+\cdots+\frac{1}{p-1}+\frac{1}{p}=\frac{r}{ps},$$

we obtain

$$s_{p-2}=\frac{(p-1)!}{s}\cdot\frac{r-s}{p}\equiv 0 \pmod{p^2}.$$

The integer $(p-1)!/s$ is prime to p, whence

$$r-s\equiv 0 \pmod{p^3}.$$

II. *Solution by Colin Blyth, Jr., Student, University of North Carolina.* We have at once

$$\sum_{a=1}^{p-1} a^{-1}=(r-s)/ps,\ (s,p)=1.$$

But $\sum_{a=1}^{p-1} a^{-1}\equiv 0 \pmod{p^2}$ for all prime >3 is a known theorem. (Wolstenholme's. L. E. Dickson, *History of the Theory of Numbers*, v. 1, p. 96, gives fifteen references, including this MONTHLY, problem 216 [1915, 103].) Finally $(r-s)/ps\equiv 0 \pmod{p^2}$ implies $r-s\equiv 0 \pmod{p^3}$, which is the desired result.

Also solved by P. T. Bateman, Alfred Brauer, D. H. Browne, P. A. Clement, R. E. Crane, Free Jamison, Y. S. Luan, Leo Moser, C. D. Olds, Chih-yi Wang, and the Proposer.

Special Properties of 1947

4269 [1947, 480]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find all numbers $N=abcd$ of four distinct digits, zero excluded, such that the sum

$$ab+ac+ad+bc+bd+cd$$

of products of the digits two at a time shall equal the sum

$$a^2+b^2+c^2+d^2$$

of the squares of the digits. For which among these is it true that the sum of the two-digit numbers ab and cd equals

$$a^2+c^2+d^2?$$

Solution by C. W. Trigg, Los Angeles City College.

1) If four digits are to satisfy the relation $\Sigma a^2=\Sigma ab$, then all must be odd, all must be even, or three must be odd and one even.

$$(\textstyle\sum a)^2=\sum a^2+2\sum ab=3\sum a^2. \tag{2}$$

Therefore $\Sigma a \equiv 0 \pmod 3$, and

$$\sum a^2 \equiv 0 \pmod 3. \tag{3}$$

4) Thus one digit is a multiple of 3 and the others are congruent to each other (mod 3).

There are only two sets of four distinct digits satisfying conditions (1) and (4). These two, 1347 (for which $\Sigma a^2 = \Sigma ab = 75$) and 1479 (for which $\Sigma a^2 = \Sigma ab = 147$), may be permuted to give twenty-four numbers each. These permutations constitute the only values of N.

If $10a+b+10c+d=a^2+c^2+d^2=M$, then $9(a+c)+\Sigma a+b^2=\Sigma a^2$. Therefore b is a multiple of 3. For the 1347 set, $b=3$, $\Sigma a^2=75$ and $M=66$, so that $d=3$, which is impossible. For the 1479 set, $b=9$, $M=66$, so that $d=7$. Hence the two integers which meet both conditions of the problem are 4917 and 1947, which latter is the year of the proposal.

Also solved by Colin Blyth, Jr., J. S. Frame, N. G. Gunderson, Free Jamison, Roger Lessard, Ou Li, Leo Moser, M. A. Rousseau, Chih-Yi Wang, and the Proposer.

Editorial Note. The above solution is adequate for the proposed problem about digits. Considerable interest, however, has developed in regard to a general solution of the equation

$$a^2 + b^2 + c^2 + d^2 = ab + ac + ad + bc + bd + cd. \tag{i}$$

(See Problem E. 823, [1949, 185].) Solving (i) as a quadratic in d, we have

$$2d = a + b + c + 3u$$

where

$$9u^2 = (a + b + c)^2 - 4(a^2 + b^2 + c^2 - ab - ac - bc),$$

or

$$4ab = 3u^2 + (a + b - c)^2.$$

This last equation can be put in the form

$$c = a + b + v, \qquad v^2 + 3u^2 = 4ab,$$

whence u, v are of the same parity, and may have any sign. Putting $m=(u+v)/2$, $n=(u-v)/2$, we have the result: if a, b, c, d are integers satisfying (i), then there exist integers m, n such that

$$ab = m^2 + mn + n^2, \tag{ii}$$

$$c = a + b + m - n, \qquad d = a + b + 2m + n. \tag{iii}$$

Conversely, m and n being arbitrary integers, if a and b are chosen to satisfy (ii) then they, together with c and d, given by (iii), satisfy (i).

From the known properties of numbers of the form p^2+pq+q^2 (therefore of form x^2+3y^2) we note from (ii) that a and b, apart from their highest common factor, must be also of the same form. Since a, b are any two of a, b, c, d, it follows that all four of them, except for any factors common to all, are of the form p^2+pq+q^2, where p and q are integers, positive, negative or zero.

Multiples of Irrational Numbers

4270 [1947, 549]. *Proposed by S. H. Gould, Victoria College, Toronto*

Let b be a fixed integer, $m=1, 2, \cdots$, and q irrational with $0<q<1$. Call the interval $(m, m+1)$ a gap if it does not contain a multiple of $b+q$. Prove that every set of b successive gaps contains exactly one multiple of $1+b/q$.

Solution by Guy Stevenson, University of Louisville, Kentucky. For every integer r there exists an integer x such that one or the other of the following relations holds:

$$rq < x < (r+1)q, \tag{1}$$

$$x < rq < (r+1)q < x+1. \tag{2}$$

If (1) holds, we have

$$r(b+q) < rb+x, \qquad rb+x+b < (r+1)(b+q)$$

so that the interval $I \equiv [r(b+q), (r+1)(b+q)]$ contains the b gaps corresponding to $m=rb+x, rb+x+1, \cdots, rb+x+b-1$.

On the other hand, if (2) holds, we have

$$rb+x < r(b+q), \qquad (r+1)(b+q) < rb+x+b+1$$

so that the interval I contains the $b-1$ gaps corresponding to $m=rb+x+1$, $rb+x+2, \cdots, rb+x+b-1$.

Thus (1) is the necessary and sufficient conditions that the interval I correspond to a set of b successive gaps. Now (1) implies also that

$$r(b+q) < \frac{x(b+q)}{q} < (r+1)(b+q)$$

which is the desired conclusion. It is seen also that no multiple of $1+b/q$ exists in an interval I which contains only $b-1$ gaps.

Also solved by Free Jamison, Leo Moser, and the Proposer.

Tetrahedron and Hyperbolic Set of Lines

4271 [1947, 549]. *Proposed by N. A. Court, University of Oklahoma*

The external bisectors of the three face angles of each trihedron of a given tetrahedron are coplanar. The four planes form a second tetrahedron. Show that the lines joining corresponding vertices of the two tetrahedrons form, in general, a hyperbolic group.

Solution by Ou Li, Yenching University, Peiping, China. Let $ABCD$ and

$A'B'C'D'$ be the given first and second tetrahedrons. Construct the ex-centers I_1, I_2, I_3 of the triangle ABC. Since I_2I_3 is one of the external bisectors of three face angles of the trihedron at A, it lies on the face $B'C'D'$ of the second tetrahedron $A'B'C'D'$. Similarly I_3I_1 and I_1I_2 lie respectively on the faces $C'A'D'$ and $A'B'D'$. It follows that I_1 lies on $A'D'$, I_2 on $B'D'$ and I_3 on $C'D'$. We note that AI_1, BI_2 and CI_3 are concurrent at the incenter I of the triangle ABC. Thus the sets of points A, A', I, I_1, D'; B, B', I, I_2, D'; C, C', I, I_3, D' are respectively coplanar, so that the line $D'I$ intersects AA', BB', and CC', and evidently also DD'.

Analogously, also $C'I$, $B'I$ and $A'I$ intersect AA', BB', CC' and DD'. Thus, the four lines AA', BB', CC', DD' which are met by $A'I$, $B'I$, $C'I$, $D'I$ form, in general, a hyperbolic group of lines.

Also solved by the Proposer who points out that if the tetrahedron $ABCD$ is isodynamic, then AA', BB', CC', DD' are concurrent.

A Summation Trick

4272 [1947, 549]. *Proposed by A. L. Epstein, Asbury Park, N. J.*

Given the sequence n_i where $n_1=a$, $n_2=b$, and $n_{i+2}=n_{i+1}+n_i$, $i=1, 2, \cdots$. Show that

$$\sum_{i=1}^{4k-2} n_i = r_k n_{2k+1}$$

and determine the form of r_k. (This is suggested by the familiar trick in which the subject selects n_1, n_2, and computes n_3, n_4, $\cdots$, n_7. He tells the operator the result n_7, and continues the calculation through n_{10}. When he adds the ten numbers he finds the operator has anticipated his result, which must be $11n_7$.)

Solution by A. B. Farnell, University of Colorado. Assuming a solution of the difference equation

$$m_{i+2} - m_{i+1} - m_i = 0$$

of the form $m_i = cq^i$, it is found that the general solution is given by

$$m_i = c_1p^i + c_2q^i,$$

where $p=(1+\sqrt{5})/2$ and $q=(1-\sqrt{5})/2$ are the roots of $x^2-x-1=0$. Hence, remembering that $q=-1/p$ and $p^2-p-1=0$, we have

$$\begin{aligned}\sum_{i=1}^{4k-2} n_i &= c_1p(p^{4k-2}-1)/(p-1) + c_2q(q^{4k-2}-1)/(q-1)\\ &= c_1p^{2k+1}(p^{2k-1}+q^{2k-1}) + c_2q^{2k+1}(q^{2k-1}+p^{2k-1})\\ &= (p^{2k-1}+q^{2k-1})(c_1p^{2k+1}+c_2q^{2k+1}) = (p^{2k-1}+q^{2k-1})n_{2k+1},\end{aligned}$$

which is the desired result with

$$r_k = p^{2k-1} + q^{2k-1} = 2^{1-2k}[(1+\sqrt{5})^{2k-1} + (1-\sqrt{5})^{2k-1}]$$
$$= 2^{2-2k}\sum_{s=0}^{k-1}\binom{2k-1}{2s}5^s.$$

Also solved by Joseph Bram, H. C. Davis, M. S. Klamkin, Roger Lessard, Ou Li and Hsien-yü-Hsü, J. F. Locke, Leo Moser, S. T. Parker, J. Rosenbaum, Guy Stevenson, F. Underwood, and the Proposer.

Editorial Note. Stevenson shows that $r_k = N_{2k-1}$, where N_j is the particular case of n_j resulting when $a=1$, $b=3$. Thus r_k satisfies the difference equation $r_{k+2} = 3r_{k+1} - r_k$ with $r_1 = 1$, $r_2 = 4$ (and $r_3 = 11$ as in the cited example.)

Rational Functions of $\cos^n\theta$ and $\sin^n\theta$

4273 [1947, 550]. *Proposed by I. S. Cohen, University of Pennsylvania*

Prove that for any positive odd integer n, $\cos\theta$ and $\sin\theta$ are rational functions of $\cos^n\theta$ and $\sin^n\theta$ with rational coefficients. Find explicit expressions in the case $n=3$.

Solution of the Proposer. Denote $\cos\theta$ and $\sin\theta$ by c and s respectively, and place $p=c^n$, $q=s^n$. We then have $c^2+s^2=1$ and must prove that c and s are contained in the field $F(p, q)$ of all rational functions of p and q with coefficients in the field F of rational numbers. In other words we must show that

$$F(c, s) = F(p, q). \tag{1}$$

We note first that $F(p) \subseteq F(c) \subseteq F(c, s)$. Now clearly

$$[F(c, s):F(c)] = 2,$$

where the brackets denote, as usual, the relative field degree. Also we have $[F(c):F(p)] = n$, so that $[F(c, s):F(p)] = 2n$. On the other hand, $F(p) \subseteq F(p, q) \subseteq F(c, s)$, so that (1) is now equivalent to

$$[F(p, q):F(p)] = 2n. \tag{2}$$

To prove this, let $f(x, y)$ be the irreducible polynomial over F such that $f(p, q)=0$, and let C be the curve $f(x, y)=0$. Thus C is the curve whose general point is (p, q)—that is, it has the parametrization $x=c^n$, $y=s^n$. It should be noted that the real part of this curve satisfies the equation $x^{2/n}+y^{2/n}=1$, and is thus a generalization of the familiar hypocycloid of four cusps.

We determine the degree of $f(x, y)$ by finding the number of intersections of C with the line $y=0$. There is no intersection at infinity, for p and q infinite implies the same for c or s, hence $p/q=(c/s)^n = \pm i$, so that C meets the line at infinity only in the circular points. The line $y=0$ thus meets C only in the points $(\pm 1, 0)$. To find the intersection multiplicity at $(1, 0)$, we seek to introduce s as local parameter. Thus

$$q = s^n, \qquad p = c^n = (1-s^2)^{n/2} = 1 - \tfrac{1}{2}ns^2 + \cdots .$$

Since n is odd it follows that s is, in fact, the local parameter. (If n were even, it would be s^2.) Thus q vanishes to the nth order, and the intersection multiplicity at $(1, 0)$ is n. Since it is the same at $(-1, 0)$, the total intersection multiplicity is $2n$, which is therefore the order of C. The degree of $f(x, y)$ is therefore $2n$, and since the point at infinity on the y-axis is not on C the degree of $f(x, y)$ in y is also $2n$. Thus q satisfies over $F(p)$ an irreducible equation of degree $2n$, and (2) follows.

Editorial Note. The Proposer employs results from the theory of algebraic curves to obtain an explicit form for $c \equiv g(p, q)/h(p, q)$, where g and h are polynomials. The following elementary procedure, suggested by C. R. Phelps, is quicker and produces the result in simpler form.

We have

$$s^2 = 1 - c^2, \quad p = c^3, \quad q = s^3,$$
$$q^2 = (1 - c^2)^3 = 1 - 3c^2 + 3pc - p^2,$$
$$cq^2 = c - 3p + 3pc^2 - p^2c.$$

Considering the last two equations as linear equations in c and c^2 and solving for c, we find

$$c = p(2 + p^2 + q^2)/(2p^2 - q^2 + 1).$$

s is found by interchange of p and q. The extension of the method for larger values of n is obvious.

Divergent Series

4278 [1948, 34]. *Proposed by Peter Ungar, Budapest, Hungary*

Construct two divergent series, $\sum a_k$ and $\sum b_k$ with $a_1 \geqq a_2 \geqq \cdots \geqq 0$, $b_1 \geqq b_2 \geqq \cdots \geqq 0$, but such that if $c_k = \min(a_k, b_k)$, $c_k > 0$, then $\sum c_k$ is convergent.

Solution by A. Novikoff, Stanford University. We consider any two series of positive terms, one convergent and one divergent, whose general terms, $C(n)$ and $D(n)$ respectively, tend monotonically to zero and such that $D(n) > C(n)$.

Since $\sum D(n)$ is divergent, corresponding to each integer n we can find another integer $\phi(n) > n$ defined by the relation

$$\sum_{j=n}^{\phi(n)} D(j) \geqq \epsilon,$$

where ϵ is any fixed positive number. We now define the sequence of integers $\{n_k\}$ recursively by the conditions $n_1 = 1$, $D(n_{k+1}) < C(\phi(n_k))$. It is clear that $n_k < \phi(n_k) < n_{k+1}$.

Now a solution to the problem is given by the sequences:

$$\{a_n\}; \quad C(n_1), C(n_1 + 1), \cdots, C(\phi(n_1)), D(n_2), D(n_2 + 1), \cdots, D(\phi(n_2)),$$
$$C(n_3), \cdots, C(\phi(n_3)), D(n_4), \cdots;$$

$$\{b_n\};\quad D(n_1), D(n_1+1), \cdots, D(\phi(n_1)), C(n_2), C(n_2+1), \cdots,$$
$$C(\phi(n_2)), D(n_3), \cdots, D(\phi(n_3)), C(n_4), \cdots.$$

It is clear that $\sum a_n$ and $\sum b_n$ both diverge, since each contains infinitely many stretches of terms adding up to ϵ or more, and that $\sum c_n$, where $c_n = \min(a_n, b_n)$ is convergent by comparison with $\sum C(n)$.

Also solved by Joshua Barlaz, M. T. Bird, Colin Blyth, M. K. Fort, Jr., Michael Golomb, William Gustin, H. J. Hamilton, Fritz Herzog, M. S. Klamkin, N. H. Kuiper, Norman Miller, S. T. Parker, George Piranian, W. Seidel, Albert Wilansky, F. E. Wood, and the Proposer.

Editorial Note. Novikoff extended his solution to show how to construct k divergent series of monotonically decreasing positive terms such that the minorant of all k series converges, but the minorant of every set of $k-1$ of the series diverges.

Many of the solvers showed that any convergent series of monotonically decreasing terms could be taken for $\sum c_n$ .The Proposer proved that, given divergent $\sum a_n$, the necessary and sufficient conditions that divergent $\sum b_n$ can be constructed such that $\sum c_n$ converges is that $\overline{\lim}\ (1/na_n) = \infty$.

A Problem of Pursuit

4280 [1948, 100]. *Proposed by Joseph Rosenbaum, Milford School, Milford, Connecticut.*

The points $A_1, A_2, \cdots, A_n$ are pursuing one another cyclically, the speed of A_i being proportional to the distance A_iA_{i+1}. If at time $t=0$ the points are the vertices of a given polygon, find the paths described by the points.

Solution by J. B. Reynolds, Salt Lake City, Utah. The motion must satisfy the set of vector equations

$$(1)\qquad \begin{aligned} k_i \frac{d\mathbf{r}_i}{dt} &= \mathbf{r}_{i+1} - \mathbf{r}_i,\ i=1, 2, \cdots, n-1 \\ k_n \frac{d\mathbf{r}_n}{dt} &= \mathbf{r}_1 - \mathbf{r}_n, \end{aligned}$$

where $\mathbf{r}_i$ is the radius vector to the point A_i and k_i are constant.

The cartesian coördinates of A_i must then satisfy the identities

$$(2)\qquad Lx_i = x_i, \qquad Ly_i = y_i,$$

in which L is the operator

$$(k_{i-1}D+1)(k_{i-2}D+1)\cdots(k_1D+1)(k_nD+1)\cdots(k_iD+1),$$

where $D \equiv d/dt$. The equations (2) have the solutions

$$x_i = \sum_{j=1}^{n} c_i^j e^{\lambda_j t}, \qquad y_i = \sum_{j=1}^{n} d_i^j e^{\lambda_j t},$$

the λ_j being (distinct) roots of

$$(3) \qquad (k_n\lambda + 1)(k_{n-1}\lambda + 1) \cdots (k_1\lambda + 1) = 1.$$

To evaluate the constants c_i^j, note that the relation

$$\sum_{j=1}^{n} k_i c_i^j \lambda_j e^{\lambda_j t} = \sum_{j=1}^{n} c_{i+1}^j e^{\lambda_j t} - \sum_{j=1}^{n} c_i^j e^{\lambda_j t}$$

must hold for all values of t. This implies that

$$(4) \qquad c_{i+1}^j = (k_i\lambda_j + 1)c_i^j.$$

Let the initial values of the x_i be $x_1^0, x_2^0, \cdots, x_n^0$; then the relations

$$(5) \qquad \sum_{j=1}^{n} c_i^j = x_i^0$$

must hold.

The equations (5) together with the recurrence relations (4) are sufficient to determine the constants c_i^j. Analogous relations hold for the d_i^j.

In the particular case $k_1 = k_2 = \cdots = k_n = k$, equation (3) becomes $(k\lambda+1)^n = 1$ with $\lambda_j = (\omega_j - 1)/k$ where the ω_j are the nth roots of unity. In this case (4) becomes

$$(6) \qquad c_{i+1}^j = \omega_j c_i^j.$$

Nothing in the solution thus far requires that the A_i be coplanar. In the three dimensional case, the analysis for z_i proceeds exactly as for x_i, y_i. Note also from (1) that $\sum x_i$, $\sum y_i$ remain constant so that the centroid of the set A_i is fixed.

Consider further the special case where the initial polygon is regular. For convenience let the vertices of the polygon coincide with the termini of the vectors representing the nth roots of unity in the complex plane. Let

$$(7) \qquad \omega_1 = e^{2\pi i/n}, \qquad \omega_{i+1} = (\omega_1)^{i+1}, \qquad \omega_n = \omega_0 = 1,$$

and note that

$$(8) \qquad \sum_{j=1}^{n} b_i^j = \sum_{j=1}^{n} (c_i^j + id_i^j) = x_i^0 + iy_i^0 = \omega_{i-1}.$$

The relations given in (6), (7) and (8) determine $b_1^1 = 1$, $b_i^j = 0$ $(j > 1)$, so that the final solution is

$$x_1 + iy_1 = e^{(e^{2\pi i/n} - 1)t/k}.$$

In polar coördinates,

$$\rho_1 = e^{-(1-\cos 2\pi/n)t/k}, \qquad \theta_1 = \frac{t}{k} \sin \frac{2\pi}{n}.$$

To obtain the equations of motion of A_{i+1}, add $2\pi/n$ to θ_i.

Also solved by the Proposer.

Integers with Digits 0 and 1

4281 [1948, 100] *Proposed by M. S. Knebelman, Washington State College*

Given an integer n. Show that an integer can always be found which contains only the digits 0 and 1 (in the decimal scale) and which is divisible by n. Is there an algorithm for finding the smallest such number?

Solution by Leo Moser, University of Manitoba. Consider the residues (mod n) of the n integers $(10^i-1)/9$, $i=1, 2, \cdots, n$. Either there is one of them which is 0, or two are the same in which case the residue of the difference of the corresponding numbers is 0. In either case we have an integer composed of a block of 1's followed by a block of 0's divisible by n. The same argument, using $(B^i-1)/(B-1)$, leads to the same result in any system of numeration, with base B.

We have thus proved that the required integer has no more than n digits Hence if we test 1, 10, 11, 100, 101, $\cdots$, $(10^n-1)/9$ for divisibility by n we will certainly find the smallest number of the required type in not more than 2^n such operations. Clearly the work will be considerably cut down by first constructing a table of residues of 10^i (mod n).

Also solved by Colin Blyth, D. H. Browne, P. A. Clement, Roger Lessard, and F. L. Miksa.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

A Source Book in Greek Science. By M. R. Cohen and I. E. Drabkin. New York, McGraw-Hill Book Co., 1948. 21+579 pages. $9.00.

A great part of the attractiveness of ancient science arises from the important fact that the reader can check its statements for himself. If an amateur astronomer wishes to repeat modern observations on the length of the year, he must have access to expensive and elaborate equipment; but to determine for himself that, as Hipparchus says, the estimate of 365 days is probably wrong by about a quarter of a day, he need only follow the advice of many ancient

authors, from the poet Hesiod onward for a thousand years; throughout this period he will find a rich association of poetry and science, impossible in modern times, when science has gone so far beyond the thoughts and observations of daily life. The study of such classics will reward him, in the words of Albert Jay Nock, with a "feeling of immense longevity." The situation is the same in mathematics. The amateur who is interested in such natural questions as whether the number of primes is infinite, or what was the original form of the basic theorems of geometry, need only turn the pages of the book under review.

There are nine divisions in the book: Mathematics, 89 pages, Astronomy 54, Mathematical Geography 39, Physics 170, Chemistry and Chemical Technology 22, Geology and Meteorology 20, Biology 73, Medicine 63, and Physiological Psychology 27. The present review is confined to the first and third of these sections, with the parenthetical remark that the other seven appear to be on an equally high level and some of them contain considerable mathematics.

The first section begins with a chapter from Proclus, on "Divisions of Mathematics, Pure and Applied"; and in suitable places appear other passages from the same writer, a fifth century Neoplatonist who wrote a valuable commentary on Euclid. There follow eleven paragraphs on the theory of numbers, including the infinitude of primes and the Eudoxian theory of irrationality. These and other passages are either quoted from standard translations or are translated for the first time by Dr. Drabkin; in each case they are accompanied by brief but excellent introductions and notes. There are three passages on Algebra; the second of these is the famous Cattle-Problem of Archimedes, and the third consists of problems from Diophantus, both determinate and indeterminate, including the one which gave rise to Fermat's well-known marginal note.

The passages on Geometry are numerous and some of them are of the highest interest. They begin with a full quotation of the valuable section in Proclus on the history of geometry up to the time of Euclid; there is a discussion, from Pappus, of analysis and synthesis; such matters as the parallel postulate and the method of exhaustion are presented in quotations of generous length; the three geometrical problems, squaring of the circle, duplication of the cube, and trisection of an angle, are adequately represented in 17 pages; there is an illuminating extract from the "Method" of Archimedes for solving problems of the integral calculus; then come passages from Apollonius on conic sections and the charming mathematical excursus from Pappus on the sagacity of bees; the section on Mathematics closes with a brief treatment of Greek trigonometry and mensuration.

The section on Mathematical Geography begins with a general discussion from Aristotle on the shape and size of the earth, followed by an account of the simple measurements of Eratosthenes whereby he determined the polar circumference as being about 23,300 miles. The section closes with various accounts, both technical and popular, on the art of map-making, with excellent illustrations of maps of the ancient world according to Hecataeus (*ca.* 500 B.C.),

Eratosthenes (*ca.* 250 B.C.) and Ptolemy (*ca.* 150 A.D.); the methods given for representing the inhabited earth on a plane surface rank high in mathematical interest.

From this partial account it will be clear that the Source Book is heartily recommended to any reader who is interested, from whatever point of view, in the history of Greek mathematics. Its only fault, one may say, is that it is not more complete. Nothing is quoted from Apollonius except the prefaces to some of his books and Archimedes is represented in an almost equally meager way. The reader may well exclaim: "then how can you speak of a source book in Greek Mathematics?" But there are many extenuating circumstances. As we read in the preface, "the availability to English readers of the late Sir Thomas L. Heath's *History of Greek Mathematics* and of Ivor Thomas's translations . . . has prompted the editors to devote less space to this field in order to give additional space to fields in which the material is not so readily available." The difficulty is fundamental to all classical scholarship; if less material were available, a scholar could take account of all of it; and if more, then he would not be expected to. The present editors have dealt with the dilemma in a skilful way.

The format, printing and paper of the book are excellent; misprints are rare and unimportant, there being one such misprint, for example, on p. 562; the notes and introductory material will be a delight to the reader; and finally, the translations are on the same plane as the other features of the book.

With respect to these translations, the reviewer would like to make one query. In Dr. Drabkin's version of the passage from Pappus (p. 80), which deals with solids of revolution, reference is made only to the volume of such solids. But the Greek words seem to allow reference to both volume and surface; i.e. *ὁ μὲν τῶν τελείων ἀμφοιστικῶν λόγος* means "the ratio of the figures (whether volumes or surfaces) generated by complete revolution," and *ἔκ τε τῶν ἀμφοισμάτων* refers to the plane figures (whether areas or circumferences) which are revolved. In other words, is there not reference here to the two theorems now known as theorems of Pappus, and not to just one of them?

A minor point in translation is this: would it not be better to use *logistic* throughout, as in the note on p. 4, to translate *λογιστική* (the art of calculation), rather than *logistics*, as on p. 4 and frequently?

Taken together with a projected work on Medieval Science, the present volume forms a welcome addition to the other Source Books of the series, each of which is devoted to the progress of a single science from 1400 to 1900. Also, we are assured that the period since 1900 will not be neglected. In a special preface by the general editor of the series, we read: "a single volume, containing the most important contributions of the major sciences from 1900 to 1950, is planned for publication about 1960, and a similar volume each half century thereafter indefinitely." Such laudable far-sightedness is presumably the result of classical reading; let us hope that the present volume on Greek science will produce in all its readers a similar "feeling of longevity."

S. H. Gould

A Collection of Papers in Memory of Sir William Rowan Hamilton. Scripta Mathematica Studies, No. 2. New York, Scripta Mathematica, 1945. 82 pages. $1.00.

The material in this little volume can be separated into three categories. First there are discussions of the life of Hamilton: *Sir William Rowan Hamilton* by David Eugene Smith, which was taken from Smith's *Portraits of Eminent Mathematicians, with Brief Biographical Sketches*; and *The Life and Early Work of Sir William Rowan Hamilton* by J. L. Synge.

A second group of essays is concerned with the scientific work of Hamilton: the paper already mentioned by Synge; *Algebra's Debt to Hamilton* by C. C. MacDuffee; *An Elementary Presentation of the Theory of Quaternions* by F. D. Murnaghan; *Hamilton's Work in Dynamics and Its Influence on Modern Thought* by H. Bateman; *Hamilton's Contribution to Mechanics* by Edwin B. Wilson; *The Constancy of the Velocity of Light* by Vladimir Karapetoff. Karapetoff's paper, the longest in the collection, contains no mention of Hamilton or his work. Wilson's paper is an excerpt from a letter to the editor of *Scripta Mathematica*, less than a page in length. Bateman's paper has 106 items in the bibliography, and each of these references is related to the essay.

The third category of material is a number of personal items: a portrait, two poems by Hamilton, facsimiles of two pages from his notebooks, and an announcement concerning an Irish postage stamp issued on the occasion of the centenary of Hamilton's discovery of quaternions.

Quaternions, of course, are prominent in many of the papers. MacDuffee treats them as quadruples of real numbers, as did Hamilton, whereas Murnaghan gives a representation of quaternions as special skew-symmetric matrices of order 4.

Several of the writers give their views on the relative merits of quaternions and Hamilton's contributions to applied mathematics. And Synge compares aspects to Hamilton's work with that of Lagrange and Jacobi. These discussions help make it a lively and entertaining, as well as informative, volume.

IVAN NIVEN

Elementary Statistical Analysis. By S. S. Wilks. Princeton University Press, 1948. 11+284 pages. $2.50.

This book is intended for a one semester elementary course in statistics. As the author states in the preface, he has been teaching this material as an introduction to statistics to students in all fields of application. In accordance with such a purpose only a very modest mathematical background is presupposed, essentially just the concepts of derivative and integral, such as a student might acquire in a one term calculus course.

It is clear that on this basis a complete treatment of much of the theory of statistical inference is not possible. The common way out of this difficulty is to make such a text a collection of methods, stated as a set of directions, together

with illustrations of their use. The present book—breaking away from this unfortunate tradition—has as its goal the understanding of statistics rather than manipulative proficiency. As the foundation for such understanding the elementary (discrete) theory of probability is developed. This includes in particular the more important theorems concerned with the notions of addition, multiplication and complementation of events, and of expectation and variance of a random variable. This material, together with a discussion of the binomial and Poisson distributions and with some extensions to the continuous case, forms the central part of the book, taking up about one half of the space.

Against this background it is possible to give a satisfactory account of the fundamentals of statistical inference. Confidence intervals and tests of significance are treated, and the notions are applied to the problems of inference on the basis of a large sample, concerning a single mean and the difference of two means. The case of small samples from normal distributions is also considered, and there is a brief chapter on testing for randomness. In this part certain results from probability theory have to be stated without proof. However, the resulting loss of understanding does not seem to be too serious since the content of the theorems can be explained satisfactorily even though the proofs cannot be given. The book also contains three chapters on descriptive statistics, and a chapter on correlation and regression for two variables together with a discussion of the method of least squares. Throughout the text, each chapter is followed by a set of exercises.

A few minor points may be raised:

(i) One wonders why in the theorem asserting linearity of the operation of expectation, independence of the variables is stressed. The theorem is true, useful, and essentially no harder to prove when the variables are dependent.

(ii) It seems a pity that the power of a test is never mentioned explicitly and that in connection with estimation no mention is made of the justification of the method of least squares from the point of view of unbiased estimation.

(iii) A section on the χ^2-distribution and some of its applications would fit well within the framework of the book, and would be of great value to most of its users.

By providing an elementary course that can be taught to students in many fields of application, and in which the objective is the teaching of concepts rather than rules, Professor Wilks has made an important contribution to elementary instruction in statistics. One may also hope that his book will encourage the centralization, within universities, of elementary statistical teaching.

E. L. LEHMANN

Les Equations Differentielles de la Technique. (Cours de Mathematiques Appliquees de L'Ecole Polytechnique de L'Universite de Lausanne). By Charles Blanc. Neuchatel, Editions du Griffin, 1947. 311 pages, unbound. Fr. s. 29.50.

This book is a text on the advanced calculus level for engineering students.

Ordinary and partial linear differential equations are studied with many examples from mechanics, electricity, and heat. Most of the equations considered have constant coefficients. The book is well written. The first part is on ordinary differential equations and includes a discussion of the steady state and transient solutions with electrical and mechanical examples and use of the Laplace transform. A chapter on boundary value problems, the Green's function and Fourier series with simple examples complete Part One.

Part Two takes up the d'Alembert equation, the heat equation and Poisson's equation where examples from electricity and mechanics are considered and the treatment includes the Laplace transform, propagation of waves, fundamental singular solutions, and Green's theorem and functions.

Part Three contains a chapter on the calculus of variations, one on elliptic integrals and one on Bessel functions.

NORMAN LEVINSON

NEW BOOKS RECEIVED

Analytic Geometry. By R. Robinson. New York, McGraw-Hill Book Co., 1949. 10+147 pages. $2.25.

Intermediate Algebra for Colleges. By P. R. Rider. New York, The Macmillan Co., 1949. 12+242 pages. $2.75.

Solid Geometry. By J. S. Frame. New York, McGraw-Hill Book Co., 1948. 10+339 pages. $3.50.

First Year College Mathematics with Applications. By P. H. Daus and W. M. Whyburn. New York, The Macmillan Co., 1949. 16+495 pages. $5.00.

Infinitesimalrechnung. Vol. I (3rd Edition), Vol. II (2nd Edition). By H. Behnke. Munster, Aschendorffsche, 1947 and 1948. 5+309 pages and 5+353 pages.

Introduction to Complex Variables and Applications. By R. V. Churchill. New York, McGraw-Hill Book Co., 1948. 6+216 pages. $3.50.

Problem Book in The Theory of Functions. Vol. I. By K. Knopp. Translated by L. Bers. New York, Dover Publications, 1949. 8+126 pages. $1.85.

Lezioni di Geometria Analitica e Projettiva. Second Edition. By G. Fano and A. Terracini. Torino, Paravia, 1948. 8+642 pages.

Niet-Euklidische Meetkunde. Second Edition. By J. C. H. Gerretsen. Gorinchem, Noorduijn's, 1949. 11+212 pages.

Numerical Calculus, By W. E. Milne. Princeton University Press, 1949. 10+393 pages. $3.75.

Practical Analysis. By F. A. Willers. Translated by R. T. Beyer. New York, Dover Publications, 1948. 10+422 pages. $6.00.

Fluid Dynamics. By V. L. Streeter. New York, McGraw-Hill Book Co., 1948. 11+263 pages. $5.00.

Computation Curves for Compressible Fluid Problems. By C. L. Dailey and F. C. Wood. New York, John Wiley and Sons. 1949. 10+33 pages, plus tables. $2.00.

Psychological Statistics. By Q. McNemar. New York, John Wiley and Sons, 1949. 8+364 pages. $4.50.

Archibald Henderson, The New Crichton. Edited by S. S. Hood. New York, Beechhurst. 1949. 18+252 pages. $5.00.

Integraltafel. Part I. By W. Grobner and N. Hofreiter. Vienna, Springer-Verlag, 1949. 8+166 pages. $5.40.

Theorie und Anwendung der Unendlichen Reihen. Fourth Edition. By K. Knopp. Berlin, Springer-Verlag, 1947. 12+583 pages.

Advances in Applied Mechanics. Vol. I. Edited by R. von Mises and T. von Karman. New York, Academic Press, Inc., 1948. 8+292 pages, $6.80.

Selected Techniques of Statistical Analysis. By The Statistical Research Group, Columbia University. New York, McGraw-Hill Book Co., 1947. 14+473 pages. $6.00.

An Essay Toward a Unified Theory of Special Functions. By C. Truesdell. Princeton University Press, 1948. 182 pages. $3.00.

Tables of Bessel Functions of the First Kind of Orders Forty through Fifty-one. (Annals of the Computation Laboratory of Harvard University, No. 9). Cambridge, The Harvard University Press, 1948. 14+620 pages. $10.00.

Principles of Mechanics. Second Edition. By J. L. Synge. New York, McGraw-Hill Book Co., 1949. 16+530 pages. $5.00.

Fundamental Theory. By A. S. Eddington. Cambridge, At the University Press, 1946. 8+292 pages. $6.00.

The Psychology of Invention in the Mathematical Field. Revised Edition. By J. Hadamard. Princeton University Press, 1949. 9+145 pages. $2.50.

Children Discover Arithmetic. By C. Stern. New York, Harper and Brothers, 1949. 24+295 pages. $4.50.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS, 1947–48

Pi Mu Epsilon, Hunter College, New York

The theme of the Hunter College Chapter of *Pi Mu Epsilon* for the Fall term of 1947 was *Mathematical paradoxes*. Among the topics discussed were:

Elementary paradoxes in algebra and geometry
Plane analytic geometry in the complex field
Cardinal numbers
Transformation of the affine, euclidean, and rigid motion groups

Transfinite numbers and paradoxes of the infinite, by Sylvia Schlachter and Suzanne Levine.

The guest speaker for the initiation dinner was Prof. Saunders MacLane of the University of Chicago, who spoke on: *Mathematical centers here and abroad.*

The theme for the Spring term was *Continued fractions*. Material presented dealt with:

Convergents
Rational numbers
Solution of diophantine linear equations
Irrational numbers
Irrational quadratic algebraic numbers
Infinite series
The numerical solution of algebraic equations by continued fractions, by Anna Lemont.

The speaker for the Spring initiation was Prof. Samuel Eilenberg of Columbia University, whose talk entitled *Some aspects of topology* concluded an evening of relaxation and enjoyment.

Kappa Mu Epsilon, Alabama College

Papers presented during the year included:

Pythagoras and the Pythagoreans, by Lida True
Some topics from the theory of numbers, by Dr. Rosa Lea Jackson
Regular polygons and how they grow, by Margaret O'Gwynn
Waves and vibrations, by Frances Jones
Mathematics and the physical sciences, by Virginia Havens.

The first meeting of the year was a party to which all upper classmen majoring in Mathematics were invited. In the Spring, eight new members were initiated. A banquet followed the initiation ceremony.

Officers elected for the year 1948–49 are: President Frances Kelly; Vice-President, Margaret O'Gwynn; Secretary, Doris Williamson; Treasurer, Betty Lou Wilson.

Mathematics Society, Cooper Union

The Cooper Union *Mathematics Society* enjoyed the following program of student papers during 1947–48:

The calculus of finite differences and difference equations, by D. Jagerman
Convergence tests, by H. Hockstadt
Indeterminate equations and their applications, by M. Stern
Pythagorean primitives, by B. Lowe
Introduction to the calculus of variations, by M. Rosenfeld
Recurring series, by D. Jagerman
The Laplace transform, by R. Cox
Theory and application of the D operator, by H. Hochstadt
Congruences, by A. Kahn
Determinants and linear equations, by M. Stern

Matrix algebra, by D. Jagerman
Non-Euclidean geometry, by E. Wachspress.

The officers for 1948–49 are: President, D. Jagerman; Vice-President, H. Hochstadt; Secretary, M. Stern; Treasurer, M. Rosenfeld; Faculty Adviser, Prof. J. N. Eastham.

Kappa Mu Epsilon, The College of Wooster

The *Ohio Beta* Chapter of *Kappa Mu Epsilon*, inactive during the war, resumed activities at the close of the school year 1946–47. There are at present twenty-two undergraduate members. The following talks by members of the faculty and undergraduate body were presented during 1946–48:

Topology, by Prof. M. P. Fobes
Einstein's theory of relativity, by Prof. R. J. Stephenson
Some interesting curves, by the Chapter Initiates
Foundations of mathematics, by Dorothy Renzema
Non-Euclidean geometry, by Duncan McCune
Mathematical fun, by the Chapter Initiates.

The Wilson Mathematical Prize for the year was won by Margaret Hagen.

The officers for 1948–49 are: President, Robert Nethercut; Vice-President, Robert Shaffer; Treasurer, John Richardson; Secretary, Margaret Herr; Faculty Sponsor, Prof. M. P. Fobes.

Pi Mu Epsilon, Kansas State College

Talks presented to the *Kansas Beta* Chapter of *Pi Mu Epsilon* included:

Nomographs, by Prof. C. E. Pearce
Approximation of functions by integral means, by Dr. P. M. Young
Summability of series and integrals, by Dr. S. T. Parker.

Newly elected officers are: Director, Dr. P. M. Young; Vice-Director, Robert Cell; Secretary, Virginia Chatelain; Treasurer, Jack Northam.

The Zeno Club, Alfred University

The *Zeno Club* of Alfred University held semi-monthly meetings, with a program and discussion, followed by a social period with refreshments. Papers presented by the faculty and others were:

Nomograms, by Dr. C. E. Rhodes
Boolean algebra, by Prof. W. V. Nevins
Probabilities in genetics, by Prof. John Freund
Transfinite cardinal numbers, by Ralph Jordan
Number system to the base 2, by Leslie Shershoff
Euclidean algorithm, by Joan Berkman
Hyperbolic functions, by Stanley Graf
Topology, by Lewis Butler
Segment functions, by Prof. John Freund
Maxima and minima, by Irwin Miller
Cryptography, by Ralph Beals
Divisibility of numbers, by Dean A. E. Whitford.

The club's major projects of the year included the presentation of several books to the mathematics section of the Alfred University Library, and the sponsoring of a campus-wide competitive examination in calculus. The prizes given for this were: ten dollars, a season movie ticket, and a calculus book.

Officers for the year 1947–48 were: President, Ralph Jordan; Vice-President, Joan Berkman; Secretary-Treasurer, Mary Elizabeth Van Norman.

Pi Mu Epsilon, Carnegie Institute of Technology

On November 20, 1947, the Carnegie Institute of Technology *Mathematics Club*, which was founded by Prof. J. B. Rosenbach in 1938, was installed as the *Pennsylvania Epsilon* Chapter of *Pi Mu Epsilon*. The installation officers were Prof. Tomlinson Fort, Director-General and Prof. J. S. Gold, Secretary-Treasurer General. Prof. J. J. Stoker of New York University, an alumnus and former faculty member of Carnegie Institute of Technology, gave the installation address on the subject *Water waves on sloping beaches*.

A group of twenty members of the chapter visited the Westinghouse Electric Corporation in East Pittsburgh in order to witness a demonstration of the Westinghouse electric analog computer.

Prof. J. L. Synge was the guest of honor and principal speaker at the annual initiation and banquet. His paper, which will appear in the Carnegie Technical, was on the topic *Mathematics and Science*.

Four additional meetings were held during the year at which the following papers were presented:

Finite differences and statistics, by Mr. H. J. Weiss

The four-color problem, by Mr. C. G. Maple

Horn angles, Mr. G. H. F. Gardner

The Westinghouse analog computer, by Mr. D. L. Whitehead, of the Westinghouse Electric Corporation.

The officers for the year 1947–48 were: Director, D. L. Wallace; Vice-Director, D. R. Harris; Secretary, R. T. Siegel; Treasurer, E. P. King; Faculty Adviser, Prof. J. B. Rosenbach.

Mathematics Club, Lander College

The *Mathematics Club* of Lander College holds six regular open meetings during the school year. A social hour at which refreshments are served follows each program.

The 1947–48 programs, given by students and faculty, consisted of games, mathematical contests and puzzles, and papers on:

The life and mathematical achievements of Rene Descartes

A review of Eddington's "Space, Time, and Gravitation"

History of the development of the atomic theory.

The 1948–49 officers are: President, Odel Black; Vice-President, Emmilene Talbert; Secretary-Treasurer, Frankie Sue Dickerson; Sponsor, Prof. Mary Pettus.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

CANADIAN MATHEMATICAL CONGRESS

The second seminar of the Canadian Mathematical Congress will be held on August 16–September 10, 1949 at the University of British Columbia, Vancouver. The seminar will resemble in structure the 1947 seminar of Toronto, its arranged program including: (a) Courses of eight lectures each on research level; (b) Seminars (discussion groups); (c) Instructional courses of fifteen or more lectures at first year graduate level. The topics are Applied Mathematics and Related Fields of Pure Mathematics.

Courses (a) will be as follows: P. A. M. Dirac, University of Cambridge, England, dynamical theory of fields, classical and quantum; Antoni Zygmund, University of Chicago, topics connected with Fourier series; Laurent Schwartz, University of Nancy, theory of distributions and applications to ordinary and partial differential equations; and one other course by a distinguished applied mathematician.

Under (b) there will be two seminars, one in applied and one in pure mathematics under the direction respectively of Leopold Infeld, University of Toronto, and Gabor Szegö, Stanford University.

Instructional courses (c) will be as follows: M. S. Knebelman, Washington State College, topics from differential geometry; W. H. McEwen, University of Manitoba, the Sturm-Liouville problem; A. F. C. Stevenson, University of Toronto, electromagnetic theory; G. M. Volkoff, University of British Columbia, introduction to quantum mechanics.

The Second Canadian Mathematical Congress will be held on September 5–10, 1949 at the University of British Columbia, Vancouver. Its program is expected to include general lectures by Professors Dirac, Schwartz, Szegö, Zygmund. There will be a short lecture series by R. L. Jeffery, Queen's University, Kingston, pertaining to recent researches in the theory of integration of real functions, and by Borge Jessen, University of Copenhagen, on mean motion and almost periodic functions; A. W. Conway, Institute of Advanced Studies, Dublin, will give a short series of technical lectures together with a special lecture on W. R. Hamilton. There will be shorter research papers and discussions on research and graduate work as well as secondary school mathematics.

Various receptions and excursions are being arranged in connection with both seminar and congress. Registration fees for seminar and congress are $10 and $2, respectively. Accommodations are available at the University of British Columbia at reasonable rates.

Members of the Mathematical Association of America are cordially invited. For further information write to: Canadian Mathematical Congress, Engineer-

ing Building, McGill University, Montreal, Canada. For accommodations apply to: Professor R. D. James, Department of Mathematics, University of British Columbia, Vancouver, Canada.

INSTITUTE FOR MATHEMATICS TEACHERS

The ninth annual session of the Institute for Mathematics Teachers will be held at Duke University, August 8–19, 1949. The general theme of the Institute is "Mathematics at Work" (Arithmetic through Calculus). The program includes twenty-two lecturers from industry, business, science, and teachers of recognition. In addition to the lectures there will be nine study groups under the leadership of teachers with outstanding ability. The work of the Institute centers around the Mathematics Laboratory being established at Duke University.

The registration fee is $10.00. A complete program may be secured by writing to the Director, Professor W. W. Rankin, Mathematics Department, Duke University, Durham, North Carolina.

SYMPOSIA ON NUMERICAL ANALYSIS

The National Bureau is planning two symposia on the effective utilization of automatic digital computing machinery to be held in June, 1949 at the Bureau's Institute for Numerical Analysis in Los Angeles, California. Symposium I will pertain to construction and applications of conformal maps. Tentative dates for this symposium are June 24–25. Symposium II will cover probability methods in numerical analysis. This symposium is being arranged jointly by the Institute for Numerical Analysis and the Rand Corporation, with the assistance of the Atomic Energy Commission. It is tentatively planned for June 27–29.

Those who would be interested in attending either of these symposia may obtain further information from Dr. J. H. Curtiss, Institute for Numerical Analysis, Los Angeles 24, California.

SALE OF BOOKS

A complete set of *Rendiconti del Circolo matematico di Palermo* has been offered for sale. There are sixty-two bound volumes and some supplements. Anyone interested should negotiate directly with Signora Anselmo Lidia, Via Alessio Narbone, 56, Palermo, Italy.

SUMMER COURSES

The following institutions announce advanced courses in mathematics for the summer of 1949:

Syracuse University. June 5 to August 12: mathematics of statistics; introduction to higher algebra; college plane geometry; introduction to modern mathematics; history of mathematics; methods and materials in mathematics education aids. July 25 to September 2: advanced calculus; mathematics of statistics; introduction to higher algebra II; fundamentals of analysis; higher

mathematics for engineers and scientists II; vector analysis; functions of a real variable; differential geometry.

University of Pittsburgh. June 13 to September 2. The following nine full year courses will be presented in the twelve weeks' session, the work of a first semester from June 13 to July 22 and of a second semester from July 25 to September 2; Professor Blumberg, advanced calculus; Professor Bompiani, geometry of partial differential equations; Professor Booth, materials of mathematical physics (first six weeks) and mathematical and physical bases of x-ray crystal structure determination (second six weeks); Professor Culver, differential equations, modern algebraic theories; Mr. Gettig, functions of a complex variable; Professor Laush, functions of a real variable; Professor Michalik, mathematical theory of statistics, mathematical theory of probability. In addition the following briefer courses will be given in a six weeks' session from June 27 to August 5: Professor Foraker, algebra for teachers, geometry for teachers; Professor Knipp, solid analytic geometry, theory of equations; Professor Taylor, recreational mathematics for teachers.

PERSONAL ITEMS

Professor J. R. Musselman, Western Reserve University, was appointed a representative of the Association at the meeting of the Board of Foreign Scholarships which was held at Cleveland on March 30, 1949.

Professor J. H. Taylor, George Washington University, was appointed a delegate of the Association at the inauguration of the Very Reverend Hunter Guthrie as President of Georgetown University on May 1, 1949.

Dr. Harish-Chandra, Institute for Advanced Study, and Dr. J. A. Jenkins of Harvard University have been awarded two of the 1949–50 Frank B. Jewett Fellowships by the American Telephone and Telegraph Company.

Massachusetts Institute of Technology announces the promotions of Associate Professor Norman Levinson and Eric Reissner to professorships and the appointment of G. W. Whitehead to an assistant professorship.

University of Washington announces that Professor B. L. Van der Waerden of the University of Amsterdam will be Visiting Professor for the Summer Quarter, 1949. He will give a course in algebraic geometry and one in operational methods in mathematical physics.

Professor O. W. Albert, chairman of the Department of Mathematics of the University of Redlands, has been granted a half year's leave of absence and is a guest of the University of North Carolina where he is engaged in study at the Institute of Statistics.

Mr. T. H. Bedwell of the University of South Dakota has received an appointment as Assistant Professor of Physics at Florida State University.

Mr. S. K. Bright of Vanderbilt University has been appointed to a professorship at Austin Peay State College.

Professor H. L. Dorwart, Washington and Jefferson College, has been appointed Seabury Professor of Mathematics and Natural Philosophy at Trinity

College, Hartford, Connecticut. He will succeed Professor H. M. Dadourian who will retire on June 30, 1949 after thirty years of service.

Dr. S. W. Hahn of the University of Michigan has been appointed to an assistant professorship at Wittenberg College, effective September 1, 1949.

Assistant Professor J. G. Herriot of Stanford University has been promoted to an associate professorship.

Mr. E. W. Marchand, University of Rochester, has accepted a position as physicist with the Eastman Kodak Company of Rochester.

Professor L. F. Ollmann, chairman of the Department of Mathematics of Hofstra College, has been appointed to direct Summer Session I, 1949.

Mr. W. B. Stovall, Jr. of the University of Florida has accepted a position as statistician with the Bureau of Vital Statistics, State Board of Health, Florida.

Associate Professor F. J. Taylor of the College of St. Thomas has been promoted to a professorship.

Mr. J. S. Thompson, vice chairman of the Board of McGraw-Hill Book Company, retired on April 1, 1949.

Miss Lona L. Turner has been appointed to an instructorship at the Chicago Undergraduate Division of the University of Illinois.

Mr. W. H. Coulter of Decatur, Illinois, died on February 16, 1949.

Assistant Professor J. H. Cross of Texas Technological College died on December 8, 1948.

President-emeritus R. L. Durham of Southern Seminary and Junior College died January 1, 1949.

Dr. T. M. Focke, dean emeritus of Case Institute of Technology, died on March 2, 1949 at the age of seventy-eight years.

Mr. G. H. Selleck, retired head of the Department of Mathematics, Phillips Exeter Academy, died on March 26, 1949.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following one hundred six persons have been elected to membership by the Board of Governors on applications duly certified:

M. I. Aissen, B.S. (CCNY) Research Assistant, Stanford University, Calif.

R. E. Anderson, B.E. (Northern Illinois S.T.C.) Instructor, Northern Illinois S.T.C., DeKalb, Ill.

D. F. Atkins, M.S. (Illinois) Instructor, University of Kentucky, Lexington, Ky.

R. F. Barnard, M.S. (Washington State) Instructor, North Central High School, Spokane, Wash.

D. C. Barton, M.A. (Rochester) Instructor, Roberts Junior College, Rochester, N. Y.

Agnes J. Beckstrom, M.A. (Northwestern) Head of Department, State Teachers Col-

lege, Minot, N. D.

C. B. BELL, JR., M.S. (Notre Dame) Student, University of Notre Dame, Ind.

RUTH E. BIGGERS, M.A. (Duke) Instructor, Emory and Henry College, Emory, Va.

GERTRUDE BLANCH, Ph.D. (Cornell) Mathematician, National Bureau of Standards, Los Angeles, Calif.

H. D. BLOCK, M.S. (Iowa State) Instructor, Iowa State College, Ames, Iowa

JOSEPH BLUM, M.A. (George Washington) Research Analyst, Department of the Army, Washington, D. C.

I. S. BOAK, M.A. (N. Y. State College for Teachers) Instructor, New York State Agricultural and Technical Institute, Canton, N. Y.

E. W. BOLD, B.S. (Dayton) Graduate Fellow, St. Louis University, Mo.

J. L. BOTSFORD, Ph.D. (California Institute of Technology) Asst. Professor, San Jose State College, Calif.

S. K. BRIGHT, M.A. (Texas) Professor, Austin Peay State College, Clarksville, Tenn.

J. E. BROWN, A.B. (Georgia) Cashier, Cloister Hotel, Sea Island, Ga.

A. L. BUCHMAN, A.M. (Columbia) Teacher, Hutchinson Central High School, Buffalo, N. Y.

R. B. BUSCHMAN, Student, Reed College, Portland, Ore.

J. L. CARLSON, A.M. (Stanford) City College of San Francisco, Calif.

JEREMIAH CERTAINE, Ph.D. (Harvard) Asst. Professor, Howard University, Washington, D. C.

S. C. COBB, S.M. (Arizona) Teacher, Phillips Academy, Andover, Mass.

R. M. COHN, Ph.D. (Columbia) Instructor, Rutgers University, New Brunswick, N. J.

E. M. COOK, M.A. (Boston) Asst. Professor, Northeastern University, Boston, Mass.

MRS. LAKE C. COOPER, A.M. (Morehead State Coll.) Instructor, University of Kentucky, Lexington, Ky.

W. R. COWELL, B.S. (Kansas State) Graduate Assistant, Kansas State College, Manhattan, Kan.

H. J. DARK, Ph.D. (Peabody) Chairman of Department, David Lipscomb College, Nashville, Tenn.

RADHA C. DAS, M.S. (Cornell) Graduate Student, Cornell University, Ithaca, N. Y.

MAMIE M. DAVIS, M.A. (Louisiana State) Asst. Professor, Southeastern Louisiana College, Hammond, La.

R. F. DENISTON, A.B. (Washington Univ.) Instructor, Iowa State College, Ames, Iowa

LINDA G. DODD, B.L. (California) Teacher, Mt. Diablo Union High School, Calif.

N. A. DRAIM, M.Sc. (M.I.T.) Captain, U.S. Navy, Bethesda, Md.

K. E. DUBBERT, M.A. (Columbia) Instructor, Rochester Junior College, Minn.

J. B. ECKSTEIN, B.S. (Notre Dame) Graduate Student, University of Detroit, Mich.

RUTH B. EDDY, A.M. (Brown) Instructor, University of Connecticut, Waterbury Branch, Conn.

F. P. EGAN, B.A. (Manhattan) Instructor, Niagara University, N. Y.

R. L. EISENMAN, A.B. (Holy Cross) Instructor, Fairfield University, Conn.

R. E. EKSTROM, B.S. (Missouri) Assistant Instructor, University of Missouri, Columbia, Mo.

R. I. FIELDS, M.A. (Arizona) Asst. Professor, University of Louisville, Ky.

J. E. FREUND, M.A. (California) Asso. Professor, Alfred University, N. Y.

KATHERINE C. GARLAND, B.A. (Denver) Graduate Fellow, University of Denver, Colo.

R. W. GIBSON, Ph.D. (Illinois) Professor, William Penn College, Oskaloosa, Iowa

L. B. GINTHER, B.Engg. (Toledo) Junior Glass Technologist, Libbey-Owens-Ford, Toledo, Ohio

R. H. GLASS, M.A. (Southern College) Instructor, University of Colorado, Boulder, Colo.

W. E. GLASS, B.A. (Texas) Graduate Student, University of Texas, Austin, Tex.

N. R. GOODMAN, Student, Illinois Institute of Technology, Chicago, Ill.

H. W. GOULD, Student, University of Virginia, Charlottesville, Va.

G. R. GRAINGER, B.A. (California) Graduate Student, University of Notre Dame, Ind.

W. H. GREGORY, B.S. (Illinois) Assistant, University of Illinois, Urbana, Ill.

S. W. HAHN, Ph.D. (Duke) Instructor, University of Michigan, Ann Arbor, Mich.

H. H. HANNON, M.A. (Michigan) Asst. Professor, Western Michigan College of Education, Kalamazoo, Mich.

F. J. HARDY, B.A. (Duquesne) Graduate Assistant, Duquesne University, Pittsburgh, Pa.

CLAIRE A. HARRISON, A. B. (Northeastern) Instructor, Connors State Agricultural College, Warner, Okla.

C. A. HAYES, JR., Ph.D. (California) Asst. Professor, University of California, Davis, Calif.

J. G. HERRIOT, Ph.D. (Brown) Asst. Professor, Stanford University, Calif.

P. S. HERWITZ, B.A. (Cincinnati) Graduate Student, University of Cincinnati, Ohio.

MRS. SPENCER, E. HICKMAN, A.M. (Columbia) Head of Department, Buffalo Seminary, Buffalo, N. Y.

M. W. HNATH, B.E. (Duquesne) Graduate Assistant, Duquesne University, Pittsburgh, Pa.

L. R. HOLLAND, M.S. (Oklahoma A & M) Professor, Eastern Oklahoma A & M, Wilburton, Okla.

C. V. HOLMES, M.A. (Mississippi) Instructor, Murray State College, Ky.

MARY T. HUGGINS, A.B. (Vassar) Instructor, Stanford University, Calif.

I. L. JONES, M.A. (Missouri) Professor, Armstrong College, Berkeley, Calif.

D. B. JORDAN, Student, Hofstra College, Hempstead, N. Y.

O. C. JUELICH, Student, Hofstra College, Hempstead, N. Y.

HYMAN KAMEL, M.S. (New York) Instructor, University of Pennsylvania, Philadelphia, Pa.

A. E. KINNEY, M.A. (Columbia) Instructor, New York State Maritime Academy, Fort Schuyler, N. Y.

HELEN F. KRIEGSMAN, M.S. (Kansas S.T.C.) Instructor, Kansas S.T.C., Pittsburgh, Kan.

F. O. LANE, JR., B.S. (New Mexico) Graduate Student, University of New Mexico, Mountainair, N. M.

JACK LEVINE, Ph.D. (Princeton) Professor, North Carolina State College, Raleigh, N. C.

FRED MARER, M.A. (Southern California) Instructor, Los Angeles City College, Calif.

D. M. MARRIAN, M.A. (Columbia) Master, Gilman Country School, Baltimore, Md.

B. C. MEYER, M.S. (Brown) Graduate Student, Stanford University, Calif.

M. L. MITCHELL, A.B. (Hofstra) 200 W. Merrick, Rd., Baldwin, Long Island, N. Y.

ELSIE C. MUELLER, M.A. (Michigan) Instructor, University of Illinois, Galesburg, Ill.

R. R. MURPHY, M.A. (Oklahoma) Professor, Panhandle A & M College, Goodwell, Okla.

J. G. MURRAY, A.B. (Holy Cross) Graduate Student, Catholic University, Washington, D. C.

E. J. MUSCH, B.S. (Michigan) Graduate Student, Kent State University, Kent, Ohio

N. G. MYERS, JR. Student, Gannon College, Erie, Pa.

C. R. NEWELL, B.A. (Toronto) Instructor, Niagara University, N. Y.

CELIA NOOGER, M.A. (NYU) Teacher, New York City Board of Education, N. Y.

M. W. OLIPHANT, M.A. (Johns Hopkins) Instructor, Georgetown University, Washington, D. C.

GLORIA OLIVE, M.A. (Wisconsin) Instructor, Idaho State College, Pocatello, Idaho

S. V. PARTER, Student, Illinois Institute of Technology, Chicago, Ill.

BARTH POLLAK, Student, Illinois Institute of Technology, Chicago, Ill.

J. W. PONDS, M.S. (Howard) Asst. Professor, West Virginia State College, Institute, W. Va.

M. R. REEKS, M.S. (Stevens) Asst. Professor, Stevens Institute of Technology, Hoboken, N. J.

ALTA H. SAMUELS, M.A. (L.S.U.) Asst. Professor, University of Mississippi, University, Miss.

S. W. SAUNDERS, Ph.D. (Pittsburgh) Professor, Morgan State College, Baltimore, Md.

L. R. SCHLAUCH, B.A. (Penn State) Graduate Student, Pennsylvania State College, State College, Pa.

STEWART SCHLESINGER, Student, Illinois Institute of Technology, Chicago, Ill.

DOROTHY V. SCHRADER, M.A. (Boston) Instructor, College of Saint Teresa, Winona, Minn.

B. B. SHARPE, M.A. (Buffalo) Instructor, Millard Fillmore College, University of Buffalo, N. Y.

O. D. SMITH, B.A. (Willamette) Graduate Assitant, Oregon State College, Corvallis, Ore.

VIVIAN SPURGEON, M.A. (Peabody) Asst. Professor, Ouachita College, Arkadelphia, Ark.

K. H. STAHL, Ph.D. (Pittsburgh) Asso. Professor, University of Colorado, Boulder, Colo.

MRS. ESTHER R. STEINBERG, M.A. (Minnesota) Instructor, Macalester College, St. Paul, Minn.

MARY VIRGINIA SUNSERI, M.A. (Stanford) Acting Asst. Professor, Stanford University, Calif.

T. E. SYDNOR, M.A. (Whittier) Instructor, Pasadena City College, Calif.

H. W. SYER, M.A. (Harvard) Asst. Professor, Boston University, Mass.

J. V. TALACKO, D.Sc. (Charles Univ., Prague) Asst. Professor, Marquette University, Milwaukee, Wis.

E. P. TOVANI, E.E. (Colorado) Asst. Professor, University of Colorado, Boulder, Colo.

BILLIE B. TOWNSEND, M.S. (North Texas) Asst. Professor, L.S.U., Baton Rouge, La.

S. I. VROOMAN, B.S. (Columbia) Instructor, R. P. I., Troy, N. Y.

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CALENDAR OF FUTURE MEETINGS

Joint Meeting with American Society for Engineering Education, Troy, New York, June 20–21, 1949.

Thirty-first Summer Meeting, Boulder, Colorado, August 29–30, 1949.

Thirty-third Annual Meeting, New York City, December 30, 1949.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN

ILLINOIS

INDIANA

IOWA

KANSAS

KENTUCKY

LOUISIANA-MISSISSIPPI, Centenary College, Shreveport, Louisiana, Spring, 1950

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK, Spring 1950

MICHIGAN

MINNESOTA, University of North Dakota, Grand Forks, October 15, 1949

MISSOURI, Spring 1950

NEBRASKA

NORTHERN CALIFORNIA, Berkeley, January 28, 1950

OHIO, April, 1950

OKLAHOMA, Oklahoma City, October 14, 1949

PACIFIC NORTHWEST, University of Washington, Spring 1950

PHILADELPHIA, Haverford College, November 26, 1949

ROCKY MOUNTAIN

SOUTHEASTERN, University of Florida, Gainesville, March, 1950

SOUTHERN CALIFORNIA, Immaculate Heart College, Hollywood, March 11, 1950

SOUTHWESTERN

TEXAS, Abilene, Spring 1950

UPPER NEW YORK STATE, Syracuse University, Spring 1950

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CONCERNING TWO CLASSES OF REMARKABLE PERFECT SQUARE PAIRS*

VICTOR THÉBAULT, Tennie, Sarthe, France

1. Introduction. Open a table of integer squares. If you happen to have one which includes several systems of numeration, you will find a great variety of numbers of alluring forms and properties [1]. Thus, with base $B=7$, $4444=(55)^2$, $2424=(42)^2$, $664664004004=(665665)^2$; and with $B=10$, $7744=(88)^2$, $165165836836=(406406)^2$. We do not know any exercises more instructive or more pleasing than to seek those perfect squares in which the pattern of the digits naturally arouses the curiosity of the mind.

In investigating such squares one encounters at every step opportunities for solving Diaphantine equations and for applying many of the beautiful theorems of arithmetic. To discover new examples† of notable four-digit squares the present paper follows two distinct lines of investigation.

2. A system of numeration in which there exist simultaneously squares $abab=x^2$ **and** $baba=y^2$. There is an infinity of systems of numeration in which there exist four-digit squares of the form [2]

$$M = x^2 = abab = (ab)(B^2+1). \tag{1}$$

Since $(ab)<x<B^2+1$, evidently B^2+1 must contain a square factor. Hence we put

$$B^2+1 = nm^2, \tag{2}$$

$$ab = aB + b = np^2, \tag{3}$$

with $m>p$. Moreover, since m and n divide B^2+1, each is itself a sum of two squares [3]. When the integers m, n, p exist, M is the square of the product $mnp=x$. In the particular case $n=2$, the relation (2) reduces to Fermat's equation

$$B^2 - 2m^2 = -1, \tag{4}$$

and we have $M=(2pm)^2$. The values of B are

$$B = 7, 41, 239, 1393, 8119, \cdots, \tag{4'}$$

for which the corresponding values of m are

$$m = 5, 29, 169, 985, 5741, \cdots. \tag{4''}$$

It will be seen that the values of B, m are the N_{2p+1}, D_{2p+1} as given below in Section 3.

In a system of numeration in which there exist simultaneously four-digit

* Translated from the French by R. E. Luce.

† We have previously pointed out a good many of these, Les Récréations Mathématiques, Supplément a Mathesis, 1943, pp. 24–88.

squares x^2 and y^2, where

$$x^2 = abab = (ab)(B^2 + 1), \qquad y^2 = baba = (ba)(B^2 + 1),$$

we must have

$$(5) \qquad B^2 + 1 = nm^2, \ ab = np^2, \ ba = nq^2, \ x^2 = (mnp)^2, \ y = (mnq)^2.$$

Since ab, $ba < B^2$, the process of obtaining squares having these forms will be a matter, first, of determining systems of numeration in which there shall be, at the same time, two-digit numbers of form np^2 and nq^2 such that the sum

$$(6) \qquad ab + ba = n(p^2 + q^2) = (a + b)(B + 1).$$

Thus, when n is a sum of two squares, the same is true of $n(p^2+q^2)$ and hence each of the numbers $a+b$ and $B+1$ can be decomposed into a product of prime factors, each of which is, in its turn, a sum of two squares.

Consider further the particular case noted above, with $n=2=1^2+1^2$. In the sequence (4′) of values of B, it is necessary to exclude $B=41, 239, 8119, \cdots$ for which $B+1$ is divisible by $3, 7, \cdots$, for primes of the form $4k-1$ are not sums of two squares. But the numbers $B=7, 1393, \cdots$, for which $7+1=2^2+2^2$, $1393+1=13^2+35^2, \cdots$, fulfill the two conditions (4) and (6) for B^2+1 and $B+1$. It will suffice therefore to discover whether there exist in these systems of numeration two-digit numbers ab, ba, each of which, according to (5) is the double of a square. For $B=7$, we find

$$ab = 2\cdot 4^2 = 44 = ba, \qquad x^2 = 4444 = (55)^2 = y^2.$$

For $B=1393$, there are several examples:

$$\text{(i)} \qquad x^2 = 2\ \overline{912}\ 2\ \overline{912} = (\overline{60}\ \overline{1130})^2, \qquad y^2 = \overline{912}\ 2\ \overline{912}\ 2 = (\overline{1127}\ \overline{179})^2.$$

$$\text{(ii)} \qquad x^2 = 2\ \overline{46}\ 4\ \overline{46} = (\overline{74}\ \overline{1328})^2, \qquad y^2 = \overline{46}\ 4\ \overline{46}\ 4 = (\overline{253}\ \overline{201})^2.$$

The numbers $(2x)^2$, $(3x)^2$, $(4x)^2$ are equally suitable.

$$\text{(iii)} \qquad x^2 = 4\ \overline{478}\ 4\ \overline{478} = (\overline{77}\ \overline{1089})^2, \qquad y^2 = \overline{478}\ 4\ \overline{478}\ 4 = (\overline{816}\ 2)^2.$$

Doubtless other examples exist which may be discovered by extended study of the further values of B in (4′) and from values of B found by using $n=5, 10, 13, 17, \cdots$ with the resulting equation analogous to (4).

3. Perfect squares $aabb=(cc)^2$ and $bbaa=(dd)^2$ in a single system of numeration. For squares having the prescribed forms it is easily shown [4] that the base B and the digits a, b, c, d necessarily satisfy the relations

$$(1) \qquad a + b = B + 1,$$

$$(2) \qquad c^2 = a(B - 1) + 1, \qquad d^2 = b(B - 1) + 1,$$

$$(3) \qquad c^2 + d^2 = B^2 + 1,$$

$$(4) \qquad c^2 - d^2 = (a - b)(B - 1).$$

Moreover if B is odd, c and d are necessarily odd, and a and b are of like parity, so that one can always put $a-b=2s$. (We take $s\geqq 0$, that is $a\geqq b$. If $a<b$ we merely interchange a with b, and c with d.) By virtue of the definition of s and equation (1), the relations (2) are equivalent to

$$(B+s)^2-2c^2=(s+1)^2-2, \tag{5}$$

$$(B-s)^2-2d^2=(s-1)^2-2. \tag{6}$$

A general formula yielding solutions may be obtained* from the familiar solution of Fermat's equations

$$x^2-2y^2=\pm 1$$

in which are employed successive approximants of the continued fraction representation of $\sqrt{2}$. If N_k/D_k is the kth approximant, we have

k	1	2	3	4	5	6	7	8	9	$10\cdots$
N_k	1	3	7	17	41	99	239	577	1393	$3363\cdots$
D_k	1	2	5	12	29	70	169	408	985	$2378\cdots$

For this development the following relations are known:

$$N_{k+1}=N_k+2D_k, \qquad D_{k+1}=N_k+D_k,$$

$$2N_k=N_{k+1}-N_{k-1}, \qquad 2D_k=D_{k+1}-D_{k-1},$$

$$N_{2p}^2-2D_{2p}^2=1, \qquad N_{2p-1}^2-2D_{2p-1}^2=-1,$$

$$N_{2p}\cdot N_{2p-1}-2D_{2p}\cdot D_{2p-1}=-1, \qquad N_{2p+1}\cdot N_{2p}-2D_{2p+1}\cdot D_{2p}=1.$$

From these equations, the following identities in s are easy to obtain:

$$(N_{2p}s+N_{2p-1})^2-2(D_{2p}s+D_{2p-1})^2=(s-1)^2-2,$$

$$(N_{2p}s-N_{2p-1})^2-2(D_{2p}s-D_{2p-1})^2=(s+1)^2-2,$$

$$(N_{2p}s+N_{2p+1})^2-2(D_{2p}s+D_{2p+1})^2=(s+1)^2-2,$$

$$(N_{2p}s-N_{2p+1})^2-2(D_{2p}s-D_{2p+1})^2=(s-1)^2-2.$$

These results suggest solutions of (5) and (6) obtained by putting

$$B+s=N_{2p}s-N_{2p-1}, \qquad c=D_{2p}s-D_{2p-1};$$

$$B-s=N_{2p}s-N_{2p+1}, \qquad d=D_{2p}s-D_{2p+1};$$

$$B'+s=N_{2p'}s+N_{2p'+1}, \qquad c'=D_{2p'}s+D_{2p'+1};$$

$$B'-s=N_{2p'}s+N_{2p'-1}, \qquad d'=D_{2p'}s+D_{2p'-1}.$$

We may now take $p=p'$ and equate the values of B and B' corresponding to c and d, c' and d', c' and d, thus obtaining

$$B_1=(N_{2p}-1)s-N_{2p-1}=(N_{2p}+1)s-N_{2p+1},$$

* According to notes communicated to the author by M. A. Barriol, Paris.

$$B_2 = (N_{2p} - 1)s + N_{2p+1} = (N_{2p} + 1)s + N_{2p-1},$$
$$B_3 = (N_{2p} - 1)s + N_{2p+1} = (N_{2p} + 1)s - N_{2p+1}.$$

(The fourth combination, c and d', gives s a negative value.) For B_1 and B_2 we find $2s = N_{2p+1} - N_{2p-} = 2N_{2p}$; and for B_3, $s = N_{2p+1}$. We have two solutions therefore for each $s = N_{2p}$ and one for each $s = N_{2p+1}$.

We have thus three infinite sequences of solutions:

(i) $\quad s = N_{2p}, \qquad B = (N_{2p} - 1)N_{2p} - N_{2p-1},$

$\quad c = D_{2p}N_{2p} - D_{2p-1}, \qquad d = D_{2p}N_{2p} - D_{2p+1};$

(ii) $\quad s = N_{2p}, \qquad B = (N_{2p} + 1)N_{2p} + N_{2p-1},$

$\quad c = D_{2p}N_{2p} + D_{2p+1}, \qquad d = D_{2p}N_{2p} + D_{2p-1};$

(iii) $\quad s = N_{2p+1}, \qquad B = N_{2p}N_{2p+1},$

$\quad c = D_{2p}N_{2p+1} + D_{2p+1}, \qquad d = D_{2p}N_{2p+1} - D_{2p+1}.$

The values of a and b are found from $a - b = 2s$, $a + b = B + 1$. The fo___ . . . ___ table indicates the simplest of these solutions:

s	B	c	d	a	b
3	13	11	7	10	4
7	21	19	9	18	4
17	265	199	175	150	116
17	313	233	209	174	140
41	697	521	463	390	308

Complete solutions of (5) and (6) present genuine difficulties, but there is another approach which leads easily to additional sequences of squares of the desired forms.

The general solution in integers of (3) is known [5] to be

$$B = mq + np, \qquad c = mp + nq, \qquad d = |mq - np|, \tag{7}$$

subject to the condition

$$mp - nq = 1. \tag{8}$$

Equations (2) may now be solved in terms of m, n, p, q, giving

$$a = \frac{4mnpq}{n(p + q) - m(p - q)}, \tag{9}$$

$$b = \frac{(m - n)(m + n)(q - p)(q + p)}{n(p + q) - m(p - q)}. \tag{10}$$

The necessary and sufficient conditions that a and b be integers is therefore

simply that $n(p+q)-m(p-q)$ be a divisor of $4mnpq$. If in addition

(11) $$0 < a < B, \quad 0 < b < B, \quad 0 < c < B, \quad 0 < d < B,$$

then the digits a, b, c, d determine perfect squares of the type under discussion, the base being B.

It is possible to assume additional relations between m, n, p, q in such a way that the conditions required for a in (9) to be integral are easily fulfilled. For example if $n=p-q$, then (9) reduces to $a=4mpq/(p+q-m)$. If, further, $m=p-2q$, this becomes simply $a=4mp/3$. With these values of n and m, condition (7) becomes $p^2-3pq+q^2=1$ or

$$(2p - 3q)^2 - 5q^2 = 4.$$

Since this last equation is known to have infinitely many integral solutions one obtains an unending sequence of pairs of perfect squares of the desired form. The values of B are 7, 313, 14686, $\cdots$.

Other conditions on m and n lead similarly to other sequences of satisfactory squares. The following are easy to determine:

$$n = p - q,\ m = p - 3q;\quad B = 13,\ 177,\ 2461,\ \cdots .$$
$$m = p + q,\ n = 2q - 2p;\ B = 526,\ \cdots .$$
$$m = p + q,\ n = p + 2q;\quad B = 31,\ 35491,\ \cdots .$$
$$m = p + q,\ n = p + 3q;\quad B = 13,\ 177,\ 2461,\ \cdots .$$
$$m = p + q,\ n = p + 5q;\quad B = 313,\ \cdots .$$
$$m = p + q,\ n = 5p - q;\quad B = 305,\ 4261,\ \cdots .$$
$$m = p + q,\ n = 4p - q;\quad B = 265,\ 12541,\ \cdots .$$

The following particular cases should be noted. (i) With $a=b$ so that

$$aaaa = bbbb = (cc)^2 = (dd)^2,$$

equation (6) becomes $B^2-2c=-1$, and the solutions are immediate:

$$B = N_{2p+1}, \qquad c = d = D_{2p+1}, \qquad a = b = (B + 1)/2.$$

If we put (ii) $b=4$, as we have already shown [2], we have, for any base $B=n^2+n+1$,

$$\overline{B-3}\ \overline{B-3}\ 4\ 4 = (\overline{B-2}\ \overline{B-2})^2,$$
$$4\ 4\ \overline{B-3}\ \overline{B-3} = (\overline{2n+1}\ \overline{2n+1})^2.$$

Selected examples are shown in the following table. The special cases just mentioned are excluded. As noted earlier, interchange of a and b, and of c and d does not give an essentially different solution. Although it does introduce a new set of values of m, n, p, q, such a second representation is not included.

m	n	p	q	B	a	b	c	d
4	1	5	19	81	19	63	39	71
11	19	7	4	177	133	45	153	89
19	5	4	15	305	75	231	151	265
5	13	21	8	313	140	174	209	233
25	14	9	16	526	384	143	449	274
15	4	19	71	1141	284	858	569	989
71	19	15	56	4261	1064	3198	2129	3691
34	89	144	55	14686	6528	8159	9791	10946

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1. Thébault, Les Récréations Mathématiques (Parme les nombres curieux), Supplément a Mathesis, 1943, pp. 88–120.
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3. Kraitchik, Theorie des Nombres, 1922, p. 98.
4. V. Thébault, Mathesis, 1936, Supplément, p. 5. See also this MONTHLY, problem 4237 [1948, 370].
5. L' Intermédiare des Mathématiciens, 1910, pp. 96, 118, 204.
6. Thébault, Ann. de la Soc. Scient. de Bruxelles, t. lxii, 1948, pp. 101–105.

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

L. E. BUSH,* College of St. Thomas

The following results of the ninth annual William Lowell Putnam Mathematical Competition held March 26, 1949, have been determined in accordance with the constitution of the Competition. This Competition is supported by the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, and is held under the auspices of The Mathematical Association of America.

The first prize, four hundred dollars, is awarded to the Department of Mathematics of Harvard University, Cambridge, Massachusetts. The members of the team were W. F. Stinespring, D. L. Yarmush, Ariel Zemach; to each of these a prize of forty dollars is awarded.

The second prize, three hundred dollars, is awarded to the Department of Mathematics of the University of Toronto, Toronto, Ontario, Canada. The members of the team were J. M. Kennedy, J. E. S. Moyse, J. B. Patterson; to each of these a prize of thirty dollars is awarded.

* Director of the Competition.

The third prize, two hundred dollars, is awarded to the Department of Mathematics of Carnegie Institute of Technology, Pittsburgh, Pennsylvania. The members of the team were G. L. Baldwin, R. E. Cutkosky, R. M. Drisko; to each of these a prize of twenty dollars is awarded.

The fourth prize, one hundred dollars, is awarded to the Department of Mathematics of the College of the City of New York, New York, New York. The members of the team were Herman Hanisch, D. J. Newman, H. R. Shapiro; to each of these a prize of ten dollars is awarded.

The five persons ranking highest in the examination, named in alphabetical order, were J. W. Milnor, Princeton University; D. J. Newman, College of the City of New York; W. F. Stinespring, Harvard University; D. L. Yarmush, Harvard University; Ariel Zemach, Harvard University. Each of these will receive a prize of fifty dollars.

The four succeeding persons ranking highest in the examination, named in alphabetical order, were: J. M. Kennedy, University of Toronto; J. D. Lordan, Massachusetts Institute of Technology; Gerhard Raynar, Harvard University; W. F. Reynolds, Holy Cross College. To each of these a prize of twenty dollars is awarded.

There was a tie for tenth place between R. E. Cutkosky, Carnegie Institute of Technology, and Louis Howard, Swarthmore College. A prize of twenty dollars will be divided between these contestants.

The following teams, named in alphabetical order, won honorable mention: California Institute of Technology, Pasadena, California, the members of the team being H. A. Forrester, H. G. Hegnemann, J. A. Hummel; Polytechnic Institute of Brooklyn, Brooklyn, New York, the members of the team being F. Blecher, George Giovmousis, S. Lerner; Princeton University, Princeton, New Jersey, the members of the team being C. H. Bernstein, P. H. Lord, J. W. Milnor; Swarthmore College, Swarthmore, Pennsylvania, the members of the team being Louis Howard, Paul Mangelsdorf, and Robert Norman.

Five individuals were given honorable mention. The names are listed in alphabetical order. R. M. Drisko, Carnegie Institute of Technology; E. O. Elliott, University of California at Berkeley; M. L. Minsky, Harvard University; J. E. S. Moyse, University of Toronto; H. R. Shapiro, College of the City of New York.

The following is a list of all colleges and universities which entered teams in the Competition. The list, in alphabetical order, is: American International College, Brooklyn College, California Institute of Technology, Carnegie Institute of Technology, College of the City of New York, College of St. Thomas, Columbia University, George Pepperdine College, Harvard University, Holy Cross College, Macalester College, McGill University, McMaster University, Northwest Missouri State Teachers College, Oklahoma A. & M. College, Polytechnic Institute of Brooklyn, Princeton University, Queen's University, Reed College, Saint Joseph's College (Philadelphia), Stanford University, Swarthmore College, United States Naval Academy, University of Alberta, University

of British Columbia, University of Buffalo, University of California at Berkeley, University of California at Los Angeles, University of New Hampshire, University of Oregon, University of Toronto, Ursinus College, and Washburn Municipal University.

The following additional colleges and universities entered individual contestants only: Carleton College, Duquesne University, Haverford College, Hofstra College, Massachusetts Institute of Technology, Occidental College, Purdue University, Radford College, St. Joseph College (West Hartford), St. Xavier College, University of Minnesota, University of Omaha, University of Pittsburgh, University of Saskatchewan, University of Wyoming, and Wiley College.

A total of 155 undergraduates representing 49 institutions took part in the Competition.

Participants in the Competition were given the following lists of problems.

PART I. THREE HOURS

(Answer the questions in any order and by any method. Show all of your work in logical sequence and indicate your answers clearly. No tables or other books may be used. Use the right hand pages of your examination booklet for your solutions, use the left hand pages for scratch work. Cross out any work which you do not wish to have considered. Partial credit may be given on a question, even when the solution is not completed.)

1. Answer *either* (a) or (b):

   (a) Three straight lines pass through the three points $(0, -a, a)$, $(a, 0, -a)$, and $(-a, a, 0)$, parallel to the x-axis, y-axis, and z-axis, respectively; $a > 0$. A variable straight line moves so that it has one point in common with each of the three given straight lines. Find the equation of the surface described by the variable line.

   (b) Which planes cut the surface $xy+xz+yz=0$ in (*i*) circles, (*ii*) parabolas?

2. We consider three vectors drawn from the same initial point O, of lengths a, b, and c, respectively. Let E be the parallelepiped with vertex O of which the given vectors are the edges and H the parallelepiped with vertex O of which these vectors are the altitudes. Show that the product of the volumes of E and H equals $(abc)^2$, and generalize the theorem, with proof, to n dimensions.

3. Assume that the complex numbers $a_1, a_2, \cdots, a_n, \cdots$ are all different from zero, and that $|a_r - a_s| > 1$ for $r \neq s$. Show that the series

$$\sum_{n=1} \frac{1}{a_n^3}$$

converges.

4. Given that P is a point inside a tetrahedron with vertices at A, B, C, and D, such that the sum of the distances $PA+PB+PC+PD$ is a minimum. Show

that the two angles $\angle APB$ and $\angle CPD$ are equal, and are bisected by the same straight line. What other pairs of angles must be equal?

5. How many roots of the equation $z^6+6z+10=0$ lie in each quadrant of the complex plane?

6. Prove that for every real or complex x

$$\prod_{k=1}^{\infty} \frac{1+2\cos\frac{2x}{3^k}}{3} = \frac{\sin x}{x}.$$

PART II. THREE HOURS

(Answer the questions in any order and by any method. Show all of your work in logical sequence and indicate your answers clearly. No tables or other books may be used. Use the right hand pages of your examination booklet for your solutions, use the left hand pages for scratch work. Cross out any work which you do not wish to have considered. Partial credit may be given on a question, even when the solution is not completed.)

1. Each rational number p/q (p, q relatively prime positive integers) of the open interval (0, 1) is covered by a closed interval of length $1/2q^2$, whose center is at p/q. Prove that $\sqrt{2}/2$ is not covered by any of the above closed intervals.

2. Answer either (a) or (b):
(a) Prove that

$$\sum_{n=2}^{\infty} \frac{\cos(\log\log n)}{\log n}$$

diverges.
(b) Assume that $p>0$, $a>0$, and $ac-b^2>0$, and show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dxdy}{(p+ax^2+2bxy+cy^2)^2} = \pi p^{-1}(ac-b^2)^{-1/2}.$$

3. Let K be a closed plane curve such that the distance between any two points of K is always less than 1. Show that K lies inside a circle of radius $1/\sqrt{3}$.

4. Show that the coefficients $a_1, a_2, a_3, \cdots$ in the expansion $\frac{1}{4}[1+x-(1-6x+x^2)^{1/2}]=a_1x+a_2x^2+a_3x^3+\cdots$ are positive integers.

5. Let $a_1, a_2, \cdots, a_n, \cdots$ be an arbitrary sequence of positive numbers. Show that

$$\limsup_{n\to\infty}\left(\frac{a_1+a_{n+1}}{a_n}\right)^n \geqq e.$$

6. Let C be a closed convex curve with a continuously turning tangent and let O be a point inside C. With each point P on C we associate the point $T(P)$ on

C which is defined as follows: Draw the tangent to C at P and from O drop the perpendicular to that tangent. $T(P)$ is then the point at which this perpendicular intersects the curve C.

Starting now with a point P_0 on C, we define points P_n by the formula $P_n = T(P_{n-1})$, $n \geqq 1$. Prove that the points P_n approach a limit, and characterize those points which can be limits of sequences P_n. (You may consider the facts that T is a continuous transformation and that a convex curve lies on one side of each of its tangents as not requiring proofs.)

ON GENERALIZED CAUCHY FUNCTIONAL EQUATIONS

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1. Introduction. In a previous paper* use was made of Cauchy's functional equation $f(x+y) = f(x)f(y)$ to obtain a generalization of trigonometry. In the present paper much more general functional equations are studied, and it is shown that the solutions of one of these equations can be made the basis for the definition of certain functions which have many properties analogous to those of the ordinary trigonometric functions.

2. The basic functional equations. We consider the functional equation

$$(A) \qquad F(x + y + nxy) = G(x)H(y), \qquad (n > 0)$$

and set ourselves the problem of finding the most general real, continuous not identically vanishing functions $F(x)$, $G(x)$ and $H(x)$ which are defined for all x greater than $-1/n$ and satisfy this equation (A). A solution which consists of functions which do not vanish identically will be called a non-trivial solution. We first state and prove

THEOREM I. *Every non-trivial solution of* (A) *is of the form* $G(x) = G(0)f_n(x)$, $H(x) = H(0)f_n(x)$, $F(x) = G(0)H(0)f_n(x)$ *where* $f_n(x)$ *is any non-trivial solution*† *of*

$$(B) \qquad f_n(x + y + nxy) = f_n(x)f_n(y), \qquad (n > 0)$$

defined for all x *greater than* $-1/n$, *where* $G(0) \neq 0$, $H(0) \neq 0$.

Proof. If in (A) we let first $y=0$, and then $x=0$, we obtain the equations

$$(2.1) \qquad F(x) = G(x)H(0) \qquad F(y) = G(0)H(y).$$

* Thielman, H. P. A generalization of trigonometry. Nat. Math. Mag., Vol. XI (1937) pp. 1–3.

† Cauchy treated this equation with $n=0$, and related equations. Cours d'Analyse (1821), Chapter 5. It is because of his study of these equations that his name is attached to them.

If $H(0)=0$, or $G(0)=0$, then $F(x)\equiv 0$. Since we are interested in the non-trivial solutions, $H(0)$ and $G(0)$ must be assumed to be different from zero. From (2.1) it follows that $F(0)=G(0)H(0)$, and $G(x)=F(x)/H(0)$, $H(x)=F(x)/G(0)$. Substituting these values for $G(x)$ and $H(x)$ in (A) we get

$$F(x+y+nxy) = F(x)F(y)/G(0)H(0),$$

which can be written as

$$\frac{F(x+y+nxy)}{G(0)H(0)} = \frac{F(x)}{G(0)H(0)}\cdot\frac{F(y)}{G(0)H(0)}. \tag{2.2}$$

If we let $f_n(x)=F(x)/G(0)H(0)$, we see from (2.2) that $f_n(x)$ satisfies equation (B), and since $G(x)=G(0)f_n(x)$, $H(x)=H(0)f_n(x)$, $F(x)=G(0)H(0)f_n(x)$, the theorem is proved.

The subscript n on $f_n(x)$ in equation (B) is used to indicate that this function may depend on the parameter n which occurs in the functional equation. If $n=0$ in the functional equation (B), it reduces to Cauchy's functional equation. We shall show that the limit as n goes to zero of certain continuous solutions of (B) will reduce to the continuous solutions of Cauchy's equation, that is, to A^x where A is any positive constant.

Theorem I shows that the problem of solving equation (A) can be reduced to that of solving the simpler equation (B). We proceed now to the study of this latter equation.

For equation (B) we have the

THEOREM II. *Every non-trivial, real continuous solution $f_n(x)$ of the functional equation (B) which is defined for all $x>-1/n$, is of the form $f_n(x)=(1+nx)^k$, where k is any real number.*

Proof. We note that (B) can be written as

$$f_n\left[\frac{(1+nx)(1+ny)-1}{n}\right] = f_n(x)f_n(y). \tag{2.3}$$

Let $1+nx=e^u$, $1+ny=e^v$, then the last displayed equation may be written as

$$f_n\left[\frac{e^{u+v}-1}{n}\right] = f_n\left(\frac{e^u-1}{n}\right)f_n\left(\frac{e^v-1}{n}\right).$$

Let $F(u)=f_n[(e^u-1)/n]$. Then because of the last displayed equation we get

$$F(u+v) = F(u)F(v), \tag{2.4}$$

which is Cauchy's functional equation. It is however well known that all real continuous solutions of this equation are of the form $F(u)=e^{ku}$ where k is any real number. Hence all continuous solutions of (B) are of the form $f_n[(e^x-1)/n] = e^{kx}$, or $f(t)=(1+nt)^k$.

It might be of interest to show that the limit as n goes to zero of certain

solutions of (B), that is, those for which $k=c/n$, give the non-trivial solutions of Cauchy's functional equation as it was mentioned earlier in the paper. To this end we write

$$\lim_{n\to 0} f_n(t) = \lim_{n\to 0} (1+nt)^{c/n} = \lim_{n\to 0} \left[(1+nt)^{1/nt}\right]^{ct} = e^{ct}.$$

THEOREM III. *Every real, non-trivial, continuous solution of the functional equation* (A) *which is defined for all* $x>-1/n$ *is of the form*

$$G(x) = G(0)(1+nx)^k, \qquad H(x) = H(0)(1+nx)^k,$$

$F(x)=G(0)H(0)(1+nx)^k$, *where* k *is any real number.*

Proof. This result is a direct consequence of Theorems I and II.

THEOREM IV. *All non-trivial, real, continuous solutions of the functional equation*

$$(C) \qquad g_n(x+y+nxy) = g_n(x) + g_n(y), \qquad n>0,$$

which are defined for all $x>-1/n$, *are of the form* $g_n(x)=k \log (1+nx)$, *where* k *is any real number.*

Proof. Let $f_n(x)=e^{g_n(x)}$. Then $f_n(x)$ is a non-trivial continuous solution of (B). Hence $g_n(x)=k \log (1+nx)$.

From Theorem III we obtain

THEOREM V. *All non-trivial, real, continuous solutions of the functional equation*

$$f(x+y+nxy) = g(x) + h(y) \qquad n>0$$

are of the form

$$\begin{aligned} g(x) &= k \log (1+nx) + g(0) \\ h(x) &= k \log (1+nx) + h(0) \\ f(x) &= k \log (1+nx) + g(0) + h(0) \end{aligned}$$

where k *is any real number.*

Proof. Let $F(x)=e^{f(x)}$, $G(x)=e^{g(x)}$, $H(x)=e^{h(x)}$. Then $F(x+y+nxy)=G(x) \cdot H(y)$, and the result follows from Theorem III.

Many interesting results‡ obtained on discontinuous solutions of the equations (2.4) and (C) (with $n=0$) can also be extended by means of transformations of variables to the equations (B) and (C) of this paper.

3. Functions which are analogous to trigonometric functions. Let $f_n(x)$ be any non-trivial solution of the functional equation (B). We know that $f_n(0)=1$.

‡ Hamel, Mathematische Annalen, Vol. LX (1905), p. 459; Ostrowski, Comm. Math. Helvetici, 1 (1929) pp. 157–159.

We define two functions $s_n(x)$ and $c_n(x)$ by the equations

$$(3.1)\quad s_n(x) = \frac{f_n(x) - f_n\left(\dfrac{-x}{1+nx}\right)}{2h}, \qquad c_n(x) = \frac{f_n(x) + f_n\left(\dfrac{-x}{1+nx}\right)}{2}$$

where $h \neq 0$ is a constant. We see that $s_n(0) = 0$, $c_n(0) = 1$, $s_n[(-x/1)+nx] = -s_n(x)$, $c_n(-x/1+nx) = c_n(x)$. Multiplying the first equation of (3.1) first by h, and then by $-h$, and adding each of the results to the second one of equations (3.1), we obtain

$$(3.2)\quad \begin{aligned} c_n(x) + hs_n(x) &= f_n(x), \\ c_n(x) - hs_n(x) &= f_n\left(\frac{-x}{1+nx}\right). \end{aligned}$$

Multiplying the last two equations we get, because of (B), Pythagoras' theorem,

$$(3.3)\quad c_n^2(x) - h^2 s_n^2(x) = 1.$$

It is rather interesting to note that if we let $f_n(x)$ be the particular continuous solution of (B) $(1+nx)^{h/n}$, then (3.3) shows that Pythagoras' theorem is satisfied by a one parameter family of simple algebraic functions

$$s_n(x) = \frac{(1+nx)^{h/n} - (1+nx)^{-h/n}}{2h}, \quad c_n(x) = \frac{(1+nx)^{h/n} + (1+nx)^{-h/n}}{2},$$

which, with $h = 1$, or $h = i$, continuously approach the hyperbolic, or circular functions as n goes to zero.

We next note that any solution $f_n(x)$ of (B) satisfies the functional equation

$$(3.4)\quad f\left[\frac{(1+nx)^r - 1}{n}\right] = f^r(x),$$

where r is any rational number. This result could be derived directly from (B), but it is quicker to derive it from Cauchy's equation (2.4). Let $F(x)$ be any solution of (2.4). Then $F(rx) = F^r(x)$, where r is any rational number, which is a well known result. But if $f_n(x)$ is any solution of (B), then $F(x) = f_n(e^x - 1/n)$ is a solution of (2.4), and hence

$$f_n\left(\frac{e^{rx} - 1}{n}\right) = f_{nr}\left(\frac{e^x - 1}{n}\right).$$

Setting $e^x = 1 + nx$, the last displayed equation reduces to (3.4). If we assume that $f_n(x)$ is continuous, the functional equation (3.4), holds for all real values of r.

Raising each side of the first one of the equations (3.2) to the pth power, we get

$$(3.5)\qquad [c_n(x) + hs_n(x)]^p = f_n^p(x) = f_n\left[\frac{(1+nx)^p - 1}{n}\right]$$

by (3.4). If $f_n(x)$ is assumed to be continuous this equation is true for all real values of p, otherwise it is true for all rational values only. Equation (3.5) can be written because of (3.2) as

$$(3.6)\qquad [c_n(x) + hs_n(x)]^p = c_n\left[\frac{(1+nx)^p - 1}{n}\right] + hs_n\left[\frac{(1+nx)^p - 1}{n}\right],$$

which corresponds to De Moivre's theorem. Also by (3.2) and (B) we have

$$(3.7)\qquad \begin{aligned} c_n(x+y+nxy) + hs_n(x+y+nxy) &= [c_n(x) + hs_n(x)][c_n(y) + hs_n(y)] \\ &= c_n(x)c_n(y) + hc_n(x)s_n(y) \\ &\quad + hc_n(y)s_n(x) + h^2s_n(x)s_n(y), \end{aligned}$$

$$(3.8)\qquad \begin{aligned} c_n(x+y+nxy) - hs_n(x+y+nxy) &= f_n\left[\frac{-(x+y+nxy)}{1+n(x+y+nxy)}\right] \\ &= f_n\left[\frac{-x}{1+nx}\right]f_n\left[\frac{-y}{1+ny}\right] \\ &= [c_n(x) - hs_n(x)][c_n(y) - hs_n(y)] \\ &= c_n(x)c_n(y) - hc_n(x)s_n(y) \\ &\quad - hc_n(y)s_n(x) + h^2s_n(x)s_n(y). \end{aligned}$$

Adding and subtracting the equations (3.7) and (3.8) we get the addition formulas

$$(3.9)\qquad \begin{aligned} c_n(x+y+nxy) &= c_n(x)c_n(y) + h^2s_n(x)s_n(y) \\ s_n(x+y+nxy) &= s_n(x)c_n(y) + s_n(y)c_n(x). \end{aligned}$$

We note the similarity of these formulas to those of the cosine and sine functions. If we should use additional properties of the function $f_n(x)$ such as differentiability, we could derive many more properties which would have their analogues in ordinary trigonometry, or in the theory of hyperbolic functions.

The functions $s_n(x)$ and $c_n(x)$ could have been defined in terms of the ordinary hyperbolic functions of arguments which satisfy a certain functional equation. We might have written $s_n(x) = 1/h \sinh g_n(x)$, $c_n(x) = \cosh g_n(x)$ where $g_n(x)$ is any solution of the functional equation (C). This shows that the functions $s_n(x)$, $c_n(x)$ are particular cases of $s(x) = 1/h \sinh g(x)$, $c(x) = \cosh g(x)$ where $g(x)$ is any solution of $g(x) + g(y) = g[\mu(x, y)]$. Then it follows that

$$\begin{aligned} c[\mu(x, y)] &= c(x)c(y) + h^2s(x)s(y), \\ s[\mu(x, y)] &= s(x)c(y) + c(y)s(y). \end{aligned}$$

In particular if $\mu(x, y) = x+y+nxy$, $s(x) = s_n(x)$, $c(x) = c_n(y)$. It is rather inter-

esting to note that if in this case $g(x)$ is assumed to be continuous then, by Theorem IV, $g(x)$ has to be $k \log (1+nx)$. The object of this section has been to derive a theory analogous to that of the hyperbolic and circular functions on the basis of a given functional equation. If one were interested in generalizing the results of this section one could start with an essentially different functional equation, or one could consider elements, and operations of some group or linear space, satisfying analogous functional equations. One could for example write $A(x+y+nxy)=A(x)\cdot A(y)$ where $A(x)$ might be a matrix and the operation $(\cdot)$ indicate matrix multiplication. Other generalizations that could be taken up here will suggest themselves to the reader.

ARITHMETICAL PROPERTIES OF SUMS OF POWERS

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1. Introduction. For each polynomial,

$$p(x) = x^n + a_1 x^{n-1} + \cdots + a_n = \prod_{\nu=1}^{n} (x - x_\nu),$$

we define $s_q(p) = x_1^q + \cdots + x_n^q$ $(q = 1, 2, 3, \cdots)$. It is known that if all the a_i's are integers, then so are all the s_i's. We will say that an ordered set of integers,

$$S(1), S(2), \cdots, S(n),$$

has the property P provided there exists a polynomial p, having integral coefficients, for which $S(j) = s_j(p)$ $(i=1, 2, \cdots, n)$.

The following criterion concerning the sums s_i is due to Jänichen [1]:

The set S has the property P if and only if the congruences

$$\sum_{d|m} \mu(d) S(m/d) \equiv 0 \pmod{m} \qquad (m = 1, 2, \cdots, n)$$

all hold.

2. A generalization. The purpose of the first part of this note is to prove the following generalization of Jänichen's criterion.

THEOREM 1. *The set S has the property P if and only if the congruences*

$$\sum_{d|m} f(d) S(m/d) \equiv 0 \pmod{m} \qquad (m = 1, 2, \cdots, n) \tag{1}$$

all hold, where f is an arbitrary integer-valued function satisfying the conditions

$$f(1) = \pm 1, \tag{2}$$

$$\sum_{d|m} f(d) \equiv 0 \pmod{m} \qquad (m = 1, 2, \cdots, n). \tag{3}$$

In particular, f may be an arbitrary multiplicative function that satisfies (3) whenever m is a power of a prime, such as Möbius' function μ and Euler's function ϕ [2]. The example $f(1)=1$, $\sum_{d|m} f(d)=-m$ for $m>1$, whence $f(2)=-3$, $f(3)=-4$, $f(6)=0$, shows that the conditions (2), (3) do not imply that f is multiplicative.

COROLLARY. *If f is an arbitrary integer-valued function satisfying the conditions*

$$\sum_{d|m} f(d) \equiv 0 \pmod{m} \qquad \textit{for each } m \geqq 1, \tag{3'}$$

then, for sums of powers $s_q(p)$,

$$\sum_{d|m} f(d) s_{m/d} \equiv 0 \pmod{m} \qquad \textit{for each } m \geqq 1. \tag{1'}$$

In order to prove Theorem 1 and the corollary we need the three lemmas that follow. All the functions involved in these lemmas are defined for $m=1, 2, \cdots n$, where n is an arbitrary positive integer.

LEMMA 1. *If $f(m)$, $g(m)$, $R(m)$ are three functions satisfying the conditions*

$$g(1)R(1) = \pm 1, \tag{4}$$

$$\sum_{d|m} g(d)R(m/d) \equiv 0 \pmod{m}, \tag{5}$$

$$\sum_{d|m} f(d)R(m/d) \equiv 0 \pmod{m}, \tag{6}$$

then also

$$F(m) = \sum_{d|m} f(d)g'(m/d) \equiv 0 \pmod{m}, \tag{7}$$

where g' is the Dirichlet reciprocal of g, that is, $\sum_{d|m} g(d)g'(m/d)=1, 0$ according as $m=1$ or $m>1$.

Proof. Relation (7) is evidently true for $m=1$. Suppose (7) is true for every divisor $d<m$ of m. Statement (7) implies $f(k)=\sum_{d|k} F(d)g(k/d)$. Hence

$$\begin{aligned} 0 \equiv \sum_{d|m} f(d)R(m/d) &= \sum_{d|m}\sum_{d'|d} F(d')g(d/d')R(m/d) = \sum_{d|m}\sum_{d'|d} F(m/d)g(d/d')R(d') \\ &= \sum_{d|m} F(m/d) \sum_{d'|d} g(d/d')R(d') \\ &= F(m)g(1)R(1) + \sum_{d|m,\, d<m} F(m/d) \sum_{d'|d} g(d/d')R(d') \\ &= \pm F(m) + \sum_{d|m,\, d<m} (m/d)Q(d)\cdot dQ'(d) \equiv \pm F(m) \pmod{m} \end{aligned}$$

(Q, Q' integer-valued).

LEMMA 2. *If $f(m)$, $g(m)$, $R(m)$, $S(m)$ are four functions satisfying the conditions* (4), (5), (6) *and*

$$\sum_{d|m} g(d)S(m/d) \equiv 0 \pmod{m}, \tag{8}$$

then also

$$\sum_{d|m} f(d)S(m/d) \equiv 0 \pmod{m}. \tag{7'}$$

Proof. By Lemma 1, we have

$$\begin{aligned}\sum_{d|m} f(d)S(m/d) &= \sum_{d|m} F(m/d) \sum_{d'|d} g(d/d')S(d') \\ &= \sum_{d|m} (m/d)Q(d)\cdot dQ''(d) \equiv 0 \pmod{m}\end{aligned}$$

(Q, Q'' integer-valued).

In particular, for $R \equiv \pm 1$ Lemma 2 implies

LEMMA 3. *If $f(m)$, $g(m)$, $S(m)$ are three functions satisfying the conditions*

$$g(1) = \pm 1, \tag{4'}$$

$$\sum_{d|m} g(d) \equiv 0 \pmod{m}, \tag{5'}$$

$$\sum_{d|m} f(d) \equiv 0 \pmod{m} \tag{6'}$$

and (8), *then also* (7′) *hold.*

Proof of Theorem 1 and Corollary. The theorem and corollary follow by Jänichen's criterion from Lemma 3 by first, putting μ for g and, second, putting f for g and μ for f.

CONVERSE OF THEOREM 1. *Suppose that f is a number-theoretic function such that every set S has the property P if and only if* (1) *hold. Then f satisfies* (2) *and* (3′).

Proof. (3′) is obvious taking $p(x) = x-1$, whence $S(m) \equiv 1$. To prove (2), suppose $f(1) \neq \pm 1$. Then there exists a prime π dividing $f(1)$. Now we can choose a set S which satisfies (1) but does not have the property P. For example $n = \pi$, $S(m) = 0$ for $m = 1, 2, \cdots, \pi - 1$, $S(\pi) = 1$, which implies that the congruences (1) hold for each $m \leqq \pi$. But, by Jänichen's criterion, this set S does not have the property P, since we have $\mu(1)S(\pi) + \mu(\pi)S(1) = 1 \not\equiv 0 \pmod{\pi}$.

3. The apparition of prime factors. The aim of the second part of this note is to prove some theorems on the apparition of prime factors in sequences (s_q). From the criterion of Jänichen one can, with I. Schur [3], deduce the following congruences:

$$s_{kp^{\alpha+1}} \equiv s_{kp^{\alpha}} \pmod{p^{\alpha+1}} \tag{9}$$

for every prime p and non-negative integral α. The congruences (9) can also be written in the following equivalent form:

$$s_{kp^{\alpha+\beta}} \equiv s_{kp^{\alpha}} \pmod{p^{\alpha+1}} \tag{10}$$

for every positive integral β. Indeed, (10) becomes (9) for $\beta=1$. Let (10) be true for β. Then by (9)

$$s_{kp^{\alpha+\beta+1}} \equiv s_{kp^{\alpha+\beta}} \pmod{p^{\alpha+\beta+1}},$$

and a fortiori

$$s_{kp^{\alpha+\beta+1}} \equiv s_{kp^{\alpha+\beta}} \pmod{p^{\alpha+1}},$$

Combining the last congruences with (10), supposed true for β, we have

$$s_{kp^{\alpha+\beta+1}} \equiv s_{kp^{\alpha}} \pmod{p^{\alpha+1}},$$

that is, (10) is true also for $\beta+1$, which establishes (10).

From (10) we immediately deduce the following theorems:

THEOREM 2. *If* $s_{kp^\alpha}\equiv 0$ *(mod* p^γ*), where* p *is a prime,* γ *is a positive integer,* α *is a non-negative integer and* $\gamma<\alpha+1$, *then* $s_{kp^{\alpha+\beta}}\equiv 0$ *(mod* p^γ*) for every positive integer* β.

In particular, for $\alpha=0$, $\gamma=1$ we have

THEOREM 2.1. *If* $s_k\equiv 0$ *(mod* p*), where* p *is a prime, then* $s_{kp^\beta}\equiv 0$ *(mod* p*) for every positive integer* β.

THEOREM 2.2. *If* $s_k=0$ *then* $s_{kp^\beta}\equiv 0$ *(mod* p*) for every prime* p *and every positive integer* β [4].

CONVERSE OF THEOREM 2.2. *If* $s_k\equiv 0$ *(mod* p*) for an infinitude of primes* p *then* $s_k=0$.

Proof. By (10) we have $s_{kp}\equiv s_k \pmod{p}$. Whence, by the hypothesis of the converse, $s_k\equiv 0 \pmod{p}$ for an infinitude of primes p. Thus $s_k=0$.

4. Remarks. A sequence (s_q), no term of which vanishes, does not necessarily contain all primes as factors. For example, the sequence $(V_q=\alpha^q+\beta^q)$, where α, β are roots of the equation $x^2-x-1=0$ ($V_1=1$, $V_2=3$, $V_q=V_{q-1}+V_{q-2}$) does not contain all primes as factors [5]. The question as to which primes appear and which do not appear as factors in a sequence (s_q) with no vanishing term seems to be open even in the case of (V_q).

The Theorems 2 and 2.1 can be generalized immediately by putting r for 0 in the congruences.

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Originally the special case of Theorem 1 where f is Euler's function ϕ was proved. I am indebted to the referee for pointing out the possibility of further generalization.

3. I. Schur, Arithmetische Eigenschaften der Potenzsummen einer algebraischen Gleichung, Compositio Mathematica 4 (1937), 432–44.

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HUGH JONES AND OCTAVE COMPUTATION

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Many mathematicians have at diverse times indulged themselves in the contemplation of a change in the base of the number system. So far as the author has been able to discover the first person in this country to advocate it seriously, at length, and with missionary zeal, was the Rev. Hugh Jones. This gentleman (1692–1760), graduate of Jesus College, Oxford, clergyman, mathematician, and historian, left England in 1716 shortly after securing the Master of Arts degree. In 1717 he was elected to the faculty of the College of William and Mary as successor to the chair of mathematics and natural philosophy first held by the Rev. Tanaquil Lefevre. He was in consequence the second professor of mathematics in the territory now known as the United States, antedating by ten years the appointment of Isaac Greenwood to the Hollis professorship at Harvard in 1727.

Jones was learned, fearless, aristocratic, a loyal Hanoverian, and a zealous churchman. In addition he was intellectually vigorous as is evidenced by the fact that aside from his duties as professor, chaplain of the House of Burgesses, and lecturer at Bruton Church in Williamsburg, he wrote an *Accidence to Christianity*, an *Accidence to the Mathematicks*, an *Accidence to the English Tongue* (the first English Grammar written in America), and a most interesting book upon the *Present State of Virginia*. This latter work was not only sprightly and interesting but withal an invaluable source of material for subsequent historians. In it he advanced many sound suggestions for the improvement of the colony, among them a proposal for a separate chair of history for the College of William and Mary.

One of his practically unknown pieces of research is a long manuscript, listed as British Museum Additional MS 21893, entitled *The Reasons, Rules and Uses of Octave Computation or Natural Arithmetic*. A very abbreviated resumé of the

material appeared in the Gentleman's Magazine of March, 1751, but this was known to only a handful of historians interested in the detailed history of Virginia. The original manuscript, Most humbly inscribed to the Right Honorable Earl of Macclesfield, consists of forty pages, 7×9 inches, excellently phrased and boldly executed in a chirography not unlike the familiar signature of John Hancock appended to the Declaration of Independence. There is evidence that the final preparing of the manuscript was the work of some amanuensis since it is not the handwriting of Hugh Jones and is considerably embellished throughout.

Professor Jones observes at the outset that since there is nothing of more general utility than a due adjustment of numbers, coins, measures, and weights it appears desirable to scrutinize their properties and consider methods for their uniform representation, particularly since nations differ so much in these matters that even common arithmetic has become "mysterious to Women and Youths and often troublesome to the best Artists."

The argument moves on to assért that much of the difficulty lies in the choice of ten as a base of the number system, especially since neither ten, nor its powers or products, admit of bi-partition down to unity. Furthermore neither ten, nor its half or its double, have square or cube roots that are integers, "all of which properties the common Radix of Number ought to have for convenience in applications to surfaces and solids."

The elimination of many of the difficulties is alleged to lie in the choice of eight for the "Radix of Computation." According to the author,

> For 8 its Powers and many of its Products are divisible down to Unity by halving, quartering etc. And the Proportion of its Parts to the whole are self-evident, without Perplexity to the Thought; for 1 is the half quarter; 2 the Quarter; 3 the Quarter and half Quarter; 4 the Half; 5 the Half and half Quarter; 6 the three Quarters; 7 the three Quarters and half Quarter of 8: Into which Portions a geometrical Line Surface or Solid is most easily and readily and precisely divisible; and therefore to this Natural Method Numbers ought to conform. Again 8 is a cubic Number whose Root is 2; 4 its Half is a Square whose Root is likewise 2; 16 its Double is a square Number whose whole Root is 4; and 64, besides being the Square of 8, is likewise a cubic Number whose Root is 4.
>
> So that Arithmetic by *Octaves* seems most agreeable to the Nature of Things, and therefore may be called Natural Arithmetic in Opposition to that now in Use, by Decades; which may be esteemed Artificial Arithmetic.

The above thesis is then buttressed by several pages of applications to arithmetic, geometry, and natural philosophy.

The second main division of the manuscript is entitled "Notation and Numeration." With considerable ingenuity the term "ones" is retained for the

units while the successive powers of the base are designated as ers, ests, thousets, millets, billets, etc. In other words, 8×8 is the est of 8 while $8\times 8\times 8$ is the thouset, and so on to any desired magnitude. In this connection it is the opinion of the writer that the names beginning with thouset are inconsistent since they derive from powers of ten. Mr. Jones, however, in his enthusiasm goes on to suggest that additional clarification would result by applying his suffixes to classification names, as for example, cash, casher, cashest in counting money, or ounce, ouncer, ouncest in weighing. As an example 352 yardest would signify $3\times 8^2+5\times 8+2$ yards.

Following this material there appears the technique for changing from base ten to base eight, familiar to any experienced mathematician. For the layman unacquainted with the process it must first be pointed out that in an octave number system eight distinct symbols, and only eight, namely 0, 1, 2, 3, 4, 5, 6, 7, or their equivalents, are required to write any integer whatsoever. With this brief word of explanation we may proceed directly to the examples in the manuscript.

$1 \times 8 = \text{er}$ $\quad$ $2 \times 4 = 10 = \text{er}$, where 10 means not ten but one times the base plus no units,

$2 \times 5 = 12 = \text{er}2$, $\quad$ $2 \times 6 = 14 = \text{er}4$,

$2 \times 7 = 16 = \text{er}6$, $\quad$ $2 \times 8 = 20 = 2\text{er}$,

$8 \times 8 = 100 = \text{est}$, $\quad$ $6 \times 6 = 44 = 4\text{er}4$,

$7 \times 7 = 61 = 6\text{er}1$

To identify the process outlined above the author coins the verb "octavate." For instance, to octavate the number one hundred the reasoning is as follows. The square of eight goes into one hundred once with remainder thirty-six. This remainder contains eight to the first power four times with remainder four. Hence to express one hundred to the base 8 we write the digits 144 which in somewhat longer form may be put as $100=1\times 8^2+4\times 8+4$. The inverse operation, which is termed "decimation," together with an adequate treatment for the octavation of decimal fractions will be mentioned here without consideration of the details.

The thread of the argument then turns to the advantages which an octave system could effect in the matter of coinage and in this connection the following query is set forth:

> Might not this Division be easily effected by altering the Proportion of Silver to Copper by 1/16 and calling the present Half Penny a Penny as it is already called in these Plantations? Then our present Half Crown would contain 64 (instead of 60) of these Pennies; and the Pound or Integer would be equal to 512 of them, & instead of 960, there would be esteemed 1024 of our Farthings in a Pound Sterling. May not the vast Quantity of Copper lately found in our Plantations on the Continent

> of *North America* well allow (if not require) this Alteration in the Proportion of the Value of Silver to that Metal?

In discussing the matter of physical units Mr. Jones advances several interesting notions. By hanging a weighted cord between two cycloids the concept of the isochronous pendulum is introduced, but in addition he defines the unit of linear measure as the length of such a pendulum which at sea level at Dover or Calais beats seconds. Convenient octave submultiples may then be taken as experience proves desirable. From this he then makes a suggestion which should accord him a high place among the scientific thinkers of his day, namely, to derive the unit of weight from the amount of rain water contained in a cubical box with edge equal to one eighth of the pendulum length above mentioned. This idea was to be actually established in its decimal counterpart forty-five years later when the French National Assembly took up the matter of the metric system in 1790.

In spite of the zeal and meticulousness of the author he is not unmindful of the inertia of human nature with respect to new ideas since he pauses to observe,

> But forasmuch as there seems no Probability that this will be soon, if ever, universally complied with; therefore it may not be impertinent to subjoyn some Methods that appear easiest for the Reduction of our present Denominations of Measures and Weights to an octave Uniformity in their different Kinds.

In this connection the term "Longimetry" is introduced to designate the process of measuring "mere length" and the observation is made that such current units as fathom, pace, ell, yard, foot, league, mile, furlong, and perch are in several instances already commonly divided into eighths. In particular is this true of the foot where the octave system would produce a unit, called a "Prime," equivalent to the present inch and a half. One eighth of this is designated as a "Second" and an additional subdivision by eight is suggested but is accorded no name.

The diurnal rotation of the earth is termed the "Nuethemeron," and is divided into eight "Primes," and these into eight "Seconds" (each equivalent to twenty-two and a half of the present minutes). This idea is not pursued to its logical conclusion but it would lead to a further division by sixty-four in order to secure a unit of the order of magnitude of the present second in which case it would then seem necessary to redetermine the length of the pendulum and other derived units previously mentioned.

The final section of the manuscript deals with additional utilitarian applications to land surveying, grain storage, and ship displacement after which the author polishes off the whole paper with the following paragraph which may or may not contain elements of unconscious humor:

> Much more might be added on these Subjects; but what is already advanced being more than was at first intended, and perhaps too prolix al-

ready, seeming sufficient to explain the Scheme proposed in the Gentleman's Magazine for July 1745 (p. 377) and publickly demanded in the first page of the Gent. Magaz. for March 1751 it is thought needless to expatiate at present; since, if these Lucubrations meet with favorable Reception, and require further Illustration or Proof, an Enlargement may be communicated by the Proposer.

MATHEMATICAL NOTES

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A TOPOLOGICAL PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA*

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Many theorems which were formerly thought to be purely analytic in character have in recent years been associated with topological concepts. For instance, it has been known for some time that the fundamental theorem of algebra could be derived from Brouwer's fixed point theorem. Such a derivation does not appear in the literature and it is the purpose of this note to supply a simple proof. Proofs have appeared [2, 3] which are based on the Brouwer index, on which the fixed point theorem rests.

For our purpose, Brouwer's fixed point theorem may be stated: Any continuous transformation of the circle $|z| \leqq R$ into itself has a fixed point. Here R is an arbitrary positive real number (to be specified later) and z is a complex variable. For an elementary proof of this theorem, see [1].

To derive the fundamental theorem of algebra from this result, let $f(z) = z^n + a_1z^{n-1} + \cdots + a_n$ be any complex polynomial with leading coefficient unity. We must prove that $f(z)$ has at least one zero. To this end, set $z = re^{i\theta}$, $0 \leqq \theta < 2\pi$, $R = 2 + |a_1| + \cdots + |a_n|$ and define a (non-analytic) function $g(z)$ by

$$g(z) = \begin{cases} z - f(z)/(Re^{i(n-1)\theta r}) & \text{for } |z| \leqq 1 \\ z - f(z)/(Rz^{n-1}) & \text{for } |z| \geqq 1. \end{cases}$$

The function $g(z)$ is single valued and continuous for all values of z. First, each of the two expressions given for $g(z)$ is continuous throughout the range

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specified since neither denominator becomes zero and, for $z=0$, we have $r=0$ so that $i(n-1)\theta r=0$ no matter what value θ has. Secondly, for $|z|=1$, the two expressions are identical.

For $|z|\leqq R$ we have $|g(z)|\leqq R$. First, for $|z|\leqq 1$,

$$|g(z)| \leqq |z| + \left|\frac{f(z)}{R}\right| \leqq 1 + \frac{1+|a_1|+\cdots+|a_n|}{R} \leqq 1+1 \leqq R.$$

Secondly, for $1\leqq|z|\leqq R$,

$$|g(z)| = \left|z - \frac{z}{R} - \frac{a_1+\cdots+a_n z^{1-n}}{R}\right| \leqq \left|(R-1)\frac{z}{R}\right| + \frac{|a_1|+\cdots+|a_n|}{R} \leqq R-1+\frac{R-2}{R} \leqq R.$$

Now consider the correspondence $z\to g(z)$. The last two paragraphs show that this is a continuous transformation which maps the circle $|z|\leqq R$ into itself. Thus, by Brouwer's fixed point theorem, there exists at least one value z_0 such that $g(z_0)=z_0$. But from the definition of $g(z)$, this means that $f(z_0)=0$ and $f(z)$ has at least one zero.

Incidentally this proof shows the existence of a zero of $f(z)$ whose absolute value does not exceed R, but this bound is of little interest since much stronger results are known. Goursat [4] gives a result which implies that *all* the zeros of $f(z)$ are less than or equal to max $(1, R-2)$ in absolute value.

References

1. R. Courant and H. Robbins, What is Mathematics?, Oxford University Press (1941) pp. 251–254.
2. Ibid. pp. 269–271.
3. S. Eilenberg and I. M. Niven, The "fundamental theorem of algebra" for quaternions, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 246–248.
4. E. Goursat, Mathematical Analysis, vol. II—Part I, Functions of a Complex Variable, Ginn & Co. (1916), p. 104.

ON THE ALGEBRAIC STRUCTURE OF KNOTS*

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1. Motivation of the postulates. By definition, a knot is a closed polygon in 3 dimensional Euclidean space. If k_1 and k_2 are knots, then their product knot $k_1\times k_2$ is the knot obtained by cutting k_1 at any point P_1 (giving loose ends e_1 and f_1), cutting k_2 at any point P_2 (giving loose ends e_2 and f_2), and joining e_1 to e_2, and f_1 to f_2. Pictorially, the product $k_1\times k_2$ is the knot of Figure 1 when k_1, k_2 are the knots of Figure 2 on the left and right, respectively. It has been shown that

* This note was motivated by a Colloquium address at the University of Michigan by Professor H. Seifert, now at the University of Heidelberg. Section 1 is essentially a summary of part of his lecture.

the product of knots is commutative and associative. Obviously the circle is the identity knot. Let c denote the circle. By definition, k is a prime knot if $k=k_1\times k_2$ implies $k_1=c$ or $k_2=c$. Let $g(k)$ denote the genus of the knot k. Then it is known that

$$g(k_1 \times k_2) = g(k_1) + g(k_2), \tag{1}$$

$$g(k) \text{ is a non-negative integer.} \tag{2}$$

The rather spectacular result has recently been shown (by Schubert) that every knot can be uniquely factored into a product of prime knots.

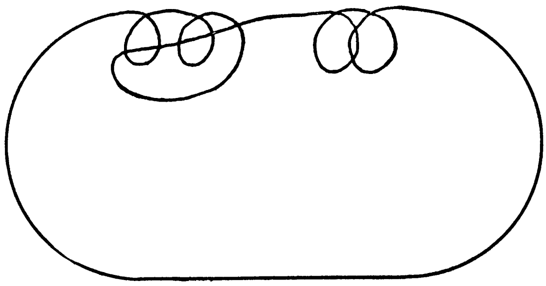

Fig. 1

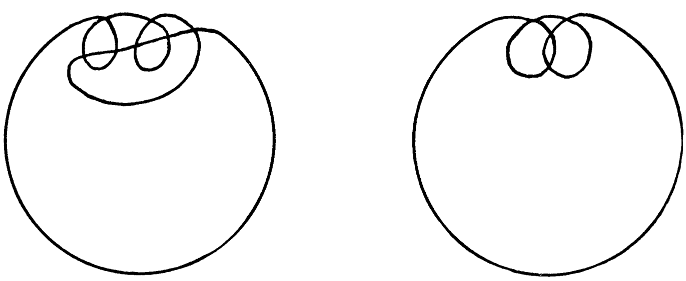

Fig. 2

2. **Postulates for knots.** From the above considerations, it follows that the set K of all knots satisfies the following abstract postulates with respect to the binary operation of knot multiplication.

Let the undefined notions be a non-empty class K of elements and a binary operation denoted by $\times$. The postulates are:

$P1$: If $k,\ k'\in K$, then $k\times k'\in K$.
$P2$: If $k,\ k',\ k\times k',\ k'\times k\in K$, then $k\times k'=k'\times k$.
$P3$: If $k,\ k',\ k'',\ k\times k',\ k'\times k''$,
$k\times(k'\times k''),\ (k\times k')\times k''\in K$,
then $k\times(k'\times k'')=(k\times k')\times k''$.
$P4$: There exists an element $c\in K$ such that if k and $c\times k\in K$, then $c\times k=k$.

Remark: If we add a postulate $P5$: If $k \in K$, then there is an element $k^{-1} \in K$ such that $k \times k^{-1} = c$; then $P1$ to $P\,5$ are a set of postulates for an abelian group.

3. Independence of the postulates. The following finite systems show the independence of postulates $P1$, $P2$, $P3$, $P4$:

$P1$

$\times$	0	1
0	0	1
1	1	2

$P2$

$\times$	0	1
0	0	1
1	0	1

$P3$

$\times$	0	1	2
0	0	1	2
1	1	1	0
2	2	0	2

$P4$

$\times$	0	1
0	0	0
1	0	0

$P1$ fails since $1 \times 1 \notin K$

$P2$ fails since $1 \times 0 \neq 0 \times 1$

$P3$ fails since $1 \times (1 \times 2) \neq (1 \times 1) \times 2$

$P4$ fails since there is no identity element.

4. Conclusions. It is well known that any algebraic system satisfying the postulates $P1$ to $P4$ (which can be called a commutative semi-group with identity element, if we choose that definition of the many extant for semi-groups which characterizes a semi-group as a closed associative system) can be embedded in an abelian group. Thus if $(K, \times)$ is the algebra of knots (i.e. K = the set of all knots, and $k_1 \times k_2$ is knot product), then $(K, \times)$ can be embedded in an abelian group $(\overline{K}, \otimes)$ such that:

1. $K \subset \overline{K}$.
2. If $k, k' \in K$, then $k \otimes k' = k \times k'$.
3. For each $k \in K$, there exists an element $k^{-1} \in \overline{K} - K$ such that $k \otimes k^{-1} = c$.

It is interesting to observe that $\overline{K} - K$ would consist of "imaginary knots" of "negative genus." The element k^{-1} can be pictured as that "knot" which *unravels* a given real knot k.

CLASSROOM NOTES

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MEAN AND ORDINARY CONVERGENCE OF A SEQUENCE OF FUNCTIONS

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In presenting the idea of mean convergence to undergraduates who are not acquainted with the Lebesgue integral the following procedure has been found to be helpful. We point out that mean convergence may apply to cases where ordinary convergence fails, and we also prove that under appropriate conditions ordinary convergence implies mean convergence.

A sequence of functions $f_n(x)$, defined in the finite closed interval (a, b), converges to $f(x)$ in the ordinary sense if

$$\lim_{n\to\infty} f_n(x) = f(x). \tag{1}$$

A sequence of functions $f_n(x)$ converges in the mean with index $p>0$ to $f(x)$ if

$$\lim_{n\to\infty} \int_a^b | f_n(x) - f(x) |^p \, dx = 0. \tag{2}$$

Throughout the discussion we assume the integration to be taken in the Riemann sense.

Convergence in the mean does not imply ordinary convergence. A familiar example* is the following. Consider the closed intervals $(0, 1/2)$, $(1/2, 1)$, $(0, 1/3)$, $(1/3, 2/3)$, $(2/3, 1)$, $(0, 1/4)$, $\cdots$. Let $f_n(x)=1$ in the nth interval and zero elsewhere. Here $\lim_{n\to\infty} f_n(x)$ does not exist. But the limit in the mean does exist since we have $\lim_{n\to\infty} \int_0^1 |f_n(x)-f(x)|^p\, dx = \lim_{n\to\infty} L_n = 0$ where L_n is the length of the nth interval and $f(x)=0$.

However ordinary convergence, uniform or non-uniform, does imply mean convergence under the conditions as stated in the following theorem.

THEOREM. *If in the finite closed interval (a, b) $f_n(x)$ is bounded in both n and x, if $f_n(x)$ and $f(x)$ are Riemann integrable, if $|f_{n+1}-f(x)| \leqq |f_n(x)-f(x)|$, and if $\lim_{n\to\infty} f_n(x)=f(x)$, then $\lim_{n\to\infty} \int_a^b |f_n(x)-f(x)|^p\, dx=0$ $(p>0)$.*

Proof. Let $F_n(x)=f_n(x)-f(x)$, and let $|f_n(x)| \leqq M/2$ so that $|f(x)| \leqq M/2$. Then we have $|F_n(x)| \leqq M$ for all x and n, and the $\lim_{n\to\infty} F_n(x)=0$. Let $I_n = \int_a^b |F_n(x)|^p\, dx$. We must show $\lim_{n\to\infty} I_n=0$. For a fixed $\epsilon>0$ and an n we have either $|F_n(x)| <\epsilon$ or $\epsilon \leqq |F_n(x)| \leqq M$. These two inequalities divide (a, b) into two sets of intervals. We may ignore the isolated singularities of $F_n(x)$ in making this subdivision since they do not affect the value of the integral. Thus we

* Titchmarsh: The Theory of Functions, Art. 12.53

have $I_n<(b-a)\epsilon^p+M^pB_n$ where B_n is the sum of the lengths of all intervals in which $|F_n(x)|\geqq\epsilon$. But the $\lim_{n\to\infty}B_n=0$ as may be seen by considering the sequence, $B_1, B_2, B_3, \cdots, B_n, \cdots$. We necessarily have $B_1\geqq B_2\geqq B_3\geqq\cdots\geqq B_n\geqq\cdots$ in view of the fact that $|F_{n+1}(x)|\leqq|F_n(x)|$. Such a positive sequence must necessarily approach a unique limit since it is monotonic, is bounded above by $(b-a)$, and is bounded below by zero. Let this limit be equal to $B\neq 0$. Then we have $|F_n(x)|\geqq\epsilon$ for all n in intervals whose sum is B. This means there are points where $\lim_{n\to\infty}|F_n(x)|\neq 0$, which is contrary to hypothesis. Thus $B=0$. Hence I_n may be made as small as desired, and the theorem is proved. The theorem may be extended to sequences which are not monotonic provided the intervals over which the sequences are not monotonic can be made as small as desired. This is actually the case in the first example below.

Example 1. The function $f_n(x)=n^c x^{1/2}\exp(-n^2x^2/4)$ in $(0, 1)$ with $f(x)=0$ illustrates the behavior of the two types of convergence. The $\lim_{n\to\infty}f_n(x)$ is zero everywhere for all c. The maximum value of $f_n(x)$ occurs at $x=1/n$ where we have $f_n(1/n)=n^{c-1/2}\exp(-1/4)$. The convergence of the sequence $f_n(x)$ is uniform for $c<1/2$. Our theorem now tells us we have mean convergence when $c\leqq 1/2$ providing we note the comment at the end of the proof above; for we have $f_n(x)\geqq f_{n+1}(x)\geqq f_{n+2}(x)\geqq\cdots$ except in an interval of length less than $1/n$. Now by direct integration we may verify mean square convergence ($p=2$), for we have $\int_0^1|f_n(x)-f(x)|^2\,dx=n^{2c-2}(1-\exp(-n^2/2))$. The table below shows the various possibilities for $\max f_n(x)$ and mean square convergence as c increases.

	$\lim\limits_{n\to\infty} f_n(1/n)$	$\lim\limits_{n\to\infty}\int_0^1 \lvert f_n(x)-f(x)\rvert^2dx$
$c<1/2$	0	0
$c=1/2$	$\exp(-1/4)$	0
$1/2<c<1$	∞	0
$c=1$	∞	1
$c>1$	∞	∞

Example 2. Let $f_n(x)=\exp(nx^2)/(1+\exp(nx^2))$ in $(-1, 1)$ where $f(0)=1/2$ and $f(x)=1$ when $x\neq 0$. Both $f_n(x)$ and $f(x)$ satisfy the hypothesis of the theorem above, and hence we have mean convergence. Here the "lim" cannot pass under the integral sign because the integrand does not converge uniformly; also, it is impossible to perform the integration by elementary methods.

We may still have mean convergence when $f_n(x)$ is unbounded or when (a, b) is an infinite interval. If we consider mean square convergence, always we must have the area bounded by $[F_n(x)]^2$ and the x-axis arbitrarily small. For example,

the function $f_n(x)$ may be unbounded in n as in Ex. 1 above when $1>c>1/2$, unbounded in x (as in the sequence $(nx)^{-1/3}$ in $(-1, 1)$ where $f(0)=\infty$ and $f(x)=0$ for $x\neq 0$), or unbounded in both x and n (as in a suitable combination of the two preceding cases). On the other hand consider $f_n(x)=(nx)^{-3/5}$. This sequence has the same limit function as $(nx)^{-1/3}$ above; the integral $\int_{-1}^{1} f_n(x)\,dx$ converges absolutely as an improper integral; but $f_n(x)$ does not converge to $f(x)$ in the mean square sense in $(-1, 1)$.

PROOFS OF THE ADDITION FORMULAE FOR SINES AND COSINES

A condensation by the editor of independent papers by

L. J. Burton, Bryn Mawr College, and E. A. Hedberg, University of South Carolina

The standard proof of the addition formulae for sines and cosines is certainly one of the least satisfactory sections of the usual trigonometry text. Its chief failings are its complexity, the artificiality of the construction required, and the limitation of the magnitudes of the angles involved. Two alternative proofs of these formulae are published below. The first, submitted by L. J. Burton, assumes a very elementary knowledge of analytic geometry. This proof has been taught in several universities for some time and probably has a long history; but

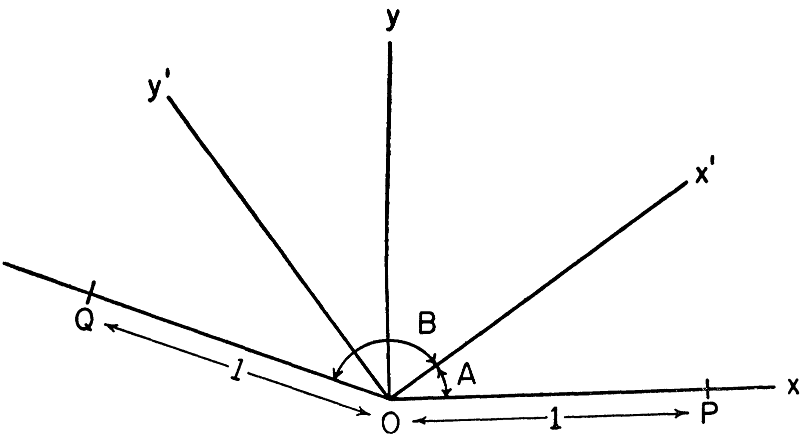

Fig. 1

since it does not appear in the popular textbooks and is unknown to most teachers, it seems desirable to make it more generally available. It has the advantage of placing no restriction on the size of the angles in addition to greater simplicity. The second, submitted by E. A. Hedberg, is based upon the laws of sines and cosines and assumes no analytic geometry. Since it makes use of triangles, it is valid only in case all of the angles involved are less than 180 degrees.

Proof by L. J. Burton: To prove that:

$$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$

Referring to Figure 1, place angle A so that its vertex is at the origin and its initial side is along the x-axis; place angle B so that its vertex is at the origin and its initial side coincides with the terminal side of angle A. Choose points P and Q at a unit distance from the origin on the initial side of A and the terminal side of B respectively. Consider also a new coordinate system with the same origin and with its x'-axis along the terminal side of A (which is also the initial side of B). The coordinates of P and Q in the two coordinate systems are seen to be:

	xy system	$x'y'$ system
P	$(1, 0)$	$[\cos(-A), \sin(-A)] = (\cos A, -\sin A)$
Q	$[\cos(A+B), \sin(A+B)]$	$(\cos B, \sin B)$

The distance PQ in the two coordinate systems is given by:

$$xy \text{ system:} \quad PQ^2 = [\cos(A+B) - 1]^2 + \sin^2(A+B) = 2 - 2\cos(A+B)$$

$$x'y' \text{ system:} \quad PQ^2 = (\cos A - \cos B)^2 + (\sin A + \sin B)^2 = 2 - 2\cos A \cos B + 2 \sin A \sin B.$$

By equating the two results we obtain the desired result. The other addition formulae are obtained from this one in the usual fashion.

Proof by E. A. Hedberg: Referring to Figure 2, from the law of cosines:

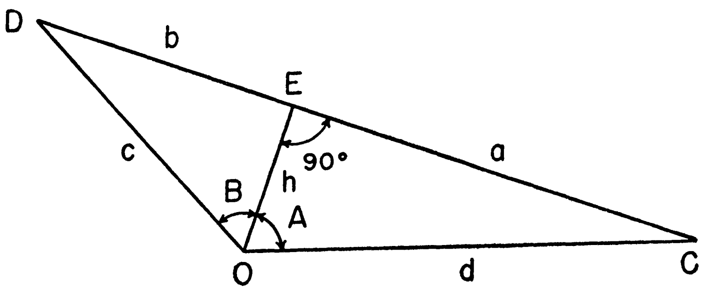

FIG. 2

$$\cos(A+B) = \frac{c^2 + d^2 - (a+b)^2}{2dc}$$

$$= \frac{(c^2 - b^2) + (d^2 - a^2) - 2ab}{2dc}$$

$$= \frac{2h^2 - 2ab}{2dc} = \frac{h^2 - ab}{dc}$$

$$= (h/d)(h/c) - (a/d)(b/c).$$

Thus, $$\cos (A + B) = \cos A \cos B - \sin A \sin B.$$

The addition formula for the sin $(A+B)$ may also be obtained from Figure 2, for by the law of sines:

$$\begin{aligned}\sin (A + B) &= \frac{(a + b) \sin D}{d} \\ &= (a/d) \sin D + b(\sin D/d) \\ &= (a/d) \sin D + b(\sin C/c) \\ &= (a/d) \sin D + (b/c) \sin C \\ &= \sin A \cos B + \cos A \sin B.\end{aligned}$$

The subtraction formulae can be obtained from these or directly from a modification of Figure 2 and a similar technique.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 876. *Proposed by H. L. Lee, University of Tennessee*

The product of four consecutive terms of an arithmetic progression of integers plus the fourth power of the common difference is a perfect square but in no case a perfect fourth power.

E 877. *Proposed by L. J. Burton, Bryn Mawr College*

Each of three arithmetic progressions continues indefinitely in both directions, and each has a difference which is an integer. Prove that if there is a term common to each pair of progressions then there is a term common to all three progressions.

E 878. *Proposed by Kaidy Tan, Anglo-Chinese College, Amoy, China*

Let S be the incenter of the right triangle ABC, and X the point of contact of the hypotenuse BC with the incircle. With center X and radius XS describe the circle cutting BS, CS at M and N respectively. Let AD be the altitude on

BC. Show that M, N are the incenters of the right triangles ABD and ACD respectively.

E 879. *Proposed by Joseph Langr, Prague, Czechoslovakia*

Let S_1, S_2, S_3 be the midpoints of three concurrent cevians of triangle ABC. Let S_2S_3, S_3S_1, S_1S_2, meet the sides BC, CA, AB in A_1, B_1, C_1; A_2, B_2, C_2; A_3, B_3, C_3 respectively. Show that (1) A_2, A_3; B_3, B_1; C_1, C_2 are isotomic points on the segments BC, CA, AB, (2) A_1, B_2, C_3 are collinear, (3) A_2, A_3, B_3, B_1, C_1, C_2 lie on a conic.

E 880. *Proposed by Peter Ungar, University College, London, England*

Let n points be given in the plane, not all on a straight line. The shortest closed route connecting them is a simple polygon.

SOLUTIONS

The Sum of a Series

E 844 [1949, 31]. *Proposed by Orrin Frink, Pennsylvania State College*

Sum the series

$$1 + 1/5! + 1/10! + 1/15! + \cdots + 1/(5n - 5)! + \cdots .$$

I. *Solution by W. Fulks, University of Minnesota.* Consider the function

$$f(x) = \sum_{n=0}^{\infty} x^{kn}/(kn)!,$$

where k is a positive integer. We note that $f(x)$ satisfies the system $f^{(k)}(x) = f(x)$, $f(0) = 1$, $f'(0) = f''(0) = \cdots = f^{(k-1)}(0) = 0$. The unique solution of this system is

$$f(x) = (1/k) \sum_{j=1}^{k} \exp (w_k^j x),$$

where w_k is a primitive kth root of unity. Thus the required sum is, taking $x = 1$, $k = 5$,

$$S = (1/5) \sum_{j=1}^{5} \exp w_5^j.$$

II. *Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn.* Since the sum of the mth powers of the k kth roots of unity is zero unless m is a multiple of k, in which case the sum is unity, we have, from the Maclaurin expansion of e^z,

$$\sum_{j=1}^{k} \exp (w_k^j x) = k \sum_{n=0}^{\infty} x^{kn}/(kn)!,$$

where w_k is a primitive kth root of unity. The required sum, obtained by taking $x=1$, $k=5$, is then

$$S = (1/5)\sum_{j=1}^{5} \exp w_5^j.$$

Also solved by W. G. Brady, D. H. Browne, Paul Carnahan, P. L. Chessin, Ragnar Dybvik, J. L. Ericksen, Harley Flanders, H. E. Gould, Roger Lessard, D. C. B. Marsh, Leo Moser, C. S. Ogilvy, S. T. Parker, C. F. Pinzka, Alex Rosenberg, Azriel Rosenfeld, C. M. Sandwick, F. C. Smith, W. R. Talbot, an unsigned solver, and the proposer.

Several solvers easily reduced the above sum to

$$S = (1/5)\left[e + 2e^{\cos 72°}\cos(\sin 72°) + 2e^{-\cos 36°}\cos(\sin 36°)\right].$$

If we are merely interested in obtaining a numerical result, however, there is little point of transforming the original series, which converges very rapidly. Thus four terms of the series gives $S=1.00833360890$.

Smith picked up, as a by-product, the pretty summations

$$\sum_{n=0}^{\infty} (x^n \cos n\theta)/n! = e^{x\cos\theta}\cos(\sin\theta),$$

$$\sum_{n=0}^{\infty} (x^n \sin n\theta)/n! = e^{x\cos\theta}\sin(\sin\theta).$$

A Difficult Enumeration Problem

E 845 [1949, 31]. *Proposed by Joseph Rosenbaum, Hartford, Connecticut*

It is required to write the fifteen combinations of a, b, c, d in a sequence such that any two adjacent terms of the sequence shall differ by a single letter. How many such sequences are there? How can they be written down?

Partial solution by the Proposer. It is obvious that if a certain sequence of the combinations satisfies the requirement, then the corresponding sequence resulting from any permutation of a, b, c, d will also satisfy the requirement. Such a pair of sequences will be called *dependent*, and in what follows only *independent* sequences will be considered.

We shall now show how a number of the required independent sequences can be written down.

If a sequence C of the combinations satisfies the requirement, and if its first term is a single letter, then corresponding to C we shall associate the following sequence S of fifteen terms. The first term of S will be the same as the first term of C, and every subsequent term of S will be that letter in which the corresponding term of C differs from its preceding term. Thus the terms of S are single letters, and it is obvious how the sequence C can be written down when its associated S is given.

In view of the above it will suffice to show how to obtain a number of valid S sequences. For this purpose it is observed that if $S(n)$ is a sequence for n letters $a_1, a_2, \cdots, a_n$ whose corresponding sequence $C(n)$ satisfies the requirement of the problem, then the S sequence for $n+1$ letters $a_1, a_2, \cdots, a_n, a_{n+1}$, defined by

$$S(n+1) = S(n),\ a_{n+1},\ S(n), \tag{1}$$

will be a valid sequence for the $n+1$ letters. More generally, if $S_1(n)$ and $S_2(n)$ are each valid sequences for $a_1, a_2, \cdots, a_n$, then the sequence defined by

$$S(n+1) = S_1(n),\ a_{n+1},\ S_2(n) \tag{2}$$

is a valid sequence for $a_1, a_2, \cdots, a_n, a_{n+1}$.

Now, starting with the two sequences $S_1(2) = a_1, a_2, a_1$ and $S_2(2) = a_2, a_1, a_2$, even though they are not independent, there are obtained by means of relations (1) and (2) the two independent sequences

$$S_1(3) = a_1, a_2, a_1, a_3, a_1, a_2, a_1, \tag{3}$$

and

$$S_2(3) = a_1, a_2, a_1, a_3, a_2, a_1, a_2. \tag{4}$$

In addition it can be verified that

$$S_3(3) = a_1, a_2, a_3, a_2, a_1, a_2, a_3 \tag{5}$$

is also a valid sequence.

Next, in each of $S_1(3)$, $S_2(3)$, $S_3(3)$, write all the six sequences corresponding to the six permutations of a_1, a_2, a_3. Denoting the resulting sequences by A_i, B_i, C_i, $i=1, \cdots, 6$, there are obtained the eighteen (not independent) sequences $S(3)$:

$$\begin{array}{lll}
A_1 = a_1a_2a_1a_3a_1a_2a_1 & B_1 = a_1a_2a_1a_3a_2a_1a_2 & C_1 = a_1a_2a_3a_2a_1a_2a_3 \\
A_2 = a_1a_3a_1a_2a_1a_3a_1 & B_2 = a_1a_3a_1a_2a_3a_1a_3 & C_2 = a_1a_3a_2a_3a_1a_3a_2 \\
A_3 = a_2a_3a_2a_1a_2a_3a_2 & B_3 = a_2a_3a_2a_1a_3a_2a_3 & C_3 = a_2a_3a_1a_3a_2a_3a_1 \\
A_4 = a_2a_1a_2a_3a_2a_1a_2 & B_4 = a_2a_1a_2a_3a_1a_2a_1 & C_4 = a_2a_1a_3a_1a_2a_1a_3 \\
A_5 = a_3a_1a_3a_2a_3a_1a_3 & B_5 = a_3a_1a_3a_2a_1a_3a_1 & C_5 = a_3a_1a_2a_1a_3a_1a_2 \\
A_6 = a_3a_2a_3a_1a_3a_2a_3 & B_6 = a_3a_2a_3a_1a_2a_3a_2 & C_6 = a_3a_2a_1a_2a_3a_2a_1
\end{array}$$

By use of relation (2) these give the 54 independent sequences $S(4)$:

$$\begin{array}{lll}
A_1a_4A_i, & A_1a_4B_i, & A_1a_4C_i, \\
B_1a_4A_i, & B_1a_4B_i, & B_1a_4C_i, \\
C_1a_4A_i, & C_1a_4B_i, & C_1a_4C_i,
\end{array} \tag{6}$$

where $i=1, \cdots, 6$.

It is easily shown that combinations like $A_ia_4A_j$, or $A_ia_4B_j$, *etc.*, or $B_ia_4A_j$, *etc.*, or $C_ia_4A_j$, *etc.*, will not give any independent sequences in addition to those of (6).

We obtain, in this manner, 54 independent solutions to the problem. Clearly these do not include all the required solutions to the problem, because a C sequence corresponding to an S sequence has its first term consist of a single letter, and there exist C sequences whose first terms contain more than a single letter. Also, for that matter, our set of 54 S sequences does not even contain all independent S sequences. The writer has discovered a transformation which, when applied to some of the S sequences in (6), yields new independent sequences.

Editorial Note. The enumeration of permissible C sequences, particularly for the general case of n letters, seems to be difficult, and further communications will be welcome. As a somewhat similar, and perhaps equally difficult, enumeration problem we might here mention the following: Let P_1 be a permutation of n objects. In how many ways can we form a sequence of $n!$ distinct permutations, $P_1, P_2, \cdots, P_{n!}$, such that, for $1 \leq j < n!$, P_{j+1} is obtained from P_j by a single transposition?

Definition of Principal Part

E 846 [1949, 31]. *Proposed by H. J. Hamilton, Pomona College*

The following is typical of many characterizations of the principal part of an infinitesimal which are to be found in elementary calculus texts.

"If an infinitesimal consists of two or more terms of different orders, the term of lowest order is called the *principal part* of the infinitesimal."

Show that this is not definitive and give a valid definition.

Solution by the Proposer. Let α be an infinitesimal and put

$$\beta = \alpha + (\alpha^2 + \alpha^3) = (\alpha + \alpha^2) + \alpha^3.$$

Then, according to the "definition" in the problem, the principal part of β is both α and $(\alpha + \alpha^2)$, each being read off from the corresponding expression above.

A somewhat more subtle example is

$$\beta = \alpha + \alpha^2 - 2\alpha \sin^2 \alpha = \alpha \cos 2\alpha + \alpha^2,$$

in which both α and $\alpha \cos 2\alpha$ satisfy the faulty definition.

An alternative, definitive, characterization of *principal part* follows. *If α and β are simultaneous infinitesimals and if, for some positive integer n,* $\lim_{\alpha\to\infty} (\beta/\alpha^n) \equiv c$ *exists, is finite, and is not zero, then $c\alpha^n$ is called the* principal part *of β relative to α.* It may be observed that the quantity $(\beta - c\alpha^n)$ is of higher order than β.

Also solved by Leo Moser.

Area in the Ambiguous Case

E 847 [1949, 179]. Corrected Statement. *Proposed by Albert Newhouse, University of Houston*

Let a, b, A be the given parts of a triangle in the ambiguous case. Show that the area of the triangle is given by

$$K = \tfrac{1}{2}b \sin A\left[b \cos A \pm (a^2 - b^2 \sin^2 A)^{1/2}\right].$$

Solution by P. W. A. Raine, Newport News High School. Assuming $b \sin A < a < b$, $A < 90°$, let x be the altitude to side c and let y and z be the projections of b and a on side c. Then

$$K = \tfrac{1}{2}x[y \pm z] = \tfrac{1}{2}b \sin A\left[b \cos A \pm (a^2 - b^2 \sin^2 A)^{1/2}\right].$$

Also solved by R. V. Andree, W. R. Beck, Louis Berkofsky, Fannie Boyce, W. G. Brady, D. H. Browne, W. E. Buker, P. L. Chessin, Monte Dernham, Ragnar Dybvik, M. D. Eulenberg, S. E. Field, H. E. H. Greenleaf, Vern Hoggatt, C. H. Holton, R. T. Hood, Albert Hopkins, B. C. Horne, Jr., M. S. Klamkin, Bill Krause, Sam Kravitz, N. D. Lane, Joseph Langr, H. L. Lee, H. R. Leifer, Roger Lessard, Jere Lundholm, R. V. B. Lynch, D. C. B. Marsh, Alta McColl, Leo Moser, C. S. Ogilvy, Mary Payne, C. F. Pinzka, J. W. Ponds, C. C. Richtmeyer, Azriel Rosenfeld, C. M. Sandwick, W. E. Schmitt, N. C. Scholomiti, R. W. Shoemaker, Kirk Stewart, Adrian Struyk, Kaidy Tan, W. R. Talbot, P. D. Thomas, H. B. Thornton, C. W. Trigg, E. H. Vance, A. A. Vuylsteke, Margaret Willerding, Maud Willey, G. A. Williams, Roscoe Woods and the proposer.

E. P. Starke raised the interesting and allied problem of finding those combinations of sides and angles which determine, uniquely or ambiguously, a quadrilateral.

Numbers Decomposable into Sum of Two Abundant Numbers

E 848 [1949, 31]. *Proposed by Leo Moser, University of Manitoba*

Prove that every integer greater than 10^5 can be expressed as the sum of two abundant numbers.

Solution by the Proposer. It is well known that if $(a, b) = 1$ and $n > ab$, then there are positive integers x and y such that $ax + by = n$ (*cf.* Polya and Szegö, *Aufgaben und Lehrsatze aus der Analysis*, vol. 1, p. 4). It is further clear that if a is abundant then so is ax. Now $88 = (2^3)(11)$ and $945 = (3^3)(5)(7)$ are relatively prime abundant numbers, so that every integer greater than (88)(945) can be expressed in the form $88x + 945y$, thus proving the theorem.

Editorial Note. One naturally wonders what is the largest integer not the sum of two abundant numbers.

The two numbers 88 and 945 are the best possible for the proposer's proof inasmuch as 945 is the first odd abundant number and 88 is the smallest abundant number prime to 945. For the abundant numbers not exceeding 10^4 see Glaisher, *Number-divisor Tables*, vol. 8, British Association Mathematical Tables, Cambridge, 1940.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results found in readily accessible sources should not be proposed for this department.

Problems for Solution

4352 [1949, 414]. Corrected Statement. *Proposed by Paul Erdös, Syracuse University*

Denote by $f(n; a_1, a_2, \cdots, a_k)$ the number of positive integers $m \leqq n$ which are either divisors or multiples of one of the a's $(1 < a_i \leqq n)$. Prove that

$$f(n; a_1, a_2, \cdots, a_k) \leqq f(n; 2, 3, \cdots, p_k),$$

where $2, 3, \cdots, p_k$ are the first k primes.

4355. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Solve the equation

$$5m^2 + 2m + 1 = n^2$$

in integers and show that, provided $m \equiv -1 \bmod 3$, there exists in every system of numeration of base $B = 3m+1$ at least one pair of perfect squares having the form $aabb = (cc)^2$, $bbaa = (dd)^2$. There are infinitely many such systems of numeration. (See also the Proposer's paper, *Concerning two classes of remarkable perfect square pairs*, in this issue of the MONTHLY.)

4356. *Proposed by P. A. Piza, San Juan, Puerto Rico*

Prove the relations:

$$\text{(a)} \qquad x^{2n+1} - (x-1)^{2n+1} = \sum_{a=0}^{n} \left[\binom{n+a}{2a+1} + \binom{n+1+a}{2a+1} \right] (x^2 - x)^{n-a}$$

$$\text{(b)} \qquad x^{2n+2} - (x-1)^{2n+2} = (2x-1) \sum_{a=0}^{n} \binom{n+1+a}{2a+1} (x^2 - x)^{n-a}.$$

4357. *Proposed by R. D. Stalley, Stanford University*

Reduce the problem of summing the series

$$\sum_{k=1}^{\infty} k^{-n} x^k,$$

where n is a positive integer $\geqq 2$ to numerical integration of a function over a finite range. (Compare the Proposer's paper, *A Generalization of the Geometric Series* [1949, 325].)

4358. *Proposed by Paul Erdös, Syracuse University*

Let $a_1 < a_2 < \cdots$ be a sequence of integers with the property that there does not exist an infinite subsequence of the a's in which no one divides another. Prove that the products

$$a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k}, \qquad 0 \leqq \alpha_i$$

have the same property.

If the a's are chosen to be the first k primes, then we obtain a well-known result due to Dickson.

4359. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If in a tetrahedron one draws lines through the vertices parallel to a given direction Δ and then locates the homothetics of the intersections of these lines with the circumsphere with respect to the centroids of the corresponding faces, in the ratio $-\frac{1}{2}$, the four points so obtained lie in a plane perpendicular to Δ and passing through the Monge point of the tetrahedron. (Compare problem 4233 whose solution appears in this issue of the MONTHLY.)

Solutions

Second Point of Lemoine

4204 [1946, 278]. Corrected Statement. *Proposed by Victor Thébault, Tennie, Sarthe, France.*

In a tetrahedron $T \equiv ABCD$, the tangents of the half-angles $\alpha, \beta, \gamma, \delta$ at the vertices A, B, C, D of the cones inscribed in the trihedral angles (A), (B), (C), (D), are proportional to the barycentric coördinates of the second point of Lemoine of the tetrahedron $T' \equiv A'B'C'D'$ having for vertices the points of contact of the inscribed sphere with the faces BCD, CDA, DAB, ABC.

*Solution by the Proposer.** We shall designate by a, a', b, b', c, c', the dihedral angles of edges BC, DA, CA, DB, AB, DC as well as the lengths of these edges; by A, B, C, D the areas of the faces BCD, CDA, DAB, ABC; and by V the volume of T.

1. We first prove the theorem: The barycentric coördinates of the second point of Lemoine L of a tetrahedron (with this tetrahedron as a reference tetrahedron) are inversely proportional to the normal coördinates of L with respect to the tangential tetrahedron.†

Consider the tetrahedron $T \equiv ABCD$, its circumsphere (O, R), the antiparallel section $A''B''C''$ in the trihedral angle (D) which contains L, and the point D_1 in which DL meets (O, R) again.

$$DL \cdot DD_1 = 2Rt = a'b'c'/k$$

* Translated by W. E. Byrne, Virginia Military Institute.

† V. Thébault, Bull. de la Soc. math. de France, 1948, p. 101.

where k is a proportionality factor,§ and x, y, z, t are the distances from L to the faces of the tangential tetrahedron of T. We have therefore

$$ab'c'x = a'bc'y = a'b'cz = abct.$$

As the normal coördinates of L with respect to T are proportional to (R_1, R_2, R_3, R_4), where R_1 is the radius of the circumcircle of triangle BCD, etc., the barycentric coördinates of L with respect to T are proportional to $(ab'c', a'bc', a'b'c, abc)$, and the theorem follows.

2. If we apply this theorem to $T' \equiv A'B'C'D'$, the normal coördinates (x, y, z, t) with respect to T of the second Lemoine point L' of T' are proportional to

$$\cos \tfrac{1}{2}a \cos \tfrac{1}{2}b' \cos \tfrac{1}{2}c', \qquad \cos \tfrac{1}{2}a' \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c',$$
$$\cos \tfrac{1}{2}a' \cos \tfrac{1}{2}b' \cos \tfrac{1}{2}c, \qquad \cos \tfrac{1}{2}a \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c.$$

3. If r is the radius of the circumsphere of $A'B'C'D'$, we obtain||

$$\tan \alpha = r/AB' = \sin \tfrac{1}{2}b \sin \tfrac{1}{2}c \sin \sphericalangle CAB/\cos \tfrac{1}{2}a'.$$

Furthermore¶

$$V = Dh_a/3, \qquad D = \tfrac{1}{2}bc \sin \sphericalangle CAB, \qquad \sin b = bh_bh_a/6V = 3bV/2BD.$$

Hence $\sin \sphericalangle CAB = 2D/bc = 9V^2/2B \cdot C \cdot D \cdot \sin b \sin c$, and

$$\tan \alpha = \frac{9V^2 A \cos \frac{1}{2}a \cos \frac{1}{2}b' \cos \frac{1}{2}c'}{8A \cdot B \cdot C \cdot D \cos \frac{1}{2}a \cos \frac{1}{2}a' \cos \frac{1}{2}b \cos \frac{1}{2}b' \cos \frac{1}{2}c \cos \frac{1}{2}c'}$$

and

$$\frac{Ax}{\tan \alpha} = \frac{By}{\tan \beta} = \frac{Cz}{\tan \gamma} = \frac{Dt}{\tan \delta}.$$

Note. We obtain thus a complete analogy between the triangle and the tetrahedron. The point L' could be called the Gergonne point of T. R. Bouvaist has indicated other interesting properties of L' in *Mathesis*, Supplément, t. 54, p. 17.

Homothetic Tetrahedrons

4233 [1947, 49]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Parallel lines, of arbitrary direction, through the vertices A', B', C', D' of a tetrahedron $A'B'C'D'$ intersect in A_1, B_1, C_1, D_1 the faces BCD, CDA, DAB, ABC of a homothetic tetrahedron $ABCD$. If k is the homothetic ratio, V the volume of $ABCD$, and V' that of $A_1B_1C_1D_1$, then

$$V_1 = -k^2(2k+1)V.$$

§ R. Bouvaist, Mathesis, t. 55, 1945–46, p. 352.

|| Weber-Wellstein, Enzyklopädie der Elementar-Mathematik, Bd. II, 3 Aut., p. 429.

¶ N. A. Court, Modern Solid Geometry, p. 88.

*Solution by the Proposer.** We choose the homothetic center O of the tetrahedrons $T\equiv ABCD$, $T'\equiv A'B'C'D'$ as the origin of a system of rectangular coordinate axes $Oxyz$ with Oz parallel to the direction d of the parallel lines in question. Let x_i, y_i, z_i, $(i=1, 2, 3, 4)$, designate the coördinates of A, B, C, D and kx_i, ky_i, kz_i those of A', B', C', D'. If V is the volume (with sign) of the tetrahedron T, we have $6V=|x_1y_1z_11|=X_1x_1+Y_1y_1+Z_1z_1+T_1$, where $|X_1Y_1Z_1T_1|$ is the adjoint determinant of $|x_1y_1z_11|$. The equation of the plane BCD is $X_1x+Y_1y+z_1z+T_1=0$. The coördinates of A_1 are kx_1, ky_1, $-(kX_1x_1+kY_1y_1+T_1)/Z_1$. It follows that $6V_1=-k^2|Z_1x_1\ \ Z_1y_1\ \ kX_1x_1+kY_1y_1+T_1\ \ Z_1|/Z_1Z_2Z_3Z_4 = -k^2(k\sum X_ix_i+k\sum Y_iy_i+\sum T_i)=-k^2(12kV+6V)$. Hence

$$V_1 = -k^2(2k+1)V. \tag{1}$$

As $k^3=V'/V$, elimination of k from (1) gives

$$V_1 = -\sqrt[3]{V'^2}(\sqrt[3]{V}+2\sqrt[3]{V'}). \tag{2}$$

It is assumed in the calculations that the direction d is not parallel to the faces of T.

COROLLARY 1. *Parallel lines of arbitrary direction drawn through the vertices of a tetrahedron T meet the opposite faces in A_1, B_1, C_1, D_1. The volume of tetrahedron $A_1B_1C_1D_1$ is three times that of T and of opposite orientation.*[1] *Here $A'\equiv A$, $B'\equiv B$, $C'\equiv C$, $D'\equiv D$, $k=1$, and $V_1=-3V$.*

COROLLARY 2. *Parallel lines of arbitrary direction drawn through the vertices A', B', C', D' of the tetrahedron T' meet the corresponding faces of the tetrahedron T inversely equal to T' in A_1, B_1, C_1, D_1. The volumes of T and $A_1B_1C_1D_1$ are equivalent.*[2] *In fact, $k=-1$ and $V_1=V$.*

COROLLARY 3. *If the homothetic ratio $k=-\frac{1}{2}$, A_1, B_1, C_1, D_1 are coplanar since $V_1=0$.*

Note. In *Mathesis*, t. 55, p. 380, there appears the statement of a generalization by René Blanchard (Le Havre, France) of our question 3231, as follows: Parallel lines, of arbitrary direction d, drawn through the vertices A', B', C' of a triangle $A'B'C'$ meet the sides BC, CA, AB of a homothetic triangle ABC in A_1, B_1, C_1. If k is the homothetic ratio, S the area (with sign) of ABC and S_1 the area of $A_1B_1C_1$, then $S_1=-k(k+1)S$. Our problem 4233 is a generalization of Blanchard's problem.

Rotating Elastic String

4282 [1948, 100]. *Proposed by F. W. Herlihy, Comstock, Michigan*

An elastic string (modulus λ, mass ma, unstretched length a) is confined within a straight tube to one end of which it is fastened. The tube rotates around

* Translated by W. E. Byrne, Virginia Military Academy.

[1] Malet, Mathesis, 1885, p. 94; 1890, p. 253.

[2] V. Thébault, Mathesis, t. 55, question 3272, p. 139.

that end with uniform angular velocity ω in a horizontal plane. Find the length of the string in equilibrium.

Solution by Wong Foh Pao, La Universitato Utopia, Shanghai, China. Suppose the tube is perfectly smooth and so narrow that it can just contain the string.

Let ds be the element of the string after stretching and ρ be the density of ds, and let ds_0 be the original length of ds. (s is measured from the fixed end.) Because the mass is unaltered by stretching,

$$\rho ds = m ds_0, \tag{1}$$

and from Hooke's law,

$$\lambda(ds - ds_0)/ds_0 = T,$$

where T is the tension of the string at the position of ds in equilibrium. Since the string is in equilibrium the resultant force acting on ds must be zero,

$$dT/ds = -\rho s\omega^2. \tag{3}$$

Elimination of ρ and s_0 between (1), (2) and (3) gives

$$\frac{dT}{ds} = -\frac{ms\omega^2}{1 + T/\lambda}. \tag{4}$$

An easy integration gives

$$2T + T^2/\lambda = -ms^2\omega^2 + 2H + H^2/\lambda, \tag{5}$$

where the constant of integration is so chosen that $T = H$ when $s = 0$.

s and ρ can be eliminated between (1), (2), (3) and (5) with the result

$$\frac{dT}{ds_0} = -\frac{\sqrt{m}\,\omega}{\sqrt{\lambda}}\sqrt{2\lambda(H - T) + H^2 - T^2},$$

whereupon integration gives

$$\cos^{-1}\frac{\lambda + T}{\lambda + H} = \frac{\sqrt{m}\,\omega}{\sqrt{\lambda}}\, s_0. \tag{6}$$

Here the constant of integration has been chosen so that $T = H$, $s_0 = 0$.

If we assume the length of the string after stretching is L, and put $s = L$ and $T = 0$ (that is at the free end of the string), then (5) becomes

$$mL^2\omega^2 = 2H + H^2/\lambda. \tag{7}$$

In (6) put $s_0 = a$, $T = 0$ (at the free end) to get

$$\frac{\lambda}{\lambda + H} = \cos\frac{a\sqrt{m}\,\omega}{\sqrt{\lambda}}. \tag{8}$$

Finally, elimination of H between (7) and (8) gives the desired result

$$L = \frac{1}{\omega}\sqrt{\frac{\lambda}{m}}\tan\frac{\sqrt{m}\,\omega a}{\sqrt{\lambda}}\cdot$$

Note that ω cannot exceed $\sqrt{\lambda}\,\pi/2a\sqrt{m}$. From (8), if $\omega=\sqrt{\lambda}\,\pi/2a\sqrt{m}$, since $\lambda\neq 0$, $H=\infty$ and from (5) we know that the maximum value of T is H; hence breakage must occur at the fixed end before ω reaches the value $\sqrt{\lambda}\,\pi/2a\sqrt{m}$.

Also solved by I. E. Highberg, Mary H. Payne, F. Underwood, and the Proposer.

Curves Along Which $\bar{z}$ Coincides With An Analytic Function

4283 [1948, 100]. *Proposed by E. P. Starke, Rutgers University*

The conjugate $\bar{z}$ of z, considered as a function of z, is nowhere analytic. Nevertheless, if C is an arbitrary circle or line, there exists a function $f(z)$ such that at every finite point of $C, f(z)$ is analytic and equal to $\bar{z}$. Consider also other curves for which a function exists having the same property.

Solution by Fritz Herzog, Michigan State College. If C is the circle with center at any point a and with radius r, let $f(z)=\bar{a}+r^2/(z-a)$. Then for $|z-a|=r$ we have $f(z)=\bar{a}+\overline{z-a}=\bar{z}$.

If C is the line having slope $\tan\alpha(0<\alpha<\pi)$ and passing through the real point x_0, let $f(z)=x_0+e^{-2i\alpha}(z-x_0)$. Then for $z=x_0+e^{i\alpha}t$, with real t, we have $f(z)=x_0+e^{-i\alpha}t=\bar{z}$.

If C is the line parallel to the real axis and passing through the point iy_0 (with real y_0) let $f(z)=z-2iy_0$. Then for $z=x+iy_0$, with real x, we have $f(z)=x-iy_0=\bar{z}$.

In general, let C be any analytic arc, i.e., let C be the mapping of a (finite or infinite) interval $[\alpha, \beta]$ of the real axis by means of an analytic function $\phi(z)$, which is assumed to be schlicht in a sufficiently small (complex) neighborhood of $[\alpha, \beta]$. Then the function $\psi(z)=\overline{\phi(\bar{z})}$ is analytic in a neighborhood of $[\alpha, \beta]$ and maps $[\alpha, \beta]$ on $\overline{C}$, the reflection of C about the real axis. Let $\phi^{-1}(z)$ be the inverse function of $\phi(z)$ and put $f(z)=\psi(\phi^{-1}(z))$. Then $f(z)=\bar{z}$ for z on C.

As an example, consider the parabola whose vertex is at $z=-1$ and whose focus is at $z=0$. (Every other parabola in the plane can be mapped into the given one by a linear mapping $Az+B$.) In this case, in the above notation, $\phi(z)=(z+i)^2$, and the above procedure leads to the function $f(z)=(z^{1/2}-2i)^2$, which is single-valued and analytic in a sufficiently small neighborhood of the given parabola; $z^{1/2}$ is to be chosen as $+i$ at $z=-1$.

Editorial Note. If the analytic arc C is given parametrically by $x=x(t)$, $y=y(t)$, a differential equation for the determination of $f(z)$ is easy to obtain. The desired function $f(z)$ is to be such that

$$f(z) = x(t) - iy(t) \tag{1}$$

whenever $z=x+iy$ is a point on C. Furthermore

$$f'(z) = \frac{x'(t) - iy'(t)}{x'(t) + iy'(t)} \tag{2}$$

holds true so long as z remains on C. The required differential equation results from the elimination of t between (1) and (2).

Euler's Constant

4286 [1948, 165]. *Proposed by H. F. Sandham, Trinity College, Dublin, Ireland.*

Prove that

$$\int_0^\infty \frac{\cos x^2 - \cos x}{x} dx = \tfrac{1}{2}\gamma,$$

where γ is Euler's constant.

Solution by M. S. Klamkin, Brooklyn Polytechnic Institute, Brooklyn, N. Y. Starting with the known relation for γ,

$$\gamma = \int_0^1 (1 - \cos y)\frac{dy}{y} - \int_1^\infty \cos y \frac{dy}{y}$$

(see Franklin, *Treatise on Advanced Calculus*, p. 571, eq. 174) we get

$$\left(\frac{1}{r} - \frac{1}{s}\right)\gamma = \int_0^1 (1 - \cos y)\frac{dy}{ry} - \int_0^1 (1 - \cos y)\frac{dy}{sy}$$
$$- \int_1^\infty \cos y \frac{dy}{ry} + \int_1^\infty \cos y \frac{dy}{sy}\cdot$$

Let $y = x^r$ in the first and third integrals and $y = x^s$ in the others, $r, s > 0$. Then

$$\frac{r - s}{rs}\gamma = \int_0^\infty \frac{\cos x^r - \cos x^s}{x} dx.$$

The proposed result is the special case, $r = 2$, $s = 1$.

Also solved by Robert Breusch, Hwang Cheng-Chung, F. J. Duarte, H. E. Fettis, Philip Franklin, R. O. French, W. J. Harrington, Frank Herlihy, J. G. Herriot, Fritz Herzog, S. Katz and A. M. Peiser, W. H. Marlow, Norman Miller, C. D. Olds, J. H. Simester, R. S. Smith, E. Trost, C. B. Walton, and the Proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

An Introduction to Mathematics. By A. N. Whitehead. Twelfth Impression; First American Edition. New York, Oxford University Press, 1948. 6+191 pages. $2.00.

The first printing of Whitehead's classic was mentioned briefly in volume 18 (1911) of this MONTHLY. A more adequate review by R. D. Carmichael appears on pages 282–283 of volume 20 (1913). His hope for a wide circulation for the book has been fulfilled, and it has been reprinted many times since then.

Now, thirty-five years later, a new edition has appeared under the editorial supervision of the author's nephew, J. H. C. Whitehead. The text has been completely reset in smaller type, the diagrams have been redrawn and a few minor errors have been corrected. The contents of the original 256 pages have been compressed into 191, but to compensate for this the price has been increased from the original seventy-five cents.

To readers of more modern books on the significance of Mathematics, Whitehead's *Introduction* may seem a bit old-fashioned. This is particularly apparent in its discussion of applications to physical problems. There is no mention of radio or of television or of the atomic bomb! But the discussion of purely mathematical concepts is as effective as it was when the book first appeared.

Changes in this edition have been kept at a minimum. New footnotes on pages 41, 99, and 152 and a new paragraph on page 105 serve to clarify passages which might be obscure to a non-mathematical reader. It is unfortunate that more changes were not made. For example, we are told (page 55) that Georg Cantor is still living, although the editor was careful on page 111 to refer to the "former" London and North-Western Railway.

Numerous misprints serve to confuse an unwary reader. The omission of an entire line at the bottom of page 114 renders an example meaningless. Misprints of varying degrees of seriousness occur on pages 118, 136, 146 and 157. An up-to-date reading list is given in the Bibliography, but here again an error occurs (page 188) in the initials of J. W. Young.

In spite of these minor flaws, we believe (to quote the previous review) that "the book deserves a wide circulation."

H. M. GEHMAN

Mathematics of Finance. By P. M. Hummel and C. L. Seebeck, Jr. New York, McGraw-Hill Book Co., 1948. 7+365 pages. $4.00.

A few months ago the reviewer became involved in an argument with a

non-mathematician concerning some now-forgotten point in practical finance. Calling to mind the reverence of the uninitiated for the printed word, he picked up the present book, which had remained for some time unopened on his shelves, surrounded by its many humble relatives. The argument was quickly settled, of course. At no little surprise to himself, however, the reviewer spent the next hour in reading the Hummel-Seebeck text with genuine interest and appreciation.

The excellent format of the book contributes no little to the good impression. A more important factor, nevertheless, is the clarity and simplicity of the writing. It would seem that the authors conceived of their readers as people of intelligence: too intelligent to be satisfied by confused explanations and intelligent enough to appreciate carefully stated principles. Perhaps the best chapter of the book is Chapter III (Equations of Equivalence) in which, for example, the authors define equivalence of "dated values" and go on to prove transitivity of equivalence. It is also encouraging to note that general annuities are handled by reducing them to simple annuities, thereby keeping the emphasis on general principles and avoiding a whole complex of complicated formulas.

The book seems eminently suited for students with a little knowledge of elementary algebra. It abounds in illustrative examples and problems. There are appendices on abridged multiplication, common logarithms and progressions, as well as fifteen double entry tables. Almost the only error noted occurs on page 203, where two headings in an illustrative table have been interchanged. The following list of chapter headings sufficiently indicates the contents: I. Simple Interest. II. Compound Interest. III. Equation of Equivalence. IV. Simple Annuities. V. Ordinary General Annuities. VI. Perpetuities. VII. Amortization and Sinking Funds. VIII. Bonds. IX. Depreciation. X. General Annuities—Advanced Topics. XI. Approximating Methods. XII. Life Annuities and Life Insurance.

R. H. Bruck

Analytic Geometry. By P. R. Rider. New York, The Macmillan Co., 1947. 10+383 pages. $3.25.

This text is suitable for a brief or an extended course in analytic geometry. In addition to eleven chapters on the traditional topics, the book contains a chapter on curve fitting, three chapters on space analytic geometry, ten pages of tables, an adequate index, and answers to the odd numbered problems.

As might be expected from this author, the exposition is carefully made and well-motivated. To this reviewer the arrangement of topics seems quite satisfactory. The lists of exercises are more useful than in the author's well known algebras, for here there are long lists of the simpler problems and only a reasonable number of the more difficult problems. Since, on the whole, the 203 figures are excellently done and will, by themselves, arouse considerable student interest, it is unfortunate that some of these contain misleading errors, such as Figures 167 and 170. The illustrative examples well merit the author's claim

that they have been chosen with extreme care to contribute as much as possible to student understanding of the principles of the corresponding sections.

In summary, this Rider text certainly deserves consideration by those who are looking for a fresh, adequate, teachable treatment of analytic geometry.

B. M. STEWART

Analytic Geometry. By P. K. Rees and E. D. Mouzon. New York, The Dryden Press, 1948. 28+305 pages. $2.75.

This is a textbook presenting the standard topics of analytic geometry. It is well arranged, quite legible, and its figures are quite clear.

At the beginning of the text there is included a review of the algebra, trigonometry, and geometry necessary as a background. This material is arranged according to subject matter, and again according to the article in which it is used. In the second listing the number of the article appears before the material making it easy to use for reference purposes.

The central conics are treated in parallel columns on the same pages. In this manner it is easier to note the similarities between them. The problem of determining the points of intersection of two curves when their equations are given in polar form is treated much more exhaustively than in most texts. The work on loci and equations is introduced at an early stage and is carried out in good detail. In this connection it is felt that a more thorough treatment might have been given to the sketching of the graphs of polar equations. For example, no mention is made of the methods of finding tangents at the pole, and the polar equations of the conics are assigned as a single problem without discussion.

Two chapters on solid analytic geometry are included in the text. These chapters provide an introduction to the subject and cover planes, lines, and quadric surfaces.

This text is quite suitable for the average student who requires a background in the fundamentals of analytic geometry rather than the student who would like to delve more deeply into the subject. The review at the beginning of the text is quite helpful, although a mention of radian measure might have been made. Also it is felt that some material on the sketching of surfaces other than the quadrics, and of solids bounded by two or more surfaces would have added to the usefulness of the text.

Few inaccuracies occur in the text. The most conspicuous error is in the definition of logarithms given on page 168.

J. H. BELL

Solid Geometry. By J. S. Frame. New York, McGraw-Hill Book Co., 1948. 9+339 pages. $3.50.

While this text can be used profitably in high schools, it is also a book upon which a course in solid geometry of undeniably college caliber can be based. In fact, because of the more than generous list of topics covered and their convenient arrangement, any one of several courses can be given with this

book as text. If the instructor prefers, the course can be handled in the more or less traditional fashion as a succession of definitions and theorems and proofs of theorems: the text is admirable for this purpose in that it furnishes adequate examples but leaves many of the proofs as exercises. It is possible, also, to de-emphasize theorem-proving and to give more attention to solid mensuration and to the drawing of two-dimensional representations of three-dimensional figures. A feature of the book is the emphasis placed on the last-mentioned topic. Methods and aids are presented throughout Parts I, II and III, and Part IV is devoted exclusively to projections and maps. A patented device known as the trimetric ruler is provided with the text to facilitate the construction of drawings from which three-dimensional figures can be readily visualized. By completing the text in fifty college class hours an instructor should be able to present a course which will train the student to make careful proofs of theorems, equip him with detailed knowledge of solid mensuration, and give him far greater competence in drawing than is usually the case.

Particularly gratifying are the author's definitions, which are frequently "tighter" than those usually found. Examples are his "collinear," "line segment," "congruent figures," "straight line," "on the same side of," "in the same direction," and "plane." Praiseworthy, too, is his employment of certain expressions which facilitate the statement of definitions and theorems: for example, "mediator of a line segment" takes the place of "plane which bisects and is perpendicular to the line segment."

One might wish that the author, in his laudable attempt to decrease the ambiguity resulting from the multiple meanings of such expressions as "dihedral angle," had not used the words "dihedron" and "trihedron," since their most readily suggested generalizations clash with the traditional meaning of "polyhedron." Also, two definitions of "plane perpendicular to plane" are given, one on page 36 and the second on page 51, their equivalence being stated as a theorem whose proof is set as an exercise. The obvious suggestion is that one be stated as a definition and the other as a theorem. Some might wish, also, that the appearance of Cavalieri's Theorem had been postponed as in traditional texts until after a few volume proofs had been given without it or with proven special cases of it.

The book is singularly free of typographical errors, and careful attention has been paid to grammar.

This reviewer is most happy that this text has appeared. He has used it as a reference work in a college course in solid geometry, and recommends it highly for its rigor, the breadth and detail of its coverage, the convenient arrangement of its topics, and its generous lists of well-chosen exercises.

L. D. RODABAUGH

NEW BOOKS RECEIVED

Arithmetic for Teacher-Training Classes. 3rd Edition. By E. H. Taylor and C. N. Mills. New York, Holt, 1949. 6+441 pp. $3.00.

Smithsonian Elliptic Functions Tables. Vol. 109. By G. W. Spenceley and R. M. Spenceley. Washington, D. C., Smithsonian Institution, 1947. 4+366 pp.

Freshman Mathematics. 3rd Edition. Revised by C. V. Newsom. New York, Rinehart, 1949. 16+559 pp. $5.00.

Numbers We See. By A. Riess, M. L. Hartung, and C. Mahoney. New York, Scott, Foresman, 1948. 162 p. $1.32.

Your Mathematics. By G. E. Hawkins and G. Tate. New York, Scott, Foresman, 1948. 592 pp. $2.20.

Concepts of the Calculus. By C. B. Boyer. Reprint, Foreword by R. Courant. New York, Hafner, 1949. 12+346 pp. $5.50.

The Mathematical Analysis of Logic. By George Boole. Reprint. New York, Philosophical Library, 1948. 6+82 pp. $3.75.

On the Theory of Stochastic Processes and Their Application to the Theory of Cosmic Radiation. By N. Arley. New York, Wiley, 1949. 240 pp. $5.00.

Klassische Funktionentheorie. Vol. 2. By H. Behnke and F. Sommer. Münster, Aschendorffsche, 1948. 8+234 pp. DM 6—.

Introduction to Analytic Geometry and the Calculus. By H. M. Dadourian. New York, Ronald Press, 1949. 10+246 pp. $3.25.

Five-figure Tables of Mathematical Functions. 2nd Edition. By J. B. Dale. London, Arnold, 1949. 8+121 pp. $1.50.

The Life and Works of Herbert Ellsworth Slaught. By H. J. Dark. Nashville, George Peabody College for Teachers, 1948. 8+152 pp.

Kurvenintegrale und Begründung der Funktionentheorie. By L. Heffter. Berlin, Springer, 1948. 4+48 pp. DM 5.40.

Rank Correlation Methods. By M. G. Kendall. London, Griffin, 1948. 8+160 pp. 18s.

Analytic Geometry. By A. L. Nelson, K. W. Folley and W. M. Borgman. New York, Ronald Press, 1949. 8+215 pp. $3.00.

Fehlertheorie und Ausgleichung von Rautenketten in der Nadirtriangulation. (Veröffentlichungen de Geodätischen Institutes in Potsdam, no. 1). By K. Reicheneder. Berlin, Akademie, 1949. 8+98 pp. DM 11—.

Vektorrechnung. Vol. 2. By F. K. Schmidt. Münster, Aschendorffsche, 1948. 8+244 pp. DM 6.40.

Calculus. By L. L. Smail. New York, Appleton-Century Crofts, 1949. 16+592 pp. $4.50.

Solid Analytic Geometry. By A. Albert. New York, McGraw-Hill, 1949. 10+162 pp. $3.00.

Commercial Algebra. By C. Bell and L. J. Adams. New York, Holt, 1949. 8+304 pp. $2.75.

Mathematics of Finance. By C. Bell and L. J. Adams. New York, Holt, 1949. 8+366 pp. $2.75.

You Can't Win. By E. E. Blanche. Washington, Public Affairs Press, 1949. 155 pp. $2.00.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

KAPPA MU EPSILON CONVENTION

The national mathematics fraternity, *Kappa Mu Epsilon*, held its seventh Biennial Convention at Topeka, Kansas on April 10–12, 1949 with *Kansas Delta* chapter of Washburn Municipal University acting as host. There were 184 delegates in attendance representing 35 chapters.

Addresses were given by Dr. L. M. Graves, University of Chicago, who spoke on *The role of generalization and abstraction in mathematics*, and Dr. G. B. Price, University of Kansas, who spoke on *Some famous problems of modern mathematics*. Papers were also presented by nine student members, while eleven other student papers were presented by title.

Two new chapters, Central College of Fayette, Missouri, and Mississippi Southern College of Hattiesburg, Mississippi were admitted to the fraternity. This brings the total number of chapters to forty-two.

National officers elected for 1949–51 were: President, H. Van Engen, Iowa State Teachers College; Vice-President, H. D. Larsen, Albion College; Secretary, E. Marie Hove, Hofstra College; Treasurer, L. F. Ollmann, Hofstra College; Historian, C. C. Richtmeyer, Central Michigan College. Dr. H. D. Larsen was reappointed to the editorship of the *Kappa Mu Epsilon* publication—*The Pentagon*.

CLUB REPORTS, 1947–48

Kappa Mu Epsilon, Northeastern State College

Titles of papers read at meetings of the *Oklahoma Alpha* Chapter of *Kappa Mu Epsilon* include:

Astronomy, by Dean L. P. Woods

Solution of higher equations, by James Barringer

Teaching secondary mathematics, by Prof. Vella Frazee

Discussion and solution of determinants, by Charles Brown and Thomas Summers

Exponential functions, e, by J. B. Willis

Moments of magic, by Prof. Ross Anderson, who spoke at the annual Founder's Banquet.

The officers for 1947–48 were: President, Robert Johnston; Vice-President Kermit Stuart; Treasurer, William Spicer; Secretary, Nellie Elledge.

Mathematics Club, Illinois Institute of Technology

The *Mathematics Club* of IIT reports a larger interest and attendance at the meetings during the past year during which the following papers were

presented:

Solution of equations by successive approximations, by Prof. L. R. Ford
Jordan's theorem, by Prof. Karl Menger
Dynamic families, by Dr. John De Cicco
The concept of mathematical limit, by Dr. LeRoy Wilcox
Lie theory of differential equations, by Richard Edwards
The real number system, by David Rubinfien
Group theory, by Donald Friedlen and Bob Natkin
Boolean algebra, by Marshall Kaplan and Donald Friedlen
Non-Euclidean geometry, by Malcolm Smith.

The officers are: President, Malcolm Smith; Program Chairman, Marshall Kaplan.

Mathematics Club, Case Institute of Technology

The activities of the *Case Mathematics Club* during 1947–48 included the following talks given by members:

A simple application of integral equations, by Prof. S. W. McCuskey
Some topics in number theory, By Ernest Leach
Euclid's parallel postulate, by J. E. Darraugh
Fourier series and boundary value problems, by Ernest Leach
The Pythagorean theorem, solved in integers, by Peter Stephen
A problem in differential geometry arising in the theory of radar antennae. by Prof. R. F. Rinehart.

Officers were: President, Ernest Leach; Faculty Adviser, Prof. Max Morris.

Kappa Mu Epsilon, Albion College

The *Michigan Alpha* Chapter of *Kappa Mu Epsilon* reports the following papers presented during 1947–48:

Galileo, by William Pillinger
Self-taught mathematicians, by Edward Eames
Descartes, by William Beers
Arithmetic revisited, by Prof. H. D. Larsen
How the middle ages counted, by Charles Bishop
4000 years for numerals, by William Doddrill
Evolution of our exponential notation, by Raymond Gillespie
The zero, by Lucy Richardson
Fallacies in mathematics, by Dorothy Manley.

A recognition evening for Prof. E. R. Sleight, who retired in June after 41 years at Albion College, as well as a joint picnic with the Chemistry and Physics Clubs were held.

The officers elected for 1948–49 are: President, William Schofield; Vice-President, Eugene Snell; Secretary-Treasurer, Barbara Barnes; Faculty Sponsor, Prof. H. D. Larsen.

Mathematics Club, Hunter College

The *Mathematics Club* of Hunter College had a most successful and enjoyable year with numerous well attended program meetings and social functions. The student speakers and their topics were:

Quaternions, by Marion Boykan

Mathematics and nature, by Cecile Cohen

Dedekind's theory of numbers, by Luisa Maissonet

Non-Euclidean geometry, by Ruth Friedman

Mathematics and music, by Minna Gottlieb

Finite geometry, by Elaine Traub

History of probability, by Evelyn Margoe

Cryptoanalysis, by Anita Fernbach and Florence Berg.

Faculty and guest speakers and their papers were:

Elliptic functions, Prof. J. H. Bushey

Modern ciphers, by Prof. L. S. Hill

Some unsolved problems of number theory, by Dr. Mary Dolciani, Research Scholar at the Institute for Advanced Study.

The social activities consisted of a winter and a spring dinner, a Christmas party, a theatre party, and a boat ride to Bear Mountain.

The officers of the club were: President, Marcia Geiger; Vice-President, Martha Friedler; Secretary, Luisa Maissonet; Treasurers, Ethel Diamont and Florence Miroff.

Mathematics Club, Adelphi College

In addition to the regular meetings, the *Mathematics Club* of Adelphi was privileged to hear the following guest speakers:

The teaching of mathematics and its applications, by Dr. C. N. Shuster, State Teachers College, Trenton, New Jersey

Pleasures in mathematics, by Prof. J. Ginsberg, Chairman of the Mathematics Department of Yeshiva College.

Trips were taken by the members to Brooklyn, New York, to visit the Electronized Chemical Corporation, and to Bethpage, New York, to visit the Gruman Aircraft Engineering Corporation.

Kappa Mu Epsilon, Drake University

The *Iowa Beta* Chapter of *Kappa Mu Epsilon* reports the following programs given during 1947–48:

Opportunities for students in mathematics, by Walter Potts, Jr.

The field of engineering, by Everett Gilman, a talk supplemented by two movies "Engineering" and "Electric Currents"

History and construction of magic squares, by William Chappell.

Informal social and initiation meetings were also held.

Officers serving during the year were: President, Robert Yohe; Vice-President, Walter Potts, Jr.; Secretary, Dean Williams; Treasurer, Robert Barkus; Faculty Sponsor, Prof. E. L. Canfield.

Junior Mathematical Club, University of Chicago

Meetings were held approximately every two weeks during the Autumn, Winter, and Spring quarters, alternating with the meetings of the *Senior Mathematical Club*. Papers presented to the *Junior Club* are intended primarily to be of interest to students of about beginning graduate level. The following papers were presented:

Transfinite numbers, by Prof. R. W. Barnard

Algebraic number fields, by Harley Flanders

Moral expectation, by Dr. L. J. Savage

Unique connected graphs, by Prof. J. K. Senior, Department of Chemistry

Foundations for a theory of genealogical systems, by Prof. L.R. Willcox, Illinois Institute of Technology

On some numerical methods, by Efrem H. Ostrow

Lie algebras and quantum kinematics, by Prof. I. E. Segal

Bernstein polynomials and their derivatives, by George Klein

Linear graphs and the theory of transportation, by Prof. T. C. Koopmans, Department of Economics

Continued fractions and the Stieltjes integral, by J. V. Finch

Ideals and topologies, by Dr. Edwin Hewitt

Real fields, by Isador Singer

The spectral theorem, by Prof. M. H. Stone.

At the final meeting of the year it was announced that Isador M. Singer was this year's recipient of the annual prize awarded for a paper presented by a student.

Officers for 1948–49 are: President, I. M. Singer; Treasurer, C. E. Miller; Faculty Adviser, Prof. M. H. Stone.

Mathematics Club, Regis College

Among the interesting activities of the *Mathematics Club* of Regis College were the following addresses:

Mathematics in art, by Edna Cunningham '48

What should be the content of the course in mathematics required of the graduate of a liberal arts college?, by Sister M. Leonarda

Things you have not noticed, by Barbara Lane

Harmonic motion, by Ruth P. Carell

The nine-point circle, by Phyllis Moran

Formulas for the area of a triangle, by Elinor O'Neil

A biographical sketch of Colin Maclaurin, by Katharine Healy

The map-maker's proposition, by Anne McDonnell

The evaluation of pi, by Mary T. Harrington

The geometries of Lobachevski and Riemann, by Marie Madden

Chance and chanceability, by Ann McCarthy

The d'Alembert, Euler, Bernoulli controversy, by Barbara A. Sullivan

A simplification of the second derivative test, by Anna McFarlane
Integration of parts, by Mariam Brault
An ornithological note, by Ruth P. Carell
Wheatstone bridges, Maryann Boyce
Seismology, by Phyllis Moran
The atomic bomb, by Catherine T. Walsh
Our hobby—architecture, by Louise Kelley and Barbara Lane
Pascal's triangle and negative exponents, by Anne McDonnell.

Two social events, a field trip, and representation at some New England mathematical meetings represented other activities. Members of the Club presented books to the College Library, wrote mathematical articles for publication in various journals, and published several issues of the *Mathematical Angle*.

Officers elected for 1948–49 are: President, Barbara Lane; Vice-President, Katharine Healy; Secretary, Maryann Boyce; Treasurer, Virginia Lee; Moderator, Sister M. Leonarda.

Pi Mu Epsilon, University of Oregon

The annual report of the *Oregon Alpha* chapter of *Pi Mu Epsilon* includes the following list of papers read before the club:

The Canadian Mathematical Congress, by Prof. K. S. Ghent
Proof that a trigonometric function of an angle having an integral number of seconds is an algebraic number, by Walter Gilbert
Correlation theory, by Prof. Fréchet of the University of Paris
Fourier series, by Shirley K. Anderson
Mathematics of life insurance, by Frank Howatt
A generalization of the average of two numbers, by Prof. B. H. Arnold of Oregon State College.

Ten dollars was contributed to the E. E. DeCou mathematics prize. This prize, in honor of the former head of the mathematics department, was started this year and will be given to an outstanding upper division student in mathematics.

The officers elected for 1949–50 are: Director, John E. Olson; Vice-Director, Carl Pride; Secretary-Treasurer, Gene Thompson.

Mathematics Club, McMaster University

The *Mathematics Club* of McMaster University held six regular meetings during the academic year 1947–48. Membership in the Club numbered fifty. Topics presented by members and guests were:

Life of Euler, by Lloyd Rollerson
Spiral nebulae, by Sidney Hillyer
Simple algebraic proof that the limit e exists, by Norman Lang
Calculating machines, by Eric McAllister

Two theorems concerning polyhedra, by Dick Beesack
The structure of the nucleus, by Dr. Martin Johns
Actuarial work, by Mr. D. Campbell of Crown Life Insurance Company
The Cornu spiral with applications to the field of optics, by Miss Catherine Zavitz.

Officers for the year 1948–49 are: President, Robert Mitchell; Vice-President, Eric McAllister; Secretary-Treasurer, Dick Beesack; Social Convenor, Audrey Baker; Year Representatives, Don McTavish, Frances Wardle, and Keith Rosebrugh.

Mathematics Club, University of Colorado

The *Mathematics Club* of the University of Colorado meets each two weeks at which time a topic of mathematical interest is presented. Usually the discussion is on a sufficiently elementary plane so as to be of interest to students having only calculus. Among the topics presented during 1947–48 were:
Selected topics from modern algebra, by Prof. A. B. Farnell
The foci of plane curves, by Prof. Claribel Kendall
Continued fractions, by Burrowes Hunt
Fourier series and the development of mathematics, by H. Bartram
Quaternians, by Gideon Culpepper.

Officers for 1948–49 are: President, Roy F. Reeves; Secretary-Treasurer, David DeVol; Sponsor, Prof. B. W. Jones.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

STANFORD UNIVERSITY COMPETITIVE EXAMINATION IN MATHEMATICS

The fourth Stanford University Competitive Examination in Mathematics (see this MONTHLY, vol. LIII, no. 7, pp. 406–409 (1946)) was held April 2, 1949, in 37 high schools in California; 189 students took part. The following problems were proposed:

1. Prove that no number in the sequence

$$11, 111, 1111, 11111, \cdots$$

is the square of an integer.

2. The three sides of a triangle are of lengths, l, m, and n, respectively. The numbers l, m, and n are positive integers,

$$l \leqq m \leqq n.$$

(a) Take $n=9$ and find the number of different triangles of the described kind.

(b) Take various values of n and find a general law.

3. (a) Prove the following theorem: A point lies inside an equilateral triangle and has the distances x, y, and z from the three sides respectively; h is the altitude of the triangle. Then

$$x + y + z = h.$$

(b) State precisely and prove the analogous theorem in solid geometry concerning the distances of an inner point from the four faces of a regular tetrahedron.

(c) Generalize both theorems so that they should apply to any point in the plane or space, respectively (and not only to points inside the triangle or tetrahedron). Give precise statements and, if you have time, also proofs.

The writer of the best paper, D. E. Johansen, student at Palo Alto High School, Palo Alto, California, received a $500 scholarship at Stanford University. D. B. Toy of Los Angeles, a student at the Harvard School in North Hollywood, California, and Frank Paulsen, a student at Castlemont Senior High School in Oakland, California, received "Honorable Mention."

CORRIGENDA

Professor R. C. Archibald, author of the *Outline of the History of Mathematics*, sixth edition, which was published as the second Slaught Memorial Paper, submits the following list of corrigenda:

Page 6, line -9, *for* 1948, *read* 1949
Page 9, line 15, *for* Pythagoras., *read* Pythagoras:
Page 9, line 23, *for* 650 700, 649 909, 1 008 541, *read* 13 500, 12 709, 18 541
Page 38, line 4, *for* over, *read* nearly
Page 40, line -3, *for* sec $\alpha-$tan α, *read* sec $\alpha+$tan α
Page 49, line 11, *for* tan x were, *read* tan x (x rational) were
Page 80, line -3, *for* [59], *read* [69]
Page 101, line 15, *for* M'Cay, Schoute, *read* M'Cay, Neuberg, Schoute
Page 101, line 17, *for* Mandart, Neuberg;, *read* Mandart;
Page 104, line 4, *for* death are, *read* death, are
Page 104, *for* Archibald (1875–), *read* Archibald, R. C. (1875–)
Page 107, col. 1, line -8, *for* 29, *read* 39.

PERSONAL ITEMS

Professor A. W. Boldyreff, University of New Mexico, was the representative of the Association at the inauguration of President T. L. Popejoy of the University of New Mexico on June 4, 1949.

Dean O. H. Rechard of the College of Liberal Arts, University of Wyoming, has been appointed to represent the Association at the celebration of the Seventy-Fifth Anniversary of the founding of the Colorado School of Mines at Golden on September 29, 1949.

Mr. W. D. Lambert, formerly of the U. S. Coast and Geodetic Survey, has been elected to membership in the National Academy of Science. He has also been awarded the William Bowie Medal for leadership in the study of earth sciences by the American Geophysical Union.

Professor Saunders MacLane of the University of Chicago has been elected to membership in the American Philosophical Society and the National Academy of Sciences.

Assistant Professor Wilfred Kaplan of the University of Michigan, Professor S. C. Kleene of the University of Wisconsin, Associate Professor G. W. Mackey of Harvard University, and Associate Professor J. W. Tukey of Princeton University have been awarded Guggenheim Fellowships.

Drs. E. E. Moise and M. A. Woodbury of the University of Michigan have been granted National Research Council Fellowships providing for postdoctoral study in the Institute for Advanced Study.

Cornell University makes the following announcements: The Mathematics Department is expanding research and instruction in the theory of probability and its applications with the continued support of a research contract with the Office of Naval Research; Professors Feller, Kac, Chung and Dr. Donsker are participating in this work. Professor G. Elfving of the University of Helsingfors has been appointed Visiting Professor of Mathematical Statistics for the academic years 1949–51; Professor J. L. Doob, on sabbatical leave from the University of Illinois, will spend the year 1949–50 at Cornell; Dr. Gilbert Hunt has been appointed Assistant Professor of Mathematics.

De Paul University announces the following: Professor Rufus Oldenburger chairman of the Mathematics Department, has been granted a leave of absence to devote his full time to the Woodward Governor Company, where he has been mathematician-engineer since 1942; Associate Professor John De Cicco of the Illinois Institute of Technology has been appointed Visiting Professor and Acting Chairman of the Department of Mathematics for the academic year 1949–50; Dr. C. W. Moran of Wright Junior College and Dr. Jerome Sachs of the Chicago Teachers College have been appointed Lecturers of Mathematics in the graduate school.

Kent State University reports: Assistant Professor C. L. Riggs of the University of Kentucky has been appointed to an assistant professorship; Mr. Myron Cox, Virginia Polytechnic Institute, Mr. Donald Dunning of North

Central College, Mr. D. W. Seibel, Denison University, have been appointed to graduate assistantships.

Michigan State College announces the following promotions and appointments: Associate Professor J. D. Hill has been promoted to a professorship; Assistant Professors J. H. Bell and Leo Katz have been promoted to associate professorships; Mr. J. W. Coy of the University of Michigan has been appointed to an instructorship.

Tarleton State College reports the following: Associate Professor L. G. Worthington has been promoted to the position of Professor and Head of the Department of Mathematics; Professor A. A. McSweeny and Assistant Professor Mary Marrs have been placed upon a status of modified service.

Professor D. B. Ames of Rensselaer Polytechnic Institute has been appointed Chairman of the Department of Mathematics at the University of New Hampshire.

Dr. R. V. Andree of the University of Wisconsin has accepted an assistant professorship at the University of Oklahoma.

Associate Professor W. C. Arnold of De Pauw University has been promoted to a professorship.

Dr. Nachman Aronszajn of the Centre National de la Recherche Scientifique, Paris, has been appointed to a research professorship at Oklahoma Agricultural and Mechanical College.

Professor T. B. Ashcraft, head of the Department of Mathematics of Colby College, has retired with the title of Professor Emeritus.

Professor O. F. H. Bert, Washington and Jefferson College, has retired with the title of Professor Emeritus.

Mr. Joseph Blum is now located at Los Alamos Scientific Laboratory.

Dr. Mary L. Boas has been appointed Lecturer at Wellesley College.

Professor Enrico Bompiani of the University of Rome has been appointed to a professorship at the University of Pittsburgh; he will spend half of each academic year at the University of Rome.

Mr. C. W. Cassel of the University of Dayton has accepted a position as chief of the Branch of Measurement and Analysis, Engineering Division, Clinton County Air Force Base, Wilmington, Ohio.

Mr. R. E. Edwards has been appointed Associate Actuary by the Baltimore Life Insurance Company.

Dr. H. E. Ellingson of Rosemount Research Center, University of Minnesota, has been appointed Mathematician with the Naval Ordnance Laboratory, Washington, D. C.

Dr. A. W. Goodman of Rutgers University has been appointed to an associate professorship at the University of Kentucky.

Miss Virginia M. Hall has been appointed to an instructorship at Simmons College.

Professor E. D. Hellinger of Northwestern University has been appointed Visiting Professor of Mathematics at Illinois Institute of Technology.

Professor W. R. Hutcherson of Northwestern State College has been appointed to a professorship at the University of Florida.

Dr. H. G. Landau of the Ballistic Research Laboratories, Aberdeen Proving Ground, has accepted a position as research associate with the Committee on Mathematical Biology, University of Chicago.

Professor E. H. Larguier, Spring Hill College, was granted a leave of absence for the summer term to accept a position as visiting professor at St. Louis University.

Mr. George Laush, Cornell University, has been appointed to an assistant professorship at the University of Pittsburgh.

Dr. Joseph Lehner, formerly of Hydrocarbon Research, Inc., New York has accepted a position as associate professor at the University of Pennsylvania.

Dr. Samuel Lubkin has been appointed Consultant to the Machine Development Laboratory, Applied Mathematics Laboratories, National Bureau of Standards.

Dr. R. K. Luneberg of the Institute for Mathematics and Mechanics of New York University has been appointed to an associate professorship at the University of Southern California.

Professor H. B. Mann of Ohio State University has been appointed to the Applied Mathematics Laboratories of the National Bureau of Standards.

Associate Professor C. G. Maple of North Texas State College has accepted an associate professorship at Iowa State College.

Professor Dorothy McCoy of Belhaven College has been appointed to a professorship at Wayland College.

Assistant Professor Paul Meier of Lehigh University is now on the staff of the Philadelphia Tuberculosis and Health Association in the position of Research Secretary.

Dr. E. P. Miles, Jr., of Duke University has been appointed to an associate professorship at Alabama Polytechnic Institute.

Rear Admiral R. E. Nelson (U.S.N. retired), professor of naval science at Dartmouth, has accepted a position as associate professor of mathematics at Dickinson College.

Associate Professor B. J. Pettis of Tulane University has been promoted to a professorship.

Mr. D. W. Pounder is now employed as an aerodynamicist by A. V. Roe Canada, Ltd., Toronto.

Assistant Professor F. M. Pulliam of the University of Kentucky has been appointed Assistant Professor of Mathematics and Mechanics at the United States Naval Postgraduate School.

Mr. Gordon Raisbeck is now a member of the staff of the Bell Telephone Laboratories, Murray Hill, New Jersey, in the acoustics division.

Mr. L. L. Rauch, Princeton University, has been appointed to an assistant professorship at the University of Michigan.

Assistant Professor W. P. Reid of Purdue University has accepted an as-

sistant professorship at the Air Forces Institute of Technology, Wright Field, Ohio.

Associate Professor E. K. Ritter of the United States Naval Postgraduate School has accepted a position as supervisor, Performance Analysis Group, Aeronautical Research Center, University of Michigan.

Miss Joan Robinson has been appointed to an instructorship at Wilson College.

Dr. C. E. Sealander of Ohio State University has joined the staff of Battelle Institute, Columbus, where he will be engaged in research in engineering physics.

Assistant Professor E. B. Shanks, Vanderbilt University, has been promoted to an associate professorship.

Professor Emeritus E. W. Sheldon of the University of Alberta has been appointed to an interim professorship at Acadia University.

Professor Otto Szasz of the University of Cincinnati has been granted a year's leave of absence to do research in the National Applied Mathematics Laboratories of the National Bureau of Standards, Los Angeles.

Associate Professor J. R. Vatnsdal of State College of Washington has been promoted to a professorship.

Mr. S. I. Vrooman of Rensselaer Polytechnic Institute has been promoted to an assistant professorship.

Dr. Daniel Zelinsky of the Institute for Advanced Study has been appointed to an assistant professorship at Northwestern University.

Professor A. Buhl of the University of Toulouse died on March 24, 1949.

Professor Torsten Carleman of the University of Stockholm died on January 11, 1949.

Dr. Aristide Fanti, formerly scientific librarian of the National Bureau of Standards, died on April 5, 1949.

Dr. H. V. Gummere of Philadelphia died on February 9, 1949.

Mr. E. S. Manson, formerly professor of astronomy at Ohio State University, died on January 29, 1949.

Professor Emeritus Helen A. Merrill of Wellesley College died on May 1, 1949.

Professor Emeritus L. R. Perkins of Middlebury College died April 27, 1948. He was a charter member of the Association.

Professor William Threlfall of the University of Heidelberg died on April 4, 1949.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

JOINT MEETING OF THE ASSOCIATION WITH A.S.E.E.

A meeting of the Mathematical Association of America was held at Rensselaer Polytechnic Institute, Troy, New York, on Monday and Tuesday, June 20–21, 1949, in conjunction with the annual meeting of the American Society for Engineering Education. All sessions of the Association were joint sessions with the Mathematics Division of the A.S.E.E.

About one hundred and fifty persons attended the sessions of the Association including the following eighty-eight members of the Association:

E. B. ALLEN, Rensselaer Polytechnic Institute
N. H. BALL, U. S. Naval Academy
D. H. BALLOU, Middlebury College
R. H. BARDELL, University of Wisconsin
R. A. BEAVER, N. Y. State College for Teachers at Albany
BROTHER BERNARD ALFRED, Manhattan College
W. W. BESSELL, U. S. Military Academy
J. S. BIGGERSTAFF, Rensselaer Polytechnic Institute
HARRY BIRCHENOUGH, N. Y. State College for Teachers at Albany
I. S. BOAK, N. Y. State Agricultural and Technical Institute, Canton
L. F. BORON, University of Maine
N. H. BRYAN, Clemson College
F. J. H. BURKETT, Union College
E. A. BUTLER, N. Y. State College for Teachers at Albany
W. E. BYRNE, Virginia Military Institute
J. D. CAMPBELL, Rensselaer Polytechnic Institute
J. W. CELL, North Carolina State College
L. H. CHAMBERS, U. S. Naval Academy
W. F. CHENEY, JR., University of Connecticut
W. H. H. COWLES, Pratt Institute
L. J. DECK, Muhlenberg College
ARNOLD DRESDEN, Swarthmore College
W. M. DUKE, Cornell Aeronautical Laboratory
J. N. EASTHAM, Cooper Union
CLARENCE FORD, Male High School, Louisville
L. R. FORD, Illinois Institute of Technology
R. M. FOSTER, Polytechnic Institute of Brooklyn
A. H. FOX, Union College
J. B. GARRETT, JR., Siena College
H. M. GEHMAN, University of Buffalo
B. H. GERE, Hamilton College
MICHAEL GOLDBERG, Bureau of Ordnance, Navy Department
J. R. GORMAN, U. S. Naval Academy
LILLIAN GOUGH, University of Buffalo
G. F. GUILFORD, JR., Rensselaer Polytechnic Institute
LUCILLE F. HETZELT, Syracuse University
H. K. HOLT, Union College
R. E. HUSTON, Rensselaer Polytechnic Institute
ROBERTA F. JOHNSON, Wilson College
A. W. JONES, Rensselaer Polytechnic Institute
H. K. JUSTICE, University of Cincinnati
F. E. JUSTIS, Geneva College
AIDA KALISH, Polytechnic Institute of Brooklyn
W. C. KRATHWOHL, Illinois Institute of Technology
JOHN KRONSBEIN, Evansville College
R. E. LANGER, University of Wisconsin
CAROLINE A. LESTER, N. Y. State College for Teachers at Albany
M. E. LEVENSON, Cooper Union
J. V. LIMPERT, St. Lawrence University
JUNE M. MCARTNEY, University of Buffalo
V. O. MCBRIEN, College of the Holy Cross
DIS MALY, Rensselaer Polytechnic Institute
L. L. MERRILL, Clarkson College
A. B. MEWBORN, U. S. Naval Postgraduate School
F. H. MILLER, Cooper Union
NORMAN MILLER, Queen's University
W. L. MISER, Vanderbilt University
R. K. MORLEY, Worcester Polytechnic Institute
D. S. MORSE, Union College
ABBA V. NEWTON, Vassar College

RUTH B. NOLLER, University of Buffalo
F. S. NOWLAN, University of Illinois
MARY H. PAYNE, Michigan State College
K. S. PURDIE, Virginia Military Institute
J. F. RANDOLPH, University of Rochester
W. R. RANSOM, Tufts College
G. B. ROBISON, Sampson College
I. H. ROSE, University of Massachusetts
M. F. ROSSKOPF, Syracuse University
S. G. ROTH, New York University
W. E. ROTH, University of Tulsa
EDITH R. SCHNECKENBURGER, University of Buffalo
WILLIAM A. SMITH, St. Lawrence University
I. S. SOKOLNIKOFF, University of California at Los Angeles
ELLEN C. STOKES, N. Y. State College for Teachers at Albany
R. R. STOLL, Lehigh University
J. S. TAYLOR, University of Pittsburgh
C. J. THORNE, University of Utah
BRYANT TUCKERMAN, Cornell University
S. I. VROOMAN, Rensselaer Polytechnic Institute
R. J. WALKER, Cornell University
W. G. WARNOCK, Rensselaer Polytechnic Institute
BERNICE L. WARR, General Electric Company
MARGARET C. WEEBER, Teachers College of Connecticut
P. M. WHITMAN, Johns Hopkins University
R. H. WILSON, JR., Temple University
J. H. ZANT, Oklahoma A & M College
H. M. ZERBE, Hazleton Center, Pennsylvania State College

The Mathematical Association of America held its first session on Monday afternoon in Sage Lecture Hall with Professor F. H. Miller, Chairman of the Mathematics Division, A.S.E.E., presiding. Professor L. R. Ford presided at the banquet on Monday evening in the Snack Bar. The second session was also held in Sage Lecture Hall on Tuesday afternoon with President R. E. Langer presiding. The Program Committee consisted of J. S. Taylor, Chairman, Michael Goldberg, and R. J. Walker.

FIRST SESSION

"Faulty Teaching of Mathematics," by Dr. L. B. Tuckerman, National Bureau of Standards.

"Mathematical Thinking," by Professor H. B. Phillips, Massachusetts Institute of Technology.

BANQUET

"Mathematical Obligations of the Engineer," by Professor Arnold Dresden, Swarthmore College.

SECOND SESSION

Retiring presidential address: "The Geometry of the Sliding Plane," by Professor L. R. Ford, Illinois Institute of Technology.

"The Use of Slides in the Teaching of College Mathematics," by Professor J. W. Cell, North Carolina State College.

"Real and Complex Vector Analysis," by Professor W. E. Restemeyer, University of Cincinnati.

MEETING OF THE BOARD OF GOVERNORS

The Board met on Monday evening in the Trustees Room, Pittsburgh Building. Eight members of the Board were present.

Reports of various committees were received and acted upon. Among other items of business, the Board voted that the Index of volumes 1–55 of the MONTHLY be published as a supplement to the MONTHLY and be distributed without charge to all members and subscribers. It was further noted that the expenses of editing and printing the Index be charged to the Jacob Houck Memorial Fund.

SOCIAL EVENTS

Local arrangements for the meeting were cared for by the members of the Department of Mathematics at Rensselaer Polytechnic Institute under the capable chairmanship of Professor E. B. Allen. At a brief business meeting of the Association held on Tuesday afternoon, the assembled mathematicians voted an expression of their appreciation to the authorities of the Institute, especially to Professor Allen and the other members of the local committee on arrangements.

A reception for all in attendance at the meetings was held in the Clubhouse on Monday evening. An exhibit of paintings from the Fine Arts Department of I.B.M. was on display at this time.

On Tuesday afternoon at the conclusion of the session of the Association, the visiting mathematicians were entertained at a tea in the Ball Room of the Clubhouse.

Sessions of the A.S.E.E. began on Monday and continued through Friday. In addition to the sessions mentioned above, the Mathematics Division of the A.S.E.E. met jointly with the Mechanics Division on Wednesday afternoon.

H. M. GEHMAN, *Secretary-Treasurer*

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following one hundred one persons have been elected to membership by the Board of Governors on applications duly certified:

JANET E. ABBEY, B.A. (William Smith) Instructor, University of Buffalo, N. Y.

A. D. ANDERSON, M.A. (Oregon) Student, West Branch, Iowa.

H. A. ANTOSIEWICZ, Ph.D. (Vienna) Asst. Professor, Montana State College, Bozeman, Mont.

W. H. BADGLEY, JR., M.A. (Columbia) Asst. Professor, Florence S.T.C., Ala.

R. W. BALL, Ph.D. (Illinois) Instructor, University of Washington, Seattle, Wash.

JO. M. BALLARD, B.S. (Howard) Instructor, Howard College, Birmingham, Ala.

E. F. BARTLEY, B.A. (Scranton) Asst. Professor, University of Scranton, Pa.

O. K. BATES, Sc.D. (M.I.T.) Professor, St. Lawrence University, Canton, N. Y.

W. R. BECK, M.A. (N.Y.U.) Instructor, Purdue University, Fort Wayne, Ind.

F. S. BECKMAN, A.M. (Columbia) Instructor, Pratt Institute, Brooklyn. N. Y.

A. I. BENSON, M.S. (Wisconsin) Staff Member, Los Alamos Scientific Laboratory, N. M.

L. E. BERG, M.A. (Syracuse) Asst. Professor, North Georgia College, Dahlonega, Ga.

IDA M. BERNHARD, M.A. (Texas) Supervisor of Mathematics, Southwest Texas S.T.C., San Marcos, Texas.

NANETTE R. BLACKISTON, M.A. (Columbia) Supervisor, Department of Education, Baltimore, Md.

W. R. BLANK, A.B. (Union) Instructor, Union College, Lincoln, Nebr.

J. L. BRENNER, Ph.D. (Harvard) Asst. Professor, University of California, Santa Barbara College, Calif.
N. A. BRIGHAM, Ph.D. (Pennsylvania) Asst. Professor, University of Maryland, College Park, Md.
B. F. BRYANT, M.A. (Peabody) Instructor, Vanderbilt University, Nashville, Tenn.
I. A. CARL, M.A. (Columbia) Asst. Professor, New York University, N. Y.
C. L. CARROLL, JR., Ph.D. (North Carolina) Asso. Professor, North Carolina State College, Raleigh, N.C.
H. W. CHARLESWORTH, M.A. (Colorado State College of Education) Chairman of Department, Denver Public Schools, Colo.
W. G. CLARK, Ph.D. (Kentucky) Asst. Professor, Mount Union College, Alliance, Ohio
W. H. CLEVELAND, M.S. (Alabama) Teacher, Meridian Junior College, Miss.
RIA J. CLINKSCALES, M.A. (Alabama) Instructor, University of Alabama, Montgomery, Ala.
K. L. COOKE, M.S. (Stanford) Student, Stanford University, Calif.
PATRICIA M. COWAN, B.A. (Reed) Student, Reed College, Portland, Ore.
MARY L. CUMMINGS, M.A. (Illinois) Instructor, University of Missouri, Columbia, Mo.
MORRIS DANSKY, M.A. (Michigan) Instructor, Creighton University, Omaha, Nebr.
D. B. DEKKER, Ph.D. (California) Instructor, University of Washington, Seattle, Wash.
R. D. DEPEW, M.A. (Peabody) Asst. Professor, Florence S.T.C., Ala.
THOMAS DUMONT, Lt. Comdr., Marine Inspector, U. S. Coast Guard, Albany, N. Y.
MARSHALL ELDER, M.A. (Southern California) Instructor, Los Angeles City College, Calif.
TERRELL ELLIS, M.A. (Texas Christian) Asst. Professor, North Texas State College, Denton, Texas
MR. GERALDINE GALLOWAY, M.A. (Illinois) Head of Department, Flat River Junior College, Mo.
H. E. GERTZ, B.S. (Illinois Institute of Technology) Grad. Student, Illinois Institute of Technology, Chicago, Ill.
R. T. GREGORY, M.S. (Iowa) Instructor, Florida State University, Tallahassee, Fla.
D. K. HARTMAN, M.S. (Minnesota) Instructor, University of Minnesota, Minneapolis, Minn.
R. G. HILL, Student, University of Buffalo, N. Y.
H. E. HOFFMAN, Student, University of Buffalo, N. Y.
R. E. HOGAN, B.S. (Southwest Missouri State College) Asst. Instructor, University of Missouri, Columbia Mo.
O. H. HOKE, B.S. (Franklin and Marshall) Grad. Student, University of North Carolina, Chapel Hill, N. C.
JULIUS HONIG, B.S. (Michigan) Student, University of Michigan, Ann Arbor, Mich.
A. J. KILLEBREW, M.S. (Auburn) Asso. Professor, S.T.C., Livingston, Ala.
C. E. KIRKWOOD, JR., M.S. (Georgia) Asso. Professor, Clemson A & M College, S. C.
HELENE G. KUSICK, A.B. (California) Teacher, Placer Union High School, Auburn, Calif.
L. M. LARSEN, M.A. (Nebraska) Professor, Kearney State College, Nebr.
EUGENE LEIMANIS, Dr. Rer. Nat. (Hamburg) Asst. Professor, University of British Columbia, Vancouver, B. C.
C. E. LEMKE, B.A. (Buffalo) Student, University of Buffalo, N. Y.
P. J. LEONARD, B.S. (Boston College) Grad. Student, Boston College, Mass.
E. R. LORCH, Ph.D. (Columbia) Professor, Barnard College, Columbia University, New York, N. Y.
R. L. MARCEAU, Licence en Sciences (Laval University) Instructor, University of Kansas, Lawrence, Kansas.
MILDRED M. MARTENS, M.A. (Michigan) Instructor, Woodruff Senior High School, Peoria, Ill.
D. E. MCCLEAN, B.A. (George Pepperdine) Grad. Student, University of Southern California, Los Angeles, Calif.
R. K. MCCONNELL, JR., B.S. (Pittsburgh) Instructor, New York University, N. Y.
ELOISE MCCORD, M.A. (Oregon) Instructor, University of Wichita, Kansas
D. L. MCINTOSH, M.A. (Denver) Teacher, South Denver High School, Colo.
J. C. C. MCKINSEY, Ph.D. (California) Rand Corporation, Santa Monica, Calif.
RUDOLF MEYER, B.A. (Buffalo) Student, University of Buffalo, N. Y.

L. C. MILLS, M.A. (Harvard) Teacher, Wheat Ridge High School, Colo.

MOTHER M. GERARD MOONEY, M.A. (Creighton) Teacher, Ursuline College, New Orleans, La.

H. W. NACE, M.A. (Cornell) Professor, Lawrence Institute of Technology, Detroit, Mich.

A. R. NOLSTAD, D.Ed. (Pittsburgh) Asst. Professor, North Carolina State College, Raleigh, N. C.

HELEN OLNEY, M.Sc. (Oregon) Asst. Professor, Hiram College, Ohio

M. H. PROTTER, Ph.D. (Brown) Asst. Professor, Syracuse University, N. Y.

F. M. PULLIAM, Ph.D. (Illinois) Asst. Professor, University of Kentucky, Lexington, Ky.

A. H. QUIRMBACH, M.S. (Virginia Polytechnic Institute) Asst. Professor, University of Alabama, University, Ala.

L. B. RALL, Student, College of Puget Sound, Tacoma, Wash.

R. F. REEVES, B.S., Instructor, Elec. Engg., Iowa State College, Ames, Iowa.

L. A. RINGENBERG, Ph.D. (Ohio State) Professor, Eastern Illinois State College, Charleston, Ill.

E. K. RITTER, Ph.D. (Virginia) Research Engineer, University of Michigan, Ypsilanti, Mich.

SIBYL M. ROCK, B.A. (California) Technical Consultant, Consolidated Engineering Corporation, Pasadena, Calif.

A. I. ROSENFELD, 475 W. 186th St., New York, N. Y.

E. D. ROUNDS, B.S. (Scranton) Instructor, University of Scranton, Pa.

J. E. ROWE, B.A. (Tennessee) Research Engr., Carbide and Carbon Chemical Corporation, Oak Ridge, Tenn.

W. C. ROYSTER, M.A. (Kentucky) Instructor, University of Kentucky, Lexington, Ky.

C. M. SANDWICK, B.A. (Lehigh) Teacher, Easton High School, Pa.

P. J. SCHILLO, Student, University of Buffalo, N. Y.

L. F. SCHOLL, M.A. (Buffalo) Supervisor of Math., Board of Education, Buffalo, N. Y.

D. M. SEWARD, Ph.D. (Duke) Professor, Ouachita College, Arkadelphia, Ark.

SISTER M. AGNETA, B.A. (St. Teresa) Teacher, St. Clare Academy, Sylvania, Ohio

SISTER ROSE G. CALLOWAY, Ph.D. (Catholic University) Professor, Mt. St. Mary's College, Los Angeles, Calif.

WILLIAM A. SMITH, M.A. (Syracuse) Asst. Professor, St. Lawrence University, Canton, N. Y.

R. J. SMURTHWAITE, Student, University of Buffalo, N. Y.

E. J. SPECHT, Ph.D. (Minnesota) Professor, Emmanuel Missionary College, Berrien Springs, Mich.

H. S. STANLEY, A.M. (Harvard) Asso. Professor, University of Georgia, Athens, Ga.

S. N. STONE, A.B. (Brooklyn) Grad. Student, New York University, N. Y.

R. L. SWAIN, Ph.D. (Texas) Instructor, Ohio State University, Columbus, Ohio

G. J. TRAMMELL, JR., B.S. (Tulane) Student, Tulane University, New Orleans, La.

T. E. TUTHILL, M.A. (Oberlin) Teacher, Nichols School, Buffalo, N. Y.

J. P. VAN ALSTYNE, B.S. (Hamilton) Instructor, Hamilton College, Clinton, N. Y.

E. L. VANDERBURGH, M.A. (Wyoming) Instructor, Pueblo Junior College, Colo.

E. A. VOORHEES, JR., A.B. (Maryville) Fellow Vanderbilt University, Nashville, Tenn.

W. E. VORONOVICH, Graduate (Polytechnic Institute of Riga) Lewis Machine Co., Cleveland, Ohio

SYLVIA VOPNI, M.A. (University of Washington) Instructor, Edison Technical School, Seattle, Wash.

J. F. WAGNER, M.S. (Michigan) Asst. Professor, University of Colorado, Boulder, Colo.

M. C. WALKER, B.S. (Illinois) Instructor, General Motors Institute, Flint, Mich.

MARGUERITE F. WELLS, M.A. (Alabama) Statistician, Osborn, Ohio

J. R. WESSON, B.S. (Birmingham-Southern) Teaching Fellow, Vanderbilt University, Nashville, Tenn.

MABEL WILLIAMS, M.A. (Texas) Teacher, Tyler Junior College, Texas

W. L. G. WILLIAMS, Ph.D. (University of Chicago) Professor, McGill University, Montreal, P. Q.

R. B. WINEGEART, Student, Northwestern State College, Natchitoches, La.

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association for a three-year term beginning July 1, 1949 by a mail vote of the membership of the Association in the Sections indicated:

Kansas	R. G. Sanger, Kansas State College
Missouri	G. M. Ewing, University of Missouri
Ohio	F. B. Wiley, Denison College
Pacific Northwest	M. S. Knebelman, State College of Washington
Southeastern	Tomlinson Fort, University of Georgia
Southwestern	Earl Walden, New Mexico College of A. and M. A.
Upper New York State	E. B. Allen, Rensselaer Polytechnic Institute
New England Region	R. E. Gilman, Brown University

Since the system of Regional Governors has been replaced by a system of Sectional Governors, no elections have been held (except in the New England Region) to replace the Regional Governors whose terms expired on July 1, 1949.

JANUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The eleventh annual meeting of the Northern California Section of the Mathematical Association of America was held at the University of San Francisco on Saturday, January 29, 1949. Professor G. C. Evans, Chairman of the Section, presided at the morning session, and Mr. K. J. Waider presided at the afternoon session.

The attendance was eighty-one including the following thirty-five members of the Association: H. M. Bacon, G. A. Baker, T. J. Bass, Alice Bell, M. T. Bird, A. C. Burdette, D. G. Chapman, M. A. Dernham, Roy Dubisch, Hazel Eggett, G. C. Evans, E. A. Fay, S. A. Francis, C. M. Fulton, W. R. Hanson. V. F. Ivanoff, D. H. Lehmer, Sophia McDonald, E. D. Miller, F. R. Morris, W. H. Myers, C. D. Olds, George Polya, Edris Rahn, R. M. Robinson, E. B. Roessler, Kathryn Rolfe, Sister Madeleine Rose, Irving Sussman, Gabor Szego, Edwin Tabor, K. J. Waider, L. A. Walker, Anna Pell Wheeler, A. R. Williams.

At the business meeting the following officers were elected for the coming year: Chairman, H. M. Bacon, Stanford University; Vice-Chairman, S. A. Francis, San Mateo Junior College; Secretary-Treasurer, E. B. Roessler, University of California at Davis; Representative on the *California Journal of Secondary Education*, Ruth G. Sumner, Oakland High School.

By invitation of the Section, Professor Edward W. Strong, Department of Philosophy, and Associate Dean of the College of Letters and Science, University of California, gave an hour's address during the morning session.

1. *The irrationality of a class of numbers*, by Dr. H. L. Alder, University of California at Davis, introduced by the Secretary.

A method is given by which to prove the irrationality of a class of numbers including e^m, where m is rational and not equal to zero, in such a way as to be understandable to students with only a little knowledge of differential and integral calculus and no knowledge of series expansions.

2. *With, or without, motivation?* by Professor George Polya, Stanford University.

This paper will be published in this MONTHLY.

3. *Newton's "mathematical way,"* by Professor E. W. Strong, University of California, introduced by the Chairman.

Newton's procedure in "mathematically determining all kinds of phenomena" presents two problems: (1) how does he explain "principles" as inductive generalizations from which a theoretical physicist proceeds mathematically to "estimate effects"? (2) what is his position concerning terms that are strictly mathematical?

An examination of his work in optics reveals the central role assigned to measurement in the formulation of mechanical (mathematical-physical) principles. The rules derived from measures in experimental inquiry and incorporated in the statement of laws are requisite for demonstration of phenomena. Newton insists that the certainty of demonstration in science cannot exceed the certainty of its physical principles. An exception to his argument appears to be his assertion of absolute, true, and mathematical space, time, and motion in the *Principia*; yet Newton himself is cognizant that he is here postulating a mathematical structure lacking in empirical support.

Although he devised his method of fluxions as a tool to assist in the solution of physical problems, Newton treats mathematical analysis *per se* as a logic of reasoning advancing concepts requiring no appeal to geometry or to nature for their legitimacy. The reasoning in "a method of determining quantities from the velocities of the motions of the increments with which they are generated" assigns meanings to the terms employed that are strictly mathematical. Subsequently a number of British mathematicians attempted to argue the superiority of Newton's method to Leibniz's differential calculus. Their defense tried not only to deny that Newton had ever employed infinitesimals but also rejected their use on the grounds that they "have no being either in geometry or in nature." Berkeley turned this argument against the defenders of Newton and was instrumental in prodding Robins, Simpson, and Maclaurin to investigate the foundations of the calculus in contributing to the theory of limits.

4. *Probability and nature*, by Professor Michel Loeve, University of California, introduced by the Chairman.

A few examples, chosen for their historical importance, illustrated the coming insight of probability notions and their gradual penetration into the rational models for natural phenomena. At first their domain seemed to be that of mass-phenomena such as the kinetic theory of gases, statistical mechanics, and genetics. At present, after a revolution in the understanding of nature, due to quantum mechanics, even individual microscopic phenomena are best explained in probability terms.

5. *Cayley arrays and multiplication matrices of an algebra*, by Roy Dubisch, Fresno State College.

Considering an algebra A over a field F as a linear space of order n over F consisting of 1 by n matrices $a=(\alpha_1, \cdots, \alpha_n)$, α_i in F, we define multiplication in A by $a \cdot x=(\alpha_1, \cdots, \alpha_n)(\xi_1, \cdots, \xi_n)=(\mu_1, \cdots, \mu_n)$, where $\mu_k=\sum_{i,j=1}^n \alpha_i\gamma_{ijk}\xi_j$ for $k=1, \cdots, n$. The right multiplication matrix of A is defined as $\Gamma_x=(\sum_{k=1}^n \gamma_{ijk}\xi_k)$ while the matrix formed from the Cayley array is defined to be $\Omega_x=(\sum_{k=1}^n \gamma_{ijk}\xi_k)$. We ask what algebras have $\Gamma_x=\Omega_x$ for all possible bases, and find that A has such a property if and only if A has a basis $u_1, u_2, \cdots, u_n$ with $u_su_r=\delta_su_r$ (δ_s in F, $s, r=1, \cdots, n$).

6. *A note on linear equations*, by Professor R. M. Robinson, University of California.

This note has since appeared in this MONTHLY, vol. 56, April, 1949, p. 251.

7. *Geometrical optics and the calculus of variations*, by Dr. Robert Weinstock, Stanford University, introduced by the Vice-Chairman.

It is shown how certain elementary minimum problems may be solved through the use of geometrical optics in conjunction with Fermat's principle of least time.

Further, the simple laws of geometrical optics are applied to a medium in which the index of refraction is a continuous function of one cartesian coordinate, through "slabification" and suitable passage to the limit. Since the optics problem, through Fermat's principle, is equivalent to a certain class of problems in the Calculus of Variations, the latter may be solved by optical means alone. For example, the classical brachistochrone problem is solved upon suitable choice of index of refraction as a function of one coordinate (John Bernoulli's original method of solution).

Finally, the differential equation in plane polar coordinates for the path of light in a medium whose index is a function of the radial coordinate alone is produced as a solution to another class of elementary problems in the Calculus of Variations.

E. B. ROESSLER, *Secretary*

FEBRUARY MEETING OF THE OKLAHOMA SECTION

The annual meeting of the Oklahoma Section of the Mathematical Association of America was held in connection with the convention of the Oklahoma Education Association in Oklahoma City on Friday, February 18, 1949. Professor J. E. LaFon, Chairman of the Section, presided.

Sixty-two persons attended the meeting, including the following thirty members of the Association: E. F. Allen, Arthur Bernhart, J. C. Brixey, A. H. Diamond, R. C. Dragoo, A. A. Grau, L. D. Gregory, O. H. Hamilton, Claire A. Harrison, J. O. Hassler, E. E. Heimann, L. R. Holland, W. N. Huff, H. V. Huneke, J. T. Krattiger, M. L. Lawson, Eunice Lewis, H. W. Linscheid, R. D. McDole, Dora McFarland, R. D. McKnelly, G. E. Meador, R. R. Murphy, C. J. Pipes, A. A. Ritchie, H. W. Smith, O. S. Spears, C. E. Springer, R. W. Veatch, J. H. Zart.

At the business session the following officers were elected: Chairman, R. W. Veatch, University of Tulsa; Vice-Chairman, E. E. Heimann, East Central State College; Secretary, J. C. Brixey, University of Oklahoma.

The program consisted of the following seven papers:

1. *Some aspects of teaching calculus*, by Professor Caspar Goffman, University of Oklahoma, introduced by Professor J. C. Brixey.

The speaker discussed a method of presenting the fundamental theorem of the calculus, and its applications, without using the theorem of the mean.

2. *Semi-continuous transformations*, by Professor O. H. Hamilton, Oklahoma A. and M. College.

The definitions and some of the properties of upper and lower semi-continuous transformations were discussed. Comparisons of semi-continuous with continuous transformations were made, and examples of the various types of transformations were given.

3. *Set properties*, by Mr. C. J. Pipes, University of Oklahoma.

For sets of real numbers, a set property is called a τ property if every subset of a τ set is a τ set, the union of a denumerable number of τ sets is a τ set, the continuum is not a τ set, and there is at least one non-empty τ set. A set property is called a π property if it is a τ property and if every set consisting of a single point is a π set, and it is a ρ property if it is a π property and if there are C or less ρ sets such that every ρ set is a subset of one of them, where C is the cardinal of the continuum. With these set properties various theorems on point sets can be generalized; e.g., "there is a non-denumerable set whose intersection with every ρ set is denumerable."

4. *Some metric properties of union curves on a surface*, by Mr. R. B. Deal, Jr., University of Oklahoma, introduced by Professor C. E. Springer.

The contour integral of union curvature was considered, and a generalization of the Gauss-Bonnet theorem was given. The torsion of any curve was expressed in terms of its union torsion, and for a one-parameter family of curves the union torsion was given, as well as necessary and sufficient conditions for the curves to be union curves relative to a specified rectilinear congruence. Finally, necessary and sufficient conditions were given for two surfaces to be in union correspondence.

5. *What colleges can do to improve mathematics in the public schools*, by Professor E. E. Heimann, East Central State College.

It was pointed out that as long as colleges continue to offer remedial mathematics courses without holding the high schools to account for the poor mathematical training of their graduates, the high schools may be expected to continue to shirk their responsibility. In addition it was stated that elementary teachers should not be allowed to teach arithmetic without having had some college mathematics.

6. *Binary and ternary relations*, by Professor A. A. Grau, University of Oklahoma.

In this paper, the author showed that the role played by the ternary relation $(a, b, c) = (a \cap b) \cup (a \cap c) \cup (b \cap c) = a$, denoted herein by $R(a; b, c)$, in a ternary Boolean algebra or lattice, is analogous to that of the partial order relation in a binary lattice. The relation $R(a; b, c)$ has the following properties: (1) it is symmetric in b and c; (2) it is reflexive in any pair; (3) for fixed b or c, it is a binary partial order relation; (4) $R(a; a', b)$ never holds if $a \neq b$; (5) $R(b; a, a')$ always holds; (6) if $R(x; a, b)$, $R(x; b, c)$, $R(x; a, c)$ hold simultaneously in a Boolean algebra, $x = (a; b, c)$. If properties (1)–(3) are used as postulates, (4) and (5) as the definition of complements (whenever they exist), and (6) as the definition of the ternary operation, many of the properties of the ternary operation in a Boolean algebra or lattice may be derived from the properties of $R(a; b, c)$.

7. *A property of a class of parabolic curves*, by Professor A. H. Diamond, Oklahoma A. and M. College.

The radius of curvature R of the class of parabolic curves $y = \pm x^n$, $x > 0$ and $0 < n < 1$, was examined at the origin. It was pointed out that R considered as a function of n at the origin, has the discontinuity $0 < n < 0.5$, $R = \infty$; $n = 0.5$, $R = 1$; $0.5 < n < 1$, $R = 0$.

J. C. BRIXEY, *Secretary*

MARCH MEETING OF THE SOUTHEASTERN SECTION

The annual meeting of the Southeastern Section of the Mathematical Association of America was held at the University of Alabama, University, Alabama, on Friday and Saturday, March 18–19, 1949. Major L. A. Dye, Chair-

man of the Section, presided at the Friday afternoon and Saturday morning meetings, except for the meetings of the subsections, which were presided over by Professors F. W. Kokomoor, J. M. Thomas, and M. G. Boyce. Professor F. A. Lewis, Vice-Chairman, presided on Friday evening at the barbecue held at Moundville State Park.

There were about two hundred present, including the following one hundred and six members of the Association: R. H. Ackerson, W. H. Badgley, Jr., Edna M. Ballard, D. F. Barrow, W. S. Beckwith, L. E. Berg, R. G. Blake, Floyd Bowling, M. G. Boyce, Mamie I. Braswell, J. P. Brewster, S. K. Bright, J. W. Brown, C. W. Bruce, N. R. Bryan, B. F. Bryant, Berdie E. J. Buffkin, L. P. Burton, John Capesius, Ella F. Casey, Elizabeth C. Cathey, B. G. Clark, Ria J. Clinkscales, A. C. Cohen, Jr., R. L. Coker, G. M. Conwell, J. A. Cooley, R. W. Cowan, J. C. Currie, R. D. Depew, R. D. Doner, Nelle C. Douglas, L. A. Dye, E. D. Eaves, J. D. Edwards, R. B. Folsom, Tomlinson Fort, S. T. Gormsen, W. W. Graham, E. H. Hadlock, B. F. Hadnot, E. A. Hedberg, G. W. Hess, A. T. Hind, Jr., G. B. Huff, P. M. Hummel, R. O. Hutchinson, J. A. Hyden, Rosa L. Jackson, Ayrlene M. Jones, A. J. Killebrew, F. W. Kokomoor, W. I. Layton, R. J. Levit, F. A. Lewis, J. F. Locke, G. H. Lundberg, J. D. Mancill, J. E. Martin, W. A. Martin, W. G. McGavock, S. W. McInnis, W. L. Miser, W. A. Moore, R. H. Moorman, T. F. Mulcrone, W. V. Neisius, J. D. Novak, H. A. Palmer, Eugene Park, W. V. Parker, Lillian G. Perkins, I. E. Perlin, C. G. Phipps, R. B. Plymale, A. H. Quirmbach, Alice B. Rabon, Adrienne S. Rayl, B. P. Reinsch, G. E. Reves, J. O. Reynolds, R. G. D. Richardson, H.A. Robinson, L. V. Robinson, C. L. Seebeck, Jr., E. B. Shanks, D. C. Sheldon, T. M. Simpson, Augustus Sisk, A. R. Sloan, C. B. Smith, C. D. Smith, E. L. Stanley, H. S. Stanley, R. B. Stiles, J. R. Sullivan, J. M. Thomas, E. A. Voorhees, F. A. Wallace, J. A. Ward, Betty R. Weber, W. W. Weber, W. L. Williams, R. L. Wilson, G. N. Wollan, F. L. Wren.

At the business session the following officers were elected for the coming year: Chairman, F. A. Lewis, University of Alabama; Vice-Chairman, C. G. Phipps, University of Florida; Secretary-Treasurer, H. A. Robinson, Agnes Scott College. The Section voted to hold its March 1950 meeting at the University of Florida.

The program consisted of the following papers:

1. *Applications of statistics in the transportation industry*, by W. V. Neisius, Consultant, Transportation Statistics, Georgia Power Company.

A description of the factors used in a multiple curvilinear correlation analysis, as developed by C. A. Stephenson, and applied to city transit rides, was given. Probable annual rides per capita were described very accurately in terms of average fare, miles of service offered, automobile registration, and bank debits.

2. *Nation-wide practices in the certification of teachers of mathematics*, by Professor W. I. Layton, Alabama Polytechnic Institute.

The present requirements for certification of teachers were reviewed, and recommendations

were offered for the possible improvement of the chaotic conditions now prevailing. It was suggested that six semester hours in mathematics be required as a minimum for certification of an elementary teacher, and eighteen for a secondary school teacher.

3. *The approximate evaluation of roots*, by Professor L. A. Dye, The Citadel,

In this expository paper, four standard procedures for approximating irrational roots of an equation were discussed. Three simple rules for gaining greater accuracy when using Newton's or Horner's methods were developed.

4. *The role of fields in college mathematics courses*, by Professor G. B. Huff. University of Georgia.

Professor Huff remarked it is a common complaint that college mathematics courses rarely offer the student much opportunity to formulate arguments, and suggested a possible remedy. After reviewing briefly some fundamental notions in the theory of abstract fields, he showed how these ideas recur in the usual material and thus present chances to make arguments.

5. *Mathematics and economics: a preliminary report*, by Professor C. G. Phipps, University of Florida.

In recent years, mathematics has been increasingly applied to problems in theoretical economics entirely apart from economic statistics. Unfortunately, some writers have desired to add prestige to their conclusions by giving mathematical proofs, but they have been reluctant to be bound by mathematical rigor. In order to establish this rigor, a graduate project is being set up at the University of Florida with a twofold purpose: (a) to make a complete and accurate analysis of certain topics in economics; and (b) with this analysis as a criterion, to review critically books and articles dealing with these topics.

6. *Plane areas by complex integration*, by Professor J. D. Mancill, University of Alabama.

This paper is to appear in this MONTHLY.

7. *Logarithms and exponentials in calculus*, by Professor Tomlinson Fort, University of Georgia.

The differentiation of the logarithm and exponential is a traditionally difficult matter for writers of textbooks on calculus. The fundamental difficulty is that these functions have never been defined in the mathematical career of the student for irrational values of the argument, and consequently any limiting process involving them is necessarily incomplete. However, if the theory of integration is properly presented before the introduction of logarithms and exponentials into calculus, the whole procedure is greatly simplified. Thus $\int_1^s (1/x)dx$ is a well defined function of s for irrational as well as rational values of s.

8. *Bernoulli's asterisks*, by P. A. Piza, San Juan, Puerto Rico.

In this paper, which Mr. Piza had printed for the occasion of the meeting, he makes certain deductions concerning the asterisks which appear in Bernoulli's formulae for the sums S_n of the first nth powers, as given on Page 97 of *Ars Conjectandi*. Mr. Piza demonstrates two methods whereby these formulae may be simplified and developed without employing Bernoulli numbers. He also develops a number of syzygies connecting certain S_i's.

9. *Conclusions from a survey on mathematics curricula in liberal arts colleges*, by Professor N. R. Bryan, University of Georgia.

This survey covers curricula conditions of the last ten years. One trend is to include more

differential equations as subject matter in undergraduate mathematics. Calculus is begun earlier by putting the differentiation and integration of algebraic functions in the freshman year. A much briefer course in analytic geometry is made possible by postponing such topics as polar coordinates and solid analytics to the calculus, where these topics can be freshly taught when needed.

10. *A novel approach to the addition formulae of trigonometry*, by Professor E. A. Hedberg, University of South Carolina.

This approach is designed to motivate and rationalize the development of the addition formulae of trigonometry by making use of the laws of sines and cosines. A triangle is constructed with given angles $A+B$ and $90°-A$, and the proof follows immediately.

11. *Mathematics in the New England private schools*, by Professor G. M. Conwell, University of Georgia.

Professor Conwell outlined the objectives of these schools, and told how they attempted to secure for the student an understanding and love of mathematics. These schools try to test ability rather than memory. Emphasis is placed on understanding. All students take algebra and geometry, the better ones solid geometry, trigonometry and advanced algebra. A continuous effort is made to use algebra in geometry and geometry in algebra. The best boys take a year of calculus.

12. *The need for cooperative effort in mathematics*, by Professor F. L. Wren, Peabody College.

In recent years we have witnessed a very definite increase in the demand for an effective program in the teaching of mathematics. This fact confronts in particular the Mathematical Association of America and the National Council of Teachers of Mathematics, with the distinct challenge to determine what this program should be. Is the program in our elementary and secondary schools the most effective that we can determine, or is it somewhat an accident of tradition? What is the most significant program we can shape for the training of teachers? Such questions can be answered authoritatively only through the cooperative efforts of the interested organizations.

13. *Teaching through discovery*, by Professor I. E. Perlin, Georgia Institute of Technology.

In this paper the importance of examining definitions and basic concepts critically was stressed. It was demonstrated that the best way to introduce a basic concept is, if possible, to let the student discover the method himself, with the instructor only acting as a guide along the proper paths. It was felt that this procedure might develop a way of thinking of great benefit to the scientific man.

14. *Grading mathematics papers*, by Professor D. F. Barrow, University of Georgia.

The grading of quiz papers is too often regarded as a tedious chore. Professor Barrow suggested some devices for making the task more interesting and more valuable to the students, and which might to some extent prevent instructors from falling into a rut.

15. *A new roll book system*, by W. V. Neisius, Georgia Institute of Technology.

A simplified visual index roll book was described which enables an instructor to determine graphically the student's average. The new roll book system is of great benefit in large classes where frequent quizzes are given.

16. *Sample design for the study of advertising*, by Professor C. D. Smith, University of Alabama.

Professor Smith discussed factors of significance in advertising cost. Data were exhibited for a sample pattern applied to a given area where one spends a cumulative total X to obtain an increase in net profit M on sales.

17. *A distinguished mathematician's contribution to genetics*, by Professor R. J. Levit, University of Georgia.

Shortly after the rediscovery of Mendel's law in 1900 a misconception arose regarding the distribution of hereditary traits among a population to be expected on the basis of these laws. This speaker discussed a solution of the problem by the application of the elementary theory of probability as given by the late G. H. Hardy in a letter to the editor of *Science* in 1908, out of which has grown the modern science of statistical genetics.

18. *On teaching the law of means*, by Professor B. P. Reinsch, Florida Southern College.

Professor Reinsch shared his experiences in motivating interest and understanding on the part of the student in connection with the law of the mean for derivatives. He demonstrated a method for constructing tangents to parabolas.

19. *Algebraic methods of extending the multiplication table*, by Professor L. V. Robinson, University of South Carolina.

It was shown how the simple principles of algebra may be applied to oral multiplication and division, and how interest in algebra may thereby be enhanced.

20. *The number of solutions of Diophantine systems*, by Professor J. M. Thomas, Duke University.

For real x the four symbols $[x]$, $[x]'$, $\{x\}$, $\{x\}'$ are defined respectively as the greatest integer not greater than x, the least integer not less than x, the least integer nearest x, and the greatest integer nearest x. Relations among these symbols are given, and the number of solutions of certain systems composed of Diophantine equations and inequalities is expressed in terms of them. In particular, the paper treats the system composed of a linear Diophantine equation and inequalities restricting the unknowns to segments or intervals.

21. *Elastic equilibrium for the interior of a wood disk*, by C. B. Smith, University of Florida.

The wood disk was assumed to be in a state of plane stress under the action of external forces which are applied to the boundary, and which act in the plane of the disk. A method was developed for solving the problem for a general distribution of external forces. Then the problem was carried out in detail for a special type of loading, and the results reduced to the case where the disk was of isotropic material.

22. *A refinement for linear interpolation applied to approximations of the roots of equations*, by Professor C. L. Seebeck, Jr., University of Alabama.

The refinement is additive and equal to

$$(b - x)(x - a)[f'(a) - f'(b)]/2(b - a),$$

where $b-a$ is the interpolation interval, and x is between a and b. When applied to the inverse of $y=f(x)$, a formula results for improving the approximate root of $f(x)=0$ obtained by linear interpolation. Rapid convergence toward roots was exhibited in several illustrations.

23. *On properties of particular solutions of a generalized Bessel's equation*, by Professor R. W. Cowan, University of Florida.

The equation is taken in a form so that it contains two parameters n and α such that when α is put equal to zero, the equation reduces to Bessel's equation. The general solution of the equation is obtained by the method of Frobenius. A number of properties of a particular solution $G_{n,\alpha}(x)$ are obtained including a recussion formula, orthogonality relations, and the evaluation of the integrated square. The latter two properties enable one to expand an arbitrary function in a series of $G_{n,\alpha}$ functions, each of which is multiplied by a certain exponential factor.

24. *On the numerical solution of a transcendental equation in statistics*, by Professor A. C. Cohen, Jr., University of Georgia.

In order to fit a truncated normal distribution to a given set of data, it is necessary to solve a somewhat complicated type of transcendental equation. In this paper, the fundamental equations involved in solutions previously given by Karl Pearson, Alice Lee, and R. A. Fisher are expressed in terms of more elementary transcendental functions and written in a form which facilitates their rapid computation. Using only an ordinary set of tables of areas and ordinates of the normal frequency curve, the solution can be completed by the simple method of successive graphs or interpolation.

25. *A note on the general polynomial theorem*, by Professor W. S. Beckwith, University of Georgia.

The purpose of this paper was to set up a general method for the expansion of a sum of algebraic variables raised to a power. A general theorem was developed by repeatedly employing the binomial theorem, and the trinomial theorem.

26. *On the pivotal element method for determinants and systems of linear equations*, by Professor M. G. Boyce, Vanderbilt University.

The pivotal element method of solving systems of linear equations, a systematic elimination of unknowns, has the advantages of easy checking and adaptability to machine computation. Its obvious relationship to determinant methods suggests defining determinants by their pivotal element evaluations. Such a definition is shown to define uniquely the value of a determinant, independent of the choice of pivotal elements, and to afford a ready derivation of the elementary properties of determinants.

27. *On a determinant of Sylvester*, by Professor F. A. Lewis, University of Alabama.

If the elements of a determinant V of order n are defined by $v_{rc}=E^{(r-1)(c-1)}$, where $E=e^{2\pi i/n}$, V is reducible if n is odd. Formulae were given for values of the component determinants whose elements involve only cosines and sines respectively.

28. *A formula for tan nx*, by Professor J. A. Ward, University of Georgia.

The formula

$$\tan nx = \left[\sum_{n=0}^{\infty}(-1)^r\binom{n}{2r+1}\tan^{2r+1}x\right]\left[\sum_{n=0}^{\infty}(-1)^r\binom{n}{2r}\tan^{2r}x\right]^{-1}$$

was derived. It was pointed out that the numerator and denominator have only a finite number of terms each when n is a positive integer. Otherwise they are series which converge when $|\tan x|<1$. If $|\tan x|>1$, then $\tan nx$ may be expressed in a similar manner by convergent series in $\cot x$.

29. *A generalized Vandermonde determinant*, by E. B. Shanks, Vanderbilt University.

Let $D_i(x)$ be a matrix of n rows and i columns ($n \geqq i$) such that the element in the pth row and qth column is the qth term of the binomial expansion of $(x+1)^{p-1}$ when q is not greater than p, all other elements being zero. Let S be a determinant of order n whose columns consist of r blocks, where a typical block contains the i_k columns of $D_{i_k}(a_k)$. When $i_k=1$ for all k, S is the Vandermonde determinant. The purpose of this paper was to prove that S is equal to the product of all factors of the type $(a_k-a_j)^{i_k i_j}$, $j<k$.

30. *The radical axis of a pair of conics*, by Lillian G. Perkins, University of South Carolina.

The generalization of the radical axis of a pair of circles was made for a pair of conics. The general case for any two conics was discussed, and then theorems were obtained for some special cases of ellipses, hyperbolas and parabolas.

31. *The use of projection in the proof of certain theorems of triangles*, by Professor W. V. Parker, University of Georgia.

Every triangle is an orthogonal projection of an equilateral triangle. The ellipse of minimum area circumscribing the triangle has its center at the centroid of the triangle and is the orthogonal projection of the circumscribing circle for the equilateral triangle. By using the properties which are invariant under orthogonal projection many theorems for general triangles may be proved by proving them for equilateral triangles only.

H. A. ROBINSON, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-third Annual Meeting, New York City, December 30, 1949.

International Congress of Mathematicians, Cambridge, Massachusetts, August 30–September 6, 1950.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN

ILLINOIS, Southern Illinois University, Carbondale, May 12–13, 1950

INDIANA, Wabash College, Crawfordsville, April 29, 1950

IOWA, State University of Iowa, Iowa City, April 21–22, 1950

KANSAS

KENTUCKY

LOUISIANA-MISSISSIPPI, Centenary College, Shreveport, Louisiana, Spring, 1950

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Fall, 1949

METROPOLITAN NEW YORK, Spring, 1950

MICHIGAN, March, 1950

MINNESOTA, University of North Dakota, Grand Forks, October 15, 1949

MISSOURI, Spring, 1950

NEBRASKA, Nebraska Wesleyan University, Lincoln, May 6, 1950

NORTHERN CALIFORNIA, Berkeley, January 28, 1950

OHIO, Denison University, Granville, April 22, 1950

OKLAHOMA, Oklahoma City, October 14, 1949

PACIFIC NORTHWEST, University of Washington, Seattle, June, 1950

PHILADELPHIA, Haverford College, November 26, 1949

ROCKY MOUNTAIN, University of Denver, April, 1950

SOUTHEASTERN, University of Florida, Gainesville, March, 1950

SOUTHERN CALIFORNIA, Immaculate Heart College, Hollywood, March 11, 1950

SOUTHWESTERN

TEXAS, Abilene, Spring, 1950

UPPER NEW YORK STATE, Syracuse University, Spring, 1950

WISCONSIN, Marquette University, Milwaukee, May, 1950

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A PROBLEM ON ARC TANGENT RELATIONS*

JOHN TODD, National Bureau of Standards

1. Notation. It will be convenient to use the letters $a, b, c, d, f, m, n, r, s, t, u, v, w$, with or without suffixes, to denote integers. The letter p will be used to denote prime numbers of the form $4n+1$. For any such p we denote by n_p the smallest n such that $1+n^2$ is a multiple of p. Finally we denote by (x) that value of arc tan x between 0 and $\frac{1}{2}\pi$, except when the context indicates that the brackets have their ordinary significance, for example, in $\Gamma(x+iy)$.

2. Introduction. Integral arc tangents. J. C. P. Miller [1], when engaged in the tabulation of $ln\ \Gamma(x+iy)$, used the relation

$$\ln \Gamma(x + iy + 1) = \ln (x^2 + y^2) + i \arctan (y/x) + \ln \Gamma(x + iy)$$

in order to transfer the argument to a region where an expansion in series converges with sufficient rapidity. He required, therefore, a table of arc tan (a/b) which he constructed and which is being extended by S. Johnston [1], [1a]. He observed identities of the form

$$(3) = 3(1) - (2), \qquad (7) = 2(2) - (1)$$

$$(8) = 2(1) + (3) - (5), \qquad (13) = 5(1) - (2) - (4)$$

and raised the question of the existence of such relations for general (n).

We call (n) irreducible if it cannot be expressed as a finite sum of the form

$$(2.1) \qquad (n) = \sum f_r \times (n_r)$$

where the f_r are integers** (positive or negative) and the n_r are (positive) integers less than n. If such a representation exists, as in the cases of (3), (7), (8) and (13) mentioned above, (n) will be called reducible.

The answer to Miller's question is included in the following theorem which has already been announced [12].

THEOREM A. *A condition necessary and sufficient for the reducibility of* (m) *is that all the prime factors of* $1+m^2$ *should occur among the prime factors of* $1+n^2$ *for* $n=1, 2, \cdots, m-1$.

Another condition is given in the next theorem.

THEOREM B. *A condition necessary and sufficient for the reducibility of* (m) *is that the largest prime factor of* $1+m^2$ *should be less than* $2m$.

We shall establish Theorem A (in §§6, 7) and deduce Theorem B from it

* This work was supported by the Office of Naval Research while the author was associated with the Institute for Numerical Analysis, National Bureau of Standards, while on leave from King's College, London.

** It can be deduced from Theorem A that nothing new is obtained by considering rational multipliers instead of integral ones.

(in §9). The proof of sufficiency in Theorem A will consist in establishing an algorithm which will completely reduce any reducible integral arc tangent, that is, express it in terms of the (irreducible) arc tangents of certain integers chosen from a base I:

$$1, 2, 4, 5, 6, 9, 10, 11, 12, 14, 15, 16, 19, 20, 22, \cdots .$$

It will be shown in §10 that the number of reducible arc tangents and the number of irreducible arc tangents are both infinite. Numerical evidence (obtained by the use of an unpublished table of factors of $1+n^2$ compiled by J. W. Wrench, Jr.) suggests that if $I(n)$ is the number of terms in I less than n then the ratio $I(n)/n$ has a limit about 0.71. This problem has been discussed elsewhere [13].

Two short tables of reductions are given. Their contents and construction are described in §§11, 12.

3. Rational arc tangents. D. H. Lehmer [2] has shown how to express arc cot (a/b) as a finite sum of the form*

$$\text{arc cot } (a/b) = \text{arc cot } n_0 - \text{arc cot } n_1 + \text{arc cot } n_2 - \cdots \tag{3.1}$$

where the integers n_i are obtained by an arc cotangent algorithm. This algorithm consists in the recurrence formulae

$$a_i = n_i b_i + b_{i+1} \ (0 \leqq b_{i+1} < b_i), \qquad a_{i+1} = a_i n_i + b_i$$

with the initial conditions $a_0 = a$, $b_0 = b$.

It follows that any (a/b) can be expressed as a finite sum of the form (2.1). The set I therefore constitutes a base for the rational arc tangents as well as the integral arc tangents. The problem of the existence of such a base has been studied by Gauss ([3], p. 523; [5], p. 74). He was apparently aware that any (a/b) could be expressed in terms of the "primitive arcs," that is, the arc tangents of certain rational numbers which constitute a base R:

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{4}{1}, \frac{5}{2}, \frac{6}{1}, \frac{5}{4}, \cdots .$$

The connection between the "rational base" R and the "integral base" I is indicated by the following remark: Corresponding to any p we have in I a term n_p and in R a term (a/b) where a, b are determined uniquely by the relations $a > b > 0$, $a^2 + b^2 = p$ and we can show that (a/b) can be expressed in terms of

* There is much to be said for using the arc cot as the basis for such considerations as those of the present paper. For instance, to have arc cot $n \to 0$ as $n \to \infty$ is often more convenient than arc tan $n \to \frac{1}{2}\pi$. Adoption of the arc cot might have prevented an error in the coefficient of (1) in the reduction of (1, 40333, 78718) which was fortunately detected by Professor Lehmer in the proofs of the announcement [12]. Nevertheless it has been decided to retain the original arc tan basis as with it checking is slightly easier. However extra columns have been added to the tables to give the representations of arc cot n in terms of arc cot n_i: in these the coefficients of (1) are somewhat smaller.

(n_p) and (m) with $m < n_p$.

4. The cases $m=2$ and $m=3$. The idea behind the proof of Theorem A has been used by many authors ([2], [3], [4], [5], [6], [7]). It is made clear in the following discussion of the cases $m=2$, $m=3$. The problem and the results do not appear to have been explicitly stated in the present form before although Størmer [5] has considered the solution of such equations as

$$f \times (1) = \sum_{r=1}^{s} f_r \times (m_r)$$

where s and f are fixed; his Theorem 2 ([5], p. 20) is essentially the same as Theorem A. He has given extensive tables of solutions. A similar problem has been studied by Bennett [11].

If (2) is reducible we must have $(2) = r(1)$ for some integer r. In this case the complex numbers $1+2i$ and $(1+i)^r$, having the same argument, must have a real ratio which is clearly a rational number, which we will denote by m/n. Taking moduli in the equation $(1+2i) = (m/n)(1+i)^r$ we have

$$5n^2 = 2^r m^2$$

which is manifestly impossible. Hence (2) is irreducible. If (3) is reducible we must have $(3) = r(1) + s(2)$ for some integers r, s. As before, we find

$$10n^2 = 2^r 5^s m^2.$$

This is soluble in integers. We find that the relevant solution is $r=3$, $s=1$, which gives

$$(3) = 3(1) - (2).$$

5. Auxiliary material. In the discussion of the general case we require certain results about complex or Gaussian integers, that is, complex numbers of the form $a+ib$, where a and b are integers. The results required will be found in standard textbooks (*e.g.*, [8] ch. 12, 14, 15). One complex integer is said to be a factor of another if the quotient, a complex number, is a complex integer. A complex integer e whose reciprocal e^{-1} is also a complex integer is called a unit; if e is a unit, any complex integer $a+ib$ and the complex integer $e(a+ib)$ are said to be associates. The only units among the complex integers are the numbers ± 1, $\pm i$. A complex integer is said to be prime if it is not 0, not a unit, and is divisible only by numbers associated with itself or with 1. Any complex integer can be expressed (uniquely apart from the order of the factors, the presence of units and the ambiguities between associated primes) as a product of primes. This decomposition can be carried out in a finite number of steps. A prime complex integer is either $1+i$ or one of its associates, an ordinary prime number of the form $4n+3$ or one of its associates, factors $a+ib$ of an ordinary prime number $4n+1$. It is known that the factors in the latter case are essentially unique; if one is $a+ib$ the others must be units or associated with $a-ib$.

We shall require in particular the following lemmas.

LEMMA 1. (Cf. [5], p. 8.) *If* u^2+v^2 *is a prime factor of* $1+n^2$ *then* $u+iv$ *is a factor of* $1\pm in$ *according as* u^2+v^2 *is a factor of* $nu\mp v$.

Proof. In the case $u^2+v^2=2$ it is easy to see that $u+iv$ divides both $1\pm in$. We shall not require this case in our proof and shall assume that $u^2+v^2>2$. We have

$$\frac{1+in}{u+iv}=\frac{(u\pm nv)+i(-v\pm nu)}{u^2+v^2}\cdot$$

Now $\mp n(u\pm nv)=-v(1+n^2)-(-v\pm nu)$; so u^2+v^2 is a factor of $\mp n(u\pm nv)$ if it is a factor of $-v\pm nu$. Assume u^2+v^2 is a factor of $-v\pm nu$. Then u^2+v^2 either divides n or $u\pm nv$. The first alternative is impossible for if u^2+v^2 was a factor of n it would also be a factor of n^2 which is impossible since it is a factor of $1+n^2$. It remains only to show that u^2+v^2 is a factor of $-v\pm nu$. It will be enough to show that u^2+v^2 divides the product $(-v+nu)(-v-nu)=v^2-n^2u^2=u^2+v^2-u^2(1+n^2)$, and this is obvious.

LEMMA 2. *Corresponding to any* p *there is an integer* $m<p$ *such that for an* $x=n_p\leqq\frac{1}{2}(p-1)$ *we have*

$$1+x^2=mp.$$

This is well-known (e.g. [8], p. 70).

We notice that there is an infinite number of primes of the form $4n+1$ and that every odd prime factor of a^2+b^2 (when a, b have no common factor) is of this form ([8], p. 13 and p. 297).

6. Proof of necessity. Suppose $(m)=\sum_{r=1}^{s}\pm(m_r)$ where the m_r, not necessarily all different, are all less than m. This means that

$$\arg\,(1+im)=\arg\prod_{r=1}^{s}(1\pm im_r)$$

and so the ratio

$$\frac{\prod_{r=1}^{s}(1-im_r)}{1+im}$$

is a real rational number. Since the numerator is of the form $n'+in''$ and the denominator of the form $1+im$ it is clear that the quotient is the integer n'. Taking moduli we find that

$$n'^2(1+m^2)=\prod_{r=1}^{s}(1+m_r^2)$$

from which the necessity of the condition stated follows.

EXAMPLES. In the case $m=7$, since $(7)=2(2)-(1)$ we have

$$n' = (1+2i)^2(1-i)/(1+7i) = 1.$$

In the case $m=286$, since $(286)=4(1)+(23)-(28)-(235)$ we have

$$n' = (1+i)^4(1+23i)(1-28i)(1-235i)/(1+286i) = 2^3\cdot 5\cdot 53.$$

In the case $m=313$, since $(313)=(1)-2(2)+(10)+(22)$ we have

$$n' = (1+i)(1-2i)^2(1+10i)(1+22i)/(1+313i) = 5.$$

7. Proof of sufficiency. It will be enough to express $1+im$ as the product of a real number and complex integers of the form $1+iw_r$ $(0<w_r<m)$, being given that each prime factor p of $1+m^2$ is a factor of $1+m_p^2$ for some $m_p<m$.

We begin by expressing $1+im$ as a product of prime complex integers. Take the prime factor $u+iv$ of largest modulus* which is neither of the required form nor associated with a number of the required form; suppose it occurs to the power $f\geqq 1$. We shall show how to replace this by a factor of the required form and one of smaller modulus. Suppose we have

$$1+im = (u+iv)^f(c+id). \tag{7.1}$$

Consider the following equation for a and b, where w is yet to be defined:

$$(u+iv)(a+ib) = 1+iw. \tag{7.2}$$

This gives

$$au-bv=1, \qquad av+bu=w. \tag{7.3}$$

We now observe that it is possible to choose solutions a and b of (7.3) such that $|w|<m$. Since u^2+v^2 is a prime factor of $1+m^2$ it is therefore also a factor of $1+n^2$ for some $n<m$. This implies, by virtue of our lemma, that $u+iv$ is a factor of $1\pm in$. We can certainly determine a, b such that (7.2) and (7.3) are satisfied with $w=\pm n$ and so with $|w|=n<m$.

We now proceed as follows. Multiply (7.1) across by $(a^2+b^2)^f$ to get

$$(1+im)(a^2+b^2)^f = (1+iw)^f(a-ib)^f(c+id). \tag{7.4}$$

Now take the prime factor u_1+iv_1 of largest modulus of $(a-ib)^f(c+id)$. Suppose it occurs to the power $f_1\geqq 1$ and that we have

$$(a-ib)^f(c+id) = (u_1+iv_1)^{f_1}(c_1+id_1).$$

Consider the equation for a_1, b_1

$$(u_1+iv_1)(a_1+ib_1) = 1+iw_1. \tag{7.5}$$

---

* There is only one factor of largest modulus (apart from associates) for if $u+iv$ and $u-iv$ were factors we would have u^2+v^2 a factor of 1.

We observe that it is possible to choose integers a_1, b_1 to satisfy (7.5) with a w_1 such that $|w_1| < m$. Since u_1+iv_1 is a factor either of $1+im$ or of $a-ib$ it follows that $u_1^2+v_1^2$ is a factor of $1+m^2$ or of a^2+b^2. In the first case the argument already given in the discussion of w applies; in the second, because of (7.2) a^2+b^2 is a factor of $1+w^2$ and since $|w| < m$ a similar argument applies.

Multiplying (7.4) across by $(a_1^2+b_1^2)^{f_1}$ we find

$$(1 + im)(a^2 + b^2)^f(a_1^2 + b_1^2)^{f_1} = (1 + iw)^f(1 + iw_1)^{f_1}(a_1 - ib_1)^{f_1}(c_1 + id_1).$$

Continuing in this way it is clear that we come, after a finite number of steps, to a relation

$$(1 + im)\prod (a_t^2 + b_t^2) = \prod (1 + iw_r)^{f_r}$$

from which the actual reduction for (m) follows by equating the argument of the two sides; hence

$$(m) = \sum f_r \times (w_r).$$

It may be necessary to adjust* this by addition of an appropriate multiple of $4(1)$ because of our convention that $0<(x)<\frac{1}{2}\pi$. This completes the proof of Theorem A.

8. Examples of reductions. Two examples will make clear how this reduction proceeds or comes to an end.

EXAMPLE 1. Since $1+286i=(11+6i)\ (11+20i)$ we consider the equation $(11+20i)\ (a+ib)=1+iw$. The general solution of this is

$$a = 11 - 20n, \qquad b = 6 - 11n, \qquad w = 286 - 521n.$$

We choose $n=1$, $w=-235$, and continue. Since

$$(1 + 286i)(106) = (1 - 235i)(- 9 + 5i)(11 + 6i)$$

we consider the equation $(11+6i)(a+ib)=1+iw$. The general solution of this is

$$a = - 1 + 11n, \qquad b = - 2 + 6n, \qquad w = - 28 + 132n.$$

We choose $n=0$, $w=-28$, and continue. Since

$$(1 + 286i)(106)(5) = (1 - 235i)(1 - 28i)(- 1 + 2i)(- 9 + 5i)$$

we consider the equation $(-9+5i)(a+ib)=1+iw$. The general solution of this is

* The adjustment can be carried out without reference to tables by carefully examining each stage of the reduction process. This is discussed in detail elsewhere [14].

$$a = 1 - 5n, \qquad b = -2 - 9n, \qquad w = 23 - 106n.$$

We choose $n = 0$ and have now obtained

$$(1 + 286i)(106)(5)(5) = (1 - 235i)(1 - 28i)(1 + 23i)(-1 + 2i)(1 + 2i)$$

whence, apart from multiples of $\pi = 4(1)$,

$$(286) = (23) - (28) - (235).$$

Checking numerically from tables we see that the required reduction is

$$(286) = 4(1) + (23) - (28) - (235).$$

EXAMPLE 2. Since $1+23i=(1+3i)(7+2i)$ we consider the equation $(7+2i)(a+ib)=1+wi$. The general solution of this is $a=1-2n$, $b=3-7n$, $w=23-53n$. It is clear we cannot get $|w|<23$ by choice of n and so (23) is irreducible.

9. Deduction of Theorem *B* from Theorem *A*. Suppose that every prime factor of $1+m^2$ is less than $2m$. Then, by Lemma 2, corresponding to each prime factor p of $1+m^2$, we have $n_p \leqq \frac{1}{2}(2m-1) < m$. That is, each such p occurs as a factor of $1+n^2$ for some $n=1, 2, \cdots, m-1$. Theorem A now shows that (m) is reducible.

On the other hand suppose that (m) is irreducible. We shall show that each prime factor p of $1+m^2$ is less than $2m$. For suppose we had $p \geqq 2m$. By Theorem A there is an $n_p < m$ such that $1+n_p^2$ is a multiple of p. Since $1+m^2$ is also a multiple of p, so is

$$m^2 - n_p^2 = (m - n_p)(m + n_p).$$

Now since one of these factors is less than m, the other less than $2m$, and neither vanish, we have arrived at a contradiction. This completes the proof of Theorem B.

10. The number of reducible and irreducible arc tangents. We shall now show that the number of irreducible arc tangents is infinite. If the number is finite choose a p greater than all the prime factors of the corresponding set of numbers $1+m^2$. In virtue of Lemma 2 we can choose an n such that $1+n^2$ contains p as a factor. It follows that (n) cannot be expressed in terms of the finite "base"; (n), therefore, is irreducible or has an irreducible component not in the given base.

We next show that the number of reducible arc tangents is infinite. In fact, corresponding to a given p we shall show that $(p-n_p)$ is reducible. For, by Lemma 2,

$$p - n_p < n_p < m = \frac{1 + n_p^2}{p}$$

and

$$(p - n_p) + (n_p) = \left(\frac{p - n_p + n_p}{1 - pn_p + n_p^2}\right) = \left(\frac{1}{m - n_p}\right) = 2(1) - (m - n_p)$$

so that $(p-n_p)=2(1)+(n_p-m)-(n_p)$. We complete the proof with the remark that we can certainly choose an infinite subsequence of the p for which the $p-n_p$ are all different.

EXAMPLES. If $p=13$ then $n_p=5$ and $m=2$; we find $(8)=2(1)+(3)-(5)=5(1)-(2)-(5)$.

If $p=37$ then $n_p=6$ and $m=1$; we find $(31)=2(1)+(5)-(6)$.

11. Description and construction of Table I. In Table I is set out the complete reductions of all reducible arc tangents (n), with $n \leqq 342$. The factor table used in its construction was that issued by the British Association for the Advancement of Science (Cambridge, 1935). This table covers the range up to 100,000.

There is no difficulty in extending this table further by use of the algorithm described or by the following process, which is often more convenient.

We suppose that the factorization of $1+m^2$ is known and that we have a table of n_p against p [15]. If a prime factor of $1+m^2$ is greater than $2m$ then (m) is irreducible. If not, we start from the highest factor p and build up $1+m^2$ into a product of numbers $1+n^2$ with $n<m$. For instance, $1+286^2=81797=157\cdot 521$. Since $521<2.286=572$, (286) is reducible. We note that $2\cdot 53\cdot 521=1+235^2$ and that $5\cdot 157=1+28^2$.

We now multiply across by $(2\cdot 53)^2$ to get

$$(1+286^2)(2\cdot 53)^2=2\cdot 53\cdot 157(1+235^2).$$

We next observe that $2\cdot 5\cdot 53=1+23^2$, and so

$$(1+286^2)(2\cdot 53)^2\cdot 5^2=2\cdot 5\cdot 53(1+28^2)(1+235^2)=(1+23^2)(1+28^2)(1+235^2).$$

We now know that (23), (28), (235) are the components of (286) and the signs can be determined by congruence relations similar to those given in Lemma 1. It remains to decide whether or not a multiple of $\pi=4(1)$ is to be added. This can be decided without use of tables [cf. 14].

In addition to the complete reduction of (n) for each n there is in Table I a column giving a coefficient c_n which enables the expression of arc cot n in terms of irreducible arc cotangents to be written down. In fact, if

$$(n)=\sum f_p\times(n_r)+f\times(1),$$

then

$$\operatorname{arc\,cot} n = \sum f_r \operatorname{arc\,cot} n_r + c_n \operatorname{arc\,cot} 1$$

where

$$c_n = 2 - f - 2\sum f_r.$$

TABLE I

n	c_n	(n)	n	c_n	(n)
3	1	3(1)−(2)	**182**	−3	−7(1)+5(2)+(23)
7	−1	−(1)+2(2)	**183**	0	4(1)+(3)−(4)−(14)
8	1	5(1)−(2)−(5)	**185**	0	2(1)+(28)−(33)
13	1	5(1)−(2)−(4)	**187**	−2	−2(1)+3(2)+(5)−(82)
17	−1	(1)+2(2)−(12)	**189**	1	7(1)−2(2)+(5)−(23)−(148)
18	1	3(1)−2(2)+(5)	**191**	−1	−(1)+(2)+(4)+(6)−(12)
21	0	2(1)+(4)−(5)	**192**	−1	3(1)+2(2)−(10)−(27)
30	1	7(1)−(2)−(4)−(23)	**193**	2	6(1)−3(2)−(5)+(44)
31	0	2(1)+(5)−(6)	**200**	0	−(4)+(5)+(19)
32	−1	(1)+2(2)−(9)	**203**	1	9(1)−(2)−(5)−(9)−(114)
38	0	−(2)+2(4)	**211**	0	2(1)+(14)−(15)
41	1	(1)−2(2)+2(12)	**212**	−1	−5(1)+(2)+(5)+(10)+(34)
43	1	3(1)−2(2)+(6)	**213**	1	7(1)−2(2)+(5)−(23)−(136)
46	−1	3(1)+2(2)−(12)−(27)	**216**	1	5(1)−(2)−(5)−(6)+(22)
47	0	4(1)+(2)−(4)−(5)	**217**	0	6(1)+(2)−(4)−(5)−(60)
50	0	2(1)+(9)−(11)	**228**	1	5(1)−2(2)+(6)−(53)
55	0	4(1)+(4)−(5)−(34)	**233**	1	9(1)−(2)−(5)−(11)−(34)
57	−2	−4(1)+3(2)+(5)	**237**	−1	3(1)+2(2)−(9)−(37)
68	2	8(1)−3(2)−(6)	**239**	−1	−5(1)+4(5)
70	0	−2(1)−(2)+2(5)+(12)	**241**	0	2(1)+(15)−(16)
72	−1	−3(1)+(2)+(4)+(11)	**242**	0	4(1)+(4)−(5)−(23)
73	1	7(1)−(2)−(5)−(9)	**251**	0	2(1)+(2)−2(4)+(33)
75	−1	3(1)+2(2)−(12)−(22)	**253**	1	5(1)−(2)−2(6)+(80)
76	0	2(1)+(23)−(33)	**254**	0	6(1)+(5)−(6)−(44)−(179)
83	1	5(1)−2(2)+(5)−(23)	**255**	0	−(5)+(9)+(11)
91	0	2(1)+(9)−(10)	**265**	−1	−(1)+(2)+(5)+(6)−(27)
93	1	5(1)−2(2)+(6)−(80)	**266**	1	7(1)−2(2)+(6)−(80)−(143)
98	1	7(1)−(2)−(4)−(15)	**268**	2	10(1)−2(2)−(4)−2(5)
99	1	5(1)−(2)−2(5)+(12)	**269**	−1	5(1)+2(2)−(12)−(22)−(104)
100	0	2(1)+(27)−(37)	**273**	1	(1)−2(2)+(12)+(16)
105	0	4(1)+(5)−(6)−(44)	**274**	0	2(1)+(81)−(115)
111	0	2(1)+(10)−(11)	**278**	0	4(1)+(5)−(9)−(12)
112	0	6(1)+(2)−(4)−(5)−(81)	**286**	0	4(1)+(23)−(28)−(235)
117	−1	−3(1)+(2)+2(6)	**288**	0	2(1)+(23)−(25)
119	0	2(1)+(22)−(27)	**293**	1	5(1)−2(2)+(4)−(10)
122	−2	−2(1)+3(2)+(5)−(107)	**294**	1	9(1)−(2)−(5)−(11)−(33)
123	0	−2(1)−(2)+(4)+(5)+(34)	**301**	0	−(2)+(4)+(5)+(34)−(208)
128	1	(1)−2(2)+(12)+(15)	**302**	0	−(4)+(6)+(12)
129	0	2(1)+(23)−(28)	**305**	0	2(1)+(64)−(81)
132	−1	−(1)+2(2)−(4)+(9)	**307**	2	4(2)−(5)−(12)
133	0	2(1)+(11)−(12)	**311**	0	2(1)+(37)−(42)
142	−1	−(1)+2(2)−(6)+(33)	**313**	1	(1)−2(2)+(10)+(22)
144	0	6(1)+(4)−(5)−(34)−(89)	**319**	1	5(1)−(2)−(4)−(9)+(27)
155	1	9(1)−(2)−(5)−(9)−(138)	**322**	0	4(1)+(2)−(4)−(5)−(34)+(89)
157	−1	−3(1)+(2)+(4)+(12)	**327**	−1	−(1)+(2)+2(4)−(6)
162	−1	3(1)+2(2)−(12)−(19)	**334**	0	2(1)+(53)−(63)
172	−1	3(1)+2(2)−(11)−(22)	**336**	1	5(1)−(2)−(4)−(12)+(107)
173	1	(1)−2(2)+(9)+(27)	**337**	−1	−5(1)+(2)+(5)+(9)+(60)
174	0	−2(1)−(2)+(4)+(5)+(37)	**338**	0	2(1)+(25)−(27)
177	0	6(1)+(2)−(4)−(5)−(64)	**342**	−1	−3(1)+2(2)−(5)+(28)+(44)

TABLE II

p	n_p	m	a	b	c_n	(a/b)
2	1	1	1	1	1	(1)
5	2	1	2	1	0	(2)
13	5	2	3	2	1	$3(1)-(5)$
17	4	1	4	1	0	(4)
29	12	5	5	2	0	$2(1)+(2)-(12)$
37	6	1	6	1	0	(6)
41	9	2	5	4	1	$3(1)-(9)$
53	23	2.5	7	2	1	$5(1)-(2)-(23)$
61	11	2	6	5	1	$3(1)-(11)$
73	27	2.5	8	3	1	$(1)-(2)+(27)$
89	34	13	8	5	1	$5(1)-(5)-(34)$
97	22	5	9	4	0	$2(1)+(2)-(22)$
101	10	1	10	1	0	(10)
109	33	2.5	10	3	0	$2(1)+(3)-(33)$
113	15	2	8	7	1	$3(1)-(15)$
137	37	2.5	11	4	1	$(1)-(2)+(37)$
149	44	13	10	7	1	$(1)-(5)+(44)$
157	28	5	11	6	0	$-2(1)+(2)+(28)$
173	80	37	13	2	0	$2(1)+(6)-(80)$
181	19	2	10	9	1	$3(1)-(19)$
193	81	2.17	12	7	1	$5(1)-(4)-(81)$
197	14	1	14	1	0	(14)
229	107	2.5^2	15	2	-1	$(1)+2(2)-(107)$
233	89	2.17	13	8	1	$(1)-(4)+(89)$
241	64	17	15	4	0	$-2(1)+(4)+(64)$
257	16	1	16	1	0	(16)
269	82	5^2	13	10	2	$2(1)-2(2)+(82)$
277	60	13	14	9	1	$5(1)-(5)-(60)$
281	53	2.5	16	5	1	$5(1)-(2)-(53)$
293	138	5.13	17	2	1	$7(1)-(2)-(5)-(138)$
313	25	2	13	12	1	$3(1)-(25)$
317	114	41	14	11	1	$5(1)-(9)-(114)$
337	148	5.13	16	9	1	$3(1)-(2)+(5)-(148)$
349	136	53	18	5	1	$7(1)-(2)-(23)-(136)$
353	42	5	17	8	0	$2(1)+(2)-(42)$
373	104	29	18	7	0	$4(1)+(2)-(12)-(104)$
389	115	2.17	17	10	1	$5(1)-(4)-(115)$
397	63	2.5	19	6	1	$5(1)-(2)-(63)$
401	20	1	20	1	0	(20)
409	143	2.5^2	20	3	-1	$-3(1)+2(2)+(143)$

TABLE II (Continued)

p	n_p	m	a	b	c_n	(a/b)
421	29	2	15	14	1	$3(1)-(29)$
433	179	2.37	17	12	1	$5(1)-(6)-(179)$
449	67	2.5	20	7	1	$(1)-(2)+(67)$
457	109	2.13	21	4	0	$2(1)+(5)-(109)$
461	48	5	19	10	1	$(1)-(3)+(48)$

12. Description and construction of Table II. In Table II is set out the complete reductions of (a/b) for all a, b without common factor and satisfying $a>b>0$, $a^2+b^2<500$. The table contains, in addition, for each $a^2+b^2=p\leqq 461$, the corresponding n_p and the residual factor $m=(1+n_p^2)/p$. It also gives the coefficient c_p by means of which the reduction of arc tan (a/b) can be transformed into a reduction of arc cot (a/b). If

$$(a/b) = \sum f_r \times (n_r) + f \times (1)$$

then

$$\text{arc cot } (a/b) = \sum f_r \text{ arc cot } n_r + c_p \text{ arc cot } 1.$$

This table was constructed by the following process. If a, b are integers without common factor and such that $a>b>0$ we find* non-negative integers $a_1<a$, $b_1<b$ such that

$$ab_1 - ba_1 = \pm 1.$$

We then observe that, where the ambiguous sign is the same as in the last equation,

$$\frac{(a/b) \pm (a_1a + b_1b)}{1 \mp (a/b)(a_1a + b_1b)} = \frac{\pm a(ab_1 - ba_1) \pm b(a_1a + b_1b)}{\pm b(ab_1 - ba_1) \mp a(a_1a + b_1b)} = \frac{-b_1}{a_1}.$$

Hence we have

$$(a/b) = -2(1) + (a_1/b_1) \mp (a_1a + b_1b),$$

expressing (a/b) in terms of arc tangents of integers and the arc tangent of a_1/b_1, a fraction with smaller denominator than a/b. Successive applications of this process will lead to an expression of (a/b) as a sum of arc tangents of integers; these can be expressed in terms of the irreducible arc tangents by the methods already described.

* The forthcoming volume in the series issued by the Royal Society, Mathematical Tables Committee, Farey Series of Order 1025 by E. H. Neville, will enable solutions of such equations to be obtained readily.

EXAMPLE. If $a=100$, $b=17$ then $a_1=47$, $b_1=8$. Since $8\cdot 100-17\cdot 47=1$ we have

$$(100/17) = -(4836) - (8/47) = -(4836) + 2(1) + (47/8).$$

Next if $a=47$, $b=8$ then $a_1=6$, $b_1=1$. Since $1\cdot 47-6\cdot 8=-1$, we have

$$(47/8) = (290) - (1/6) = (290) - 2(1) + (6).$$

Hence, referring to tables or otherwise determining [14] the appropriate multiple of $\pi=4(1)$ to add, we find $(100/17)=(6)+(290)-(4836)$. Each of these components can be proved irreducible.

This is a much more convenient process than obtaining the Lehmer-reduction and then completely reducing it because of the rapidity at which the numbers in Lehmer's algorithm increase. To see this we write down the decomposition (3.1) corresponding to the case just discussed:

$$\text{arc cot } (100/17) = \text{arc cot } 5 - \text{arc cot } 34 + \text{arc cot } 2513 - \text{arc cot } 22105608.$$

13. Acknowledgement. The author is indebted to E. M. Wilson of the Royal Naval Scientific Service for help in checking the reductions. This was done by combining the components according to the addition formula for the arc tangent and also numerically, using the standard tables [9], [10]. He is also indebted to J. C. P. Miller for reading an early version of the paper.

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AN APPROXIMATION TO THE QUOTIENT OF GAMMA FUNCTIONS

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1. A modification of Wallis' formula. A commonly used approximation for the middle term of the expansion of $(\frac{1}{2}+\frac{1}{2})^{2n}$ is

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n} \sim (n\pi)^{-1/2}. \tag{1}$$

This is equivalent to the product formula of Wallis

$$\frac{\pi}{2} = \lim_{n=\infty}\left(\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdots\frac{2n-2}{2n-1}\cdot\frac{2n}{2n-1}\right), \tag{2}$$

or to the approximation

$$\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} \sim n^{1/2}. \tag{3}$$

A much closer approximation, with a relative error less than $\frac{1}{2}(4n+1)^{-4}$ is obtained by replacing $n^{1/2}$ in (1) and (3) by $(n^2+\frac{1}{2}n+\frac{1}{8})^{1/4}$. We then have

$$\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} \sim \left(n^2+\frac{n}{2}+\frac{1}{8}\right)^{1/4}. \tag{4}$$

Upon squaring both members of (4), we obtain for integral n the following approximation for π, which is a modification of Wallis' formula:

$$\pi_n = \left(\frac{2}{1}\cdot\frac{4}{3}\cdots\frac{2n}{2n-1}\right)^2\left(n^2+\frac{n}{2}+\frac{1}{8}\right)^{-1/2}; \qquad \lim_{n=\infty}\pi_n = \pi. \tag{5}$$

This approximation π_n can be shown to satisfy the inequality

$$\pi - \frac{\pi}{(4n+1)^4} < \pi_n < \pi - \frac{0.74\pi}{(4n+1)^4}, \qquad \text{for } n \geqq 1. \tag{6}$$

2. The Gamma function quotient. Formula (4) is but a special case of the following excellent approximation to the quotient of two nearby values of the Gamma function:

$$\frac{\Gamma\left(n+\dfrac{1+u}{2}\right)}{\Gamma\left(n+\dfrac{1-u}{2}\right)} \sim \left(n^2+\frac{1-u^2}{12}\right)^{u/2}. \tag{7}$$

Formula (7) is an exact equality for all $n\geqq 1$, when $u=0, \pm 1, \pm 2$. Its logarithmic relative error $E_n(u)$, and a related correction factor $F_n(u)$ may be de-

fined by the equations

$$\frac{\Gamma\left(n+\frac{1+u}{2}\right)}{\Gamma\left(n+\frac{1-u}{2}\right)} = \left(n^2 + \frac{1+u^2}{12}\right)^{u/2} e^{-E_n(u)}, \tag{8}$$

$$E_n(u) = \frac{u(1-u^2)(4-u^2)}{6!n^4} F_n(u). \tag{9}$$

We shall show that $F_n(u)$ is an even function of both n and u, and that it is positive but less than 1 for $n \geqq 1$, $|u| \leqq 1$. A good estimate for $F_n(u)$ will be given in (39) which is sufficiently accurate to define $E_n(u)$ to six decimals, and is a simple rational function of n^2 and u^2. To give the reader an idea of the magnitude of $F_n(u)$ we now tabulate in table (10) certain of its values, the computation of which will be discussed in Section 4.

Table for $F_n(u)$

(10)

u \ n	1	2	3	4	∞
0	0.636	0.865	0.933	0.960	1.000
0.5	0.658	0.875	0.937	0.963	1.000
1.0	0.734	0.904	0.952	0.972	1.000
2.0	1.257	1.040	1.015	1.008	1.000

From (8), (9), and (10) it is evident that the approximation (7) is good to at least four significant figures for $n > 2$, $|u| \leqq 1$.

3. The asymptotic expansion. To obtain the approximation (7) and estimate the error factors $E_n(u)$ and $F_n(u)$ in (8) and (9), we start with the asymptotic expansion in powers of $1/n$ of a related function $G(x, n)$ and the power sum polynomials $S_\nu(x)$ which it generates. Let

$$G(x, n) = \ln \frac{\Gamma(n+x)}{n^x \Gamma(n)} = -\sum_{\nu=1}^{\infty} \frac{S_\nu(x)}{\nu(-n)^\nu}. \tag{11}$$

For integral values of x, and for $n > |x|$, we may expand $G(x, n)$ in a convergent power series in $1/n$, thus:

$$G(x, n) = \sum_{k=1}^{x-1} \ln\left(1 + \frac{k}{n}\right) = -\sum_{\nu=1}^{\infty}\left(\frac{1}{\nu(-n)^\nu} \sum_{k=1}^{x-1} k^\nu\right). \tag{12}$$

Comparing (11) and (12) we see that $S_\nu(x)$ are the power sum polynomials:

$$(13)\qquad S_\nu(x) = \sum_{k=1}^{x-1} k^\nu = \frac{x^{\nu+1}}{\nu+1} - \frac{1}{2}x^\nu + \sum_{r=1}^{[\nu/2]} \frac{(-1)^{r-1}B_r}{2r}\binom{\nu}{2r-1}x^{\nu+1-2r}$$

where the coefficients B_r are the Bernoulli numbers

$$(14)\qquad B_1 = 1/6,\ B_2 = 1/30,\ B_3 = 1/42,\ B_4 = 1/30,\ B_5 = 5/66, \text{ etc.}$$

If x is not an integer, the formulas (11) and (13) are still valid for $n > |x|$, but the expansion (12) which was used in the proof of (13) must be replaced by the somewhat more complicated asymptotic expansion (15), which is called Stirling's formula with the remainder, namely,

$$(15)\qquad \begin{aligned} &\ln \Gamma(n) = \tfrac{1}{2}\ln(2\pi) + (n - \tfrac{1}{2})\ln n - n + R(n), \quad \text{where} \\ &R(n) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{n^{2k-1}}\frac{B_k}{2k(2k-1)}. \end{aligned}$$

Thus,

$$(16)\qquad \begin{aligned} G(x, n) &= (n + x - \tfrac{1}{2})\ln(n+x) - (n - \tfrac{1}{2})\ln n + R(n+x) - R(n) \\ &= -\sum_{\nu=1}^{\infty} S_\nu(x)/\nu(-n)^\nu. \end{aligned}$$

Taking the logarithm of the left member of (7) and subtracting $u \ln n$ we obtain the function

$$(17)\qquad \begin{aligned} &G\left(\frac{1+u}{2}, n\right) - G\left(\frac{1-u}{2}, n\right) \\ &\qquad = -\sum_{\nu=1}^{\infty}\left\{S_\nu\left(\frac{1+u}{2}\right) - S_\nu\left(\frac{1-u}{2}\right)\right\}/\nu(-n)^\nu. \end{aligned}$$

This is obviously an odd function of u. But some other properties of this function, such as the fact that it is an even function of n, are most easily derived by using an exponential generating function $S(x, t)$ for the polynomials $S_\nu(x)$, namely,

$$(18)\qquad S(x, t) = \frac{e^{xt} - 1}{e^t - 1} = \sum_{\nu=0}^{\infty} S_\nu(x)\frac{t^\nu}{\nu!}\cdot$$

It may interest some readers to mention that the function $-G(x, -n)$ is the Laplace transform of the function $[S(x, t) - x]/t$. Certain difficulties with convergence are avoided by working with the exponential series (18) instead of the logarithmic series (16) to derive the properties of the power sum polynomials $S_\nu(x)$. Thus if we expand the left members of the identity

$$(19)\qquad S(1 - x, -t) + S(x, t) \equiv 1,$$

using (18), compare coefficients, and set $x=(1+u)/2$, we obtain

$$(-1)^\nu S_\nu\left(\frac{1-u}{2}\right)+S_\nu\left(\frac{1+u}{2}\right)=0, \qquad \text{for } \nu>0. \tag{20}$$

Hence the odd powers of $1/n$ drop out of the expansion (17). The polynomial coefficients for even powers of $1/n$ may be generated by using another identity, namely,

$$\begin{aligned}&\frac{1}{u}\left\{S\left(\frac{1+u}{2},\ t\right)-S\left(\frac{1-u}{2},\ t\right)\right\}\\&=\frac{\sinh\frac{1}{2}ut}{u\sinh\frac{1}{2}t}=1+\frac{u^2-1}{12}\frac{t^2}{2!}+\frac{(u^2-1)(3u^2-7)}{2\cdot 5!}\frac{t^4}{4!}+\cdots.\end{aligned} \tag{21}$$

Observing the first two terms in the expansion just given, and observing that the function (21) has the value 1 for $u=1$ and the value $\cosh\frac{1}{2}t$ for $u=2$, we may preserve these properties by using the approximating function $\cosh wt$, where $w^2=(u^2-1)/12$. The remainder after subtracting $\cosh wt$ from (21) is a function $P(u, t)$ which is small for small t and vanishes for all t when $u=\pm 1$ or ± 2. It generates certain polynomials $P_\nu(u)$ which are closely related to the function $E_n(u)$ which we set out to estimate:

$$\begin{aligned}P(u,t)&=\sum_{\nu=1}^{\infty}P_\nu(u)\frac{t^{2\nu}}{(2\nu-1)!}=\frac{\sinh\frac{1}{2}ut}{u\sinh\frac{1}{2}t}-\cosh\left(\frac{u^2-1}{12}\right)^{1/2}t\\&=\frac{(u^2-1)(u^2-4)}{6!}\frac{t^4}{3!}+\cdots\end{aligned} \tag{22}$$

where

$$P_\nu(u)=-\frac{1}{2\nu}\left(\frac{u^2-1}{12}\right)^\nu+\frac{1}{2\nu u}\left\{S_{2\nu}\left(\frac{1+u}{2}\right)-S_{2\nu}\left(\frac{1-u}{2}\right)\right\}. \tag{23}$$

We multiply both sides of (23) by $un^{-2\nu}$, and add, recalling that the odd powers of $1/n$ drop out of the expansion (17). This brings in the expression $(u/2)$ $\ln(1-w^2/n^2)$ where $w^2=(u^2-1)/12$, which suggests the approximation (7). It gives for $E_n(u)$ the expression:

$$\begin{aligned}E_n(u)&=u\sum_{\nu=1}^{\infty}P_\nu(u)n^{-2\nu}\\&=\frac{u}{2}\ln\left(1+\frac{1-u^2}{12n^2}\right)-G\left(\frac{1+u}{2},\ n\right)+G\left(\frac{1-u}{2},\ n\right).\end{aligned} \tag{24}$$

In like manner we obtain an exponential generating function $Q(u, t)$ for the polynomials $Q_\nu(u)$ which occur as coefficients of the powers of $1/n^2$ in the asymptotic expansion of the function $F_n(u)$ defined in (9). We have

$$F_n(u) = \sum_{\nu=2}^{\infty} Q_\nu(u) n^{4-2\nu} \tag{25}$$

where

$$\begin{aligned} Q(u, t) &= \sum_{\nu=1}^{\infty} Q_\nu(u) \frac{t^{2\nu}}{(2\nu - 1)!} \\ &= \frac{6!}{(1 - u^2)(4 - u^2)} \left\{ \frac{\sinh \frac{1}{2}ut}{u \sinh \frac{1}{2}t} - \cosh \left(\frac{u^2 - 1}{12} \right)^{1/2} t \right\}. \end{aligned} \tag{26}$$

It is evident from (26) that the polynomials $Q_\nu(u)$ are even functions of u, and from (25) that $F_n(u)$ is an even function of both n and u. By expanding a few terms of (26) and substituting in (25) we find

$$\begin{aligned} F_n(u) = 1 &- \frac{85 - 25u^2}{126n^2} + \frac{107 - 40u^2 + 5u^4}{144n^4} \\ &- \frac{6467 - 2523u^2 + 405u^4 - 29u^6}{4752n^6} + \cdots . \end{aligned} \tag{27}$$

The coefficients appear somewhat simpler if $v = (1 - u^2)/3$ is introduced as a new variable. Then

$$\begin{aligned} F_n(u) = f_n(v) = 1 &- \frac{20 + 25v}{42n^2} + \frac{8 + 10v + 5v^2}{16n^4} \\ &- \frac{160 + 200v + 106v^2 + 29v^3}{176n^6} + \cdots . \end{aligned} \tag{28}$$

The general expression for the polynomials $q_k(v)$ which occur as numerators of the terms in (28) is rather complicated. We state it here for completeness:

$$\begin{aligned} \frac{(-4)^k k q_k}{120} = &- \frac{v^{k-1} - (-1)^{k-1}}{3(v + 1)} + \sum_{h=0}^{k-2} \frac{(4^h - 2)B_h}{2k + 1} \binom{2k + 1}{2h} \\ &\times \left\{ \frac{(3v - 1)^{k-h} - (-1)^{k-h}}{3v} - \frac{(3v - 1)^{k-h} - (-4)^{k-h}}{3(v + 1)} \right\}. \end{aligned} \tag{29}$$

4. Computation of the correction factor $F_n(u)$. Although the terms in the alternating series (27) and (28) diminish rapidly when u is small and n is large, the νth term does not ultimately approach zero for fixed n as ν becomes infinite. Instead the series is an alternating asymptotic series, in which for fixed n sufficiently large the successive terms decrease at first and then increase without limit as ν becomes infinite.

For small n we cannot use the series (27) to evaluate the entries for $F_n(u)$ in table (10). Instead we make use of the Euler-MacLaurin summation formula applied to the function $1/n$:

$$\sum_{k=1}^{n} \frac{1}{k} = \gamma + \ln n + \frac{1}{2n} - \frac{B_1}{2n^2} + \frac{B_2}{4n^4} - \frac{B_3}{6n^6} + \cdots \tag{30}$$
$$\gamma = 0.57721\ 56649\ 01533 \cdots \text{ (Euler's constant).}$$

where B_ν are the Bernouilli numbers (14). We define the function $H(n)$ by the formula

$$\begin{aligned} H(n) &= 120\left(\frac{B_2}{4} - \frac{B_3}{6n^2} + \frac{B_4}{8n^4} - \cdots\right) \\ &= 120n^4\left\{\sum_{k=1}^{n} \frac{1}{k} - \frac{1}{2n} - \gamma - \ln n + \frac{1}{12n^2}\right\}. \end{aligned} \tag{31}$$

Then

$$\begin{aligned} &H(1) = 70 - 120\gamma = 0.73412020 \cdots, \\ &H(2) = 0.90433,\ H(3) = 0.95229,\ H(4) = 0.97200. \end{aligned} \tag{32}$$

In order to expand the generating functions (22) and (26) we need to use the fact that

$$\frac{t}{2} \operatorname{csch} \frac{t}{2} = 1 + \sum_{\nu=1}^{\infty} (1 - 2^{1-2\nu}) \frac{B_\nu}{(2\nu)!} (-t^2)^\nu. \tag{33}$$

Then by (26) and (22) we find

$$\begin{aligned} Q(0, t) &= \frac{6!}{4} P(0, t) = \frac{6!}{4}\left\{\frac{t}{2} \operatorname{csch} \frac{t}{2} - \cos \frac{t}{(12)^{1/2}}\right\} \\ &= \frac{6!}{4} \sum_{\nu=2}^{\infty} \left\{(1 - 2^{1-2\nu})B_\nu - \frac{1}{12^\nu}\right\} \frac{(-t^2)^\nu}{(2\nu)!}. \end{aligned} \tag{34}$$

Hence, by (25), (26), and (34), we obtain

$$\begin{aligned} F_n(0) &= \frac{6!}{4} \sum_{\nu=2}^{\infty} \left\{(1 - 2^{1-2\nu})B_\nu - \frac{1}{12^\nu}\right\} \frac{(-1)^\nu}{2\nu} n^{4-2\nu} \\ &= \frac{3}{2} H(n) - \frac{3}{16} H(2n) - \frac{5}{8} (144n^4)\left\{\frac{1}{12n^2} - \ln\left(1 + \frac{1}{12n^2}\right)\right\}. \end{aligned} \tag{35}$$

As an important special case we may calculate the smallest $F_n(0)$ for $n = 1$, namely

$$F_1(0) = 0.93180 - 0.31250 + 0.01736 - 0.00108 + 0.00007 = 0.63566. \tag{36}$$

Similarly, for $u = 1$, we find

$$Q(1, t) = \frac{6!}{6} P_u(1, t) = -120\left(\frac{t}{2}\coth\frac{t}{2} - 1 - \frac{t^2}{12}\right) \tag{37}$$
$$= 120 \sum_{\nu=2}^{\infty} \frac{B_\nu(-t^2)^\nu}{(2\nu)!}$$

$$F_n(1) = 120n^4 \sum_{\nu=2}^{\infty} \frac{B_\nu}{2\nu} \frac{1}{(-n^2)^\nu} = H(n). \tag{38}$$

Thus the values of $F_n(1)$ are given by (32) and (38), those of $F_n(0)$ by (35), and the other values in table (10) by similar expansions. The data of table (10) may be fitted to at least two significant figures by the following rational function

$$F_n(u) \sim \frac{n^2 + 0.33 + 0.06(u^2 - 1)/n^2}{n^2 + 1 - 0.20u^2} \qquad n \geqq 1, 0 \leqq u \leqq 2. \tag{39}$$

By using this expression for $F_n(u)$ in conjunction with the formulas (8) and (9), we can evaluate the ratio of two values of the Gamma function to at least six significant figures, provided that the arguments differ by not more than 2.

A PROBLEM IN DIFFERENCE SETS

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1. Introduction. The propositions to be proved in this paper were suggested by a result quoted in *Scripta Mathematica** from an Italian journal.† They are substantially as follows: Let a, b, c, d be a set of 4 positive integers, and form the 4 absolute differences $|a-b|, |b-c|, |c-d|, |d-a|$ in a cyclical manner. The new set thus obtained will be called the derivative of the original set. If we now form, in the same manner, the higher derivatives of the original set, we shall, after a finite number of steps, arrive at a set all terms of which vanish.‡

The main object of this paper will be to show that this property does not depend on the fact that the terms of the original set are all integers. The precise theorem to be proved is:

THEOREM I: *Let a, b, c, d be a set of four real numbers. By forming the successive derivatives (as defined above) of this set we will always, after a finite number of steps, obtain a set all terms of which are equal to zero, except in the case in which the first derivative of the original set is, apart from trivial transformations, of the form*

$$1, q, q^2, q^3$$

* Scripta Mathematica, vol. V, p. 135 (April 1938).

† Periodiche di Matematiche, vol. 17, pp. 25–30 (1937).

‡ The same is true about sets of 8, 16, 32, $\cdots$, integers (*ibidem*).

where q is the positive solution of the equation.

$$q^3 - q^2 - q - 1 = 0.$$

2. Discussion and proof of the theorem. Before entering upon the proof of Theorem I, we shall make a few general observations.

Consider $n(n \geqq 2)$ real numbers a_i arranged in a cyclical manner, *i.e.*, $\cdots a_n, a_1, a_2, \cdots, a_n, a_1 \cdots$. This arrangement will be called a cycle of n terms and denoted by Q_n. We shall write:

$$Q_n \equiv (a_1, a_2, \cdots, a_n), \quad \text{or} \quad Q_n \equiv (a_i).$$

It is immaterial which term of Q_n is regarded as the first; only the relative position of the terms in Q_n is important.

The cycle whose terms are the absolute differences of each two consecutive terms of Q_n, will be denoted by $Q_n' \equiv (a_i')$ and called the derivative of Q_n, *i.e.*,

$$Q_n' \equiv (|a_1 - a_2|, \ |a_2 - a_3|, \cdots, \ |a_{n-1} - a_n|, \ |a_n - a_1|).$$

The operation of finding the derivative will be called differentiation. The derivative of Q_n' will be denoted by Q_n'' and called the second derivative of Q_n, and so on. If after k (k finite) differentiations of Q_n we arrive at a $Q_n^{(k)}$ such that its terms are all zero, then Q_n will be called a vanishing cycle. If no such k exists, Q_n will be called a non-vanishing cycle.

If r consecutive terms are repeated in Q_n, m times in the same order, so that $n = m \cdot r$, $r \geqq 2$, $m \geqq 2$, Q_n will be called a periodic cycle; evidently all its derivatives are also periodic cycles.

When $n \geqq 3$ and the terms of Q_n form a geometric progression the last term of which equals the sum of all other terms, Q_n will be called a geometric cycle and denoted by Q_n^*, *i.e.*,

$$Q_n^* \equiv (a, aq, aq^2, \cdots, aq^{n-1}), \quad a > 0, \quad q^{n-1} = \sum_{i=0}^{n-2} q^i.$$

The last equation in q always has exactly one positive root,† $1 < q < 2$, increasing with n. The derivative $Q_n^{*\prime}$ is also a geometric cycle, since $a_i' = a_i(q-1)$ for all i (also for $i = n$, in view of the definition of q); the same is therefore true for all higher derivatives $Q_n^{*(k)}$. Hence we have the result: A geometric cycle never vanishes.

The terms a_i' of a derivative Q_n' of any cycle Q_n satisfy

$$\sum \pm a_i' = 0,$$

if we choose the proper signs. Indeed,

† It is obvious that there is a root between 1 and 2. It follows from Descartes' rule of signs that there is only one positive root.

$$\begin{aligned}\sum \pm a_i' &= + | a_1 - a_2 | \pm | a_2 - a_3 | \pm \cdots \pm | a_n - a_1 | \\ &= \pm [(a_1 - a_2) + (a_2 - a_3) + \cdots + (a_n - a_1)] = 0.\end{aligned}$$

Any cycle of two terms vanishes by no more than two differentiations. In fact

$$Q_2 \equiv (a, b); \qquad Q_2' \equiv (| a - b |, | b - a |); \qquad Q_2'' \equiv (0, 0).$$

Any Q_n, n odd, containing at least two unequal terms, never vanishes. This is seen as follows: If Q_n vanishes by exactly k differentiations, it is clear that $k>1$. Then $Q_n^{(k-1)}$ must contain only equal terms; but an odd number of equal terms cannot satisfy the condition $\sum \pm a_i^{(k)} = 0$, shown above to be necessary. Consequently $Q_n^{(k-1)}$ cannot be a derivative of any cycle, whence the contradiction.

For every n, except powers of 2, there exists an infinity of non-vanishing cycles Q_n. For odd n's, this has just been shown; if $n = 2^p \cdot r$, $r(\geqq 3)$ odd, at least all periodic cycles Q_n having 2^p periods of r unequal terms (and consequently all cycles having one such derivative) are non-vanishing.

If n is a power of 2, $(n = 2^p)$, there exist at least $p-1$ different non-vanishing cycles. These are the geometric cycle Q_n^* and the $p-2$ periodic cycles containing as periods one of the geometric cycles

$$Q_4^*, \ Q_8^*, \ \cdots, \ Q_{n/2}^*.$$

We shall now limit the discussion to cycles of 4 terms:

$$Q_4 \equiv Q \equiv (x, y, z, u).$$

LEMMA I: *If two non-consecutive terms of Q are equal, Q vanishes after not more than* 4 *differentiations.*

Proof: Let $z = x$;
then,

$$\begin{aligned} Q' &\equiv (| x - y |, | x - y |, | x - u |, | x - u |), \\ Q'' &\equiv (0, || x - y | - | x - u ||, 0, || x - y | - | x - u ||), \end{aligned}$$

or

$$\begin{aligned} Q'' &\equiv (0, t, 0, t), \\ Q^{(3)} &\equiv (t, t, t, t), \\ Q^{(4)} &\equiv (0, 0, 0, 0). \end{aligned}$$

Now take any Q_4 and find Q'. The four terms of Q' may be written a, $a+b$, $a+b+c$, $a+b+c+d$, $(a, b, c, d \geqq 0)$; this of course is not necessarily their proper order in Q'. As the four terms of Q' must satisfy $\sum \pm a_i' = 0$, there arise two possibilities; namely,

I. the sum of two terms is equal to the sum of the rest, whence in all cases $d=b$.

II. the sum of three terms is equal to the fourth, and then $d=2a+b$.

Since we also have to take into account the relative position of the terms of Q', each case gives rise to three types:

A, in which a is not adjacent to $a+b$;
B, in which a is not adjacent to $a+b+c+d$;
C, in which a is not adjacent to $a+b+c$.

Thus Q' may be of one of the six following types:

	I	II
A	$(a,\ a+b+c,\ a+b,\ a+2b+c)$	$(a,\ a+b+c,\ a+b,\ 3a+2b+c)$
B	$(a,\ a+b,\ a+2b+c,\ a+b+c)$	$(a,\ a+b,\ 3a+2b+c,\ a+b+c)$
C	$(a,\ a+b,\ a+b+c,\ a+2b+c)$	$(a,\ a+b,\ a+b+c,\ 3a+2b+c)$

LEMMA II. *If Q' is not of the type* CII, *Q vanishes by at most seven differentiations.*

Proof: If Q' is of the types I, Q'' contains two equal non-adjacent terms and vanishes, by Lemma I, after 4 differentiations. If Q' is of the types AII or BII, then in Q'' the sum of two terms is equal to the sum of the other two, *i.e.*, Q'' is here of the types Q'I, and vanishes therefore after 5 differentiations.

COROLLARY: If in Q' one term is zero (*i.e.*, if Q contains two equal consecutive terms), Q vanishes by a few differentiations, as Q' is then of the types I.

By LEMMA II and its corollary all derivatives $Q_4^{(k)}$ of a non-vanishing cycle must be monotonic and of the form

$$(a,\ a+b,\ a+b+c,\ 3a+2b+c)$$

with a, b, c positive. In this case $Q_4^{(k+1)} \equiv (b,\ c,\ 2a+b,\ 2a+2b+c)$, and necessarily $b<c<2a+b$. Hence the monotony is preserved in the same sense in all derivatives. Denoting the first derivative of a non-vanishing Q_4 by P, and the higher ones by P_1, P_2, P_3, and so on, P may be written

$$(x,\ y,\ z,\ x+y+z), \text{ where } 0<x<y<z,$$

and similarly

$$P_n \equiv (x_n,\ y_n,\ z_n,\ x_n+y_n+z_n), \qquad 0<x_n<y_n<z_n.$$

But

$$P_1 \equiv (y-x,\ z-y,\ x+y,\ y+z).$$

Denoting by M the matrix

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

we may write:

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = M \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

and generally:

$$\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = M^n \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Here are the values of some M^n which will be needed later:

$$M^2 = \begin{pmatrix} 1 & -2 & 1 \\ 1 & 2 & -1 \\ -1 & 0 & 1 \end{pmatrix}, \qquad M^3 = 2 \cdot \begin{pmatrix} 0 & 2 & -1 \\ -1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$M^5 = 2 \cdot \begin{pmatrix} 3 & 4 & -3 \\ -3 & 0 & 1 \\ 1 & -2 & 1 \end{pmatrix}, \qquad M^6 = (M^3)^2 = 2^2 \cdot \begin{pmatrix} -3 & -2 & 2 \\ 2 & -1 & 0 \\ 0 & 2 & -1 \end{pmatrix},$$

$$M^{12} = (M^6)^2 = 2^4 \cdot \begin{pmatrix} 5 & 12 & -8 \\ -8 & -3 & 4 \\ 4 & -4 & 1 \end{pmatrix}, \quad M^{24} = (M^{12})^2 = 2^8 \cdot \begin{pmatrix} -103 & 56 & 0 \\ 0 & -103 & 56 \\ 56 & 56 & -47 \end{pmatrix};$$

and generally, for $k=0, 1, 2, \cdots$,

$$M^{3 \cdot 2^k} = 2^{2k} \cdot T_k, \quad \text{where} \quad T_k = \begin{pmatrix} a_{1,k} & b_{1,k} & c_{1,k} \\ a_{2,k} & b_{2,k} & c_{2,k} \\ a_{3,k} & b_{3,k} & c_{3,k} \end{pmatrix};$$

the numbers $a_{i,k}$, $b_{i,k}$, $c_{i,k}$ are integers.

By the preceding,

$$0 < y_6 = 4(2x - y), \quad 0 < y_5 = 2(-3x + z), \quad 0 < z_6 = 4(2y - z).$$

Hence, $y < 2x$, $3x < z < 2y$, and therefore

$$x_1 = y - x < x, \qquad y_1 = z - y < y, \qquad z_1 = x + y < 3x < z.$$

Thus the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ are strictly descending, and clearly $0<x_{24}<x$, $0<y_{24}<y$. But

$$x_{24} = 2^8(-103x + 56y), \qquad y_{24} = 2^8(-103y + 56z);$$

hence, $1\cdot 8392<(y/x)<1\cdot 8394$, $1\cdot 8392<(z/y)<1\cdot 8394$. From this we easily deduce that the ratios x_1/x, y_1/y, z_1/z are less than $0\cdot 84$, and therefore less than $2^{-1/4}$. The same is true generally about x_{k+1}/x_k, *etc.* Hence, x_n/x, y_n/y, z_n/z are all less than $2^{-n/4}$.

Now consider the infinite set S of derivatives P_n, where $n=3\cdot 2^k$. As the determinant of M is

$$|M| = \begin{vmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2,$$

it follows that $|M^n| = 2^n$ and $|I_k| = 1$; consequently in every $|T_k|$ at least one minor is different from zero. Hence S contains an infinite sub-set S_1, for the terms of which at least one minor, of constant place in T_k, always differs from zero. We shall assume that in S_1, $D(a_{3,k})$, the minor of $s_{3,k}$, always differs from zero. Later on it will become clear that this assumption does not affect the generality of our proof.

For the terms x_n and y_n of every P_n belonging to S_1 we may write

$$0 < x_n < (2^{-1/4})^{3\cdot 2^k}\cdot x = x\cdot 2^{-3\cdot 2^{k-2}},$$
$$0 < y_n < 2x_n < 2x\cdot 2^{-3\cdot 2^{k-2}},$$

or

$$0 < 2^{2k}(a_1x + b_1y + c_1z) < x\cdot 2^{-3\cdot 2^{k-2}},$$
$$0 < 2^{2k}(a_2x + b_2y + c_2z) < 2x\cdot 2^{-3\cdot 2^{k-2}},$$

(writing simply a_i, b_i, c_i instead of $a_{i,k}$, $b_{i,k}$, $c_{i,k}$).

We now write

$$\begin{cases} a_1 + b_1\left(\dfrac{y}{x}\right) + c_1\left(\dfrac{z}{x}\right) = \delta_k, \\[2ex] a_2 + b_2\left(\dfrac{y}{x}\right) + c_2\left(\dfrac{z}{x}\right) = \epsilon_k, \end{cases}$$

where δ_k and ϵ_k denote positive numbers which are arbitrarily small for sufficiently large k; indeed,

$$0 < \delta_k < 2^{-7\cdot 2^{k-2}}, \qquad 0 < \epsilon_k < 2\cdot 2^{-7\cdot 2^{k-2}}.$$

(For a definite k, δ_k and ϵ_k may vary perhaps, but at any rate not outside the given intervals.)

Solving the two equations for y/x and z/x, we obtain:

$$\frac{y}{x} = \frac{c_1a_2 - c_2a_1}{b_1c_2 - b_2c_1} + \eta_k, \quad \text{where} \quad \eta_k = \frac{\delta_k c_2 - \epsilon_k c_1}{b_1c_2 - b_2c_1}.$$

The denominators are $D(a_{3,k})$, which by hypothesis is not zero; as a determinant of integers, $|D(a_{3,k})| = |b_1c_2 - b_2c_1| \geqq 1$, and

$$|\eta_k| \leqq |\delta_k c_2 - \epsilon_k c_1| \leqq \delta_k |c_2| + \epsilon_k |c_1| \leqq (\delta_k + \epsilon_k)m_k,$$

denoting by m_k the maximum of $|a_{i,k}|$, $|b_{i,k}|$, $|c_{i,k}|$. As $T_{k+1} = (T_k)^2$, the terms of T_{k+1} are of the form:

$$a_{i,k}j_{1,k} + b_{i,k}j_{2,k} + c_{i,k}j_{3,k},$$

i standing for 1, 2 or 3, and j standing for a, b or c. Hence $m_{k+1} \leqq 3(m_k)^2 = \frac{1}{3}(3m_k)^2$, and, by induction,

$$m_{k+p} \leqq \tfrac{1}{3}(3m_k)^{2p}.$$

For $k=2$, *i.e.*, $n = 3 \cdot 2^2 = 12$, we found above (see M^{12}) $m_k = 12$; therefore, for $k > 2$,

$$m_k \leqq \tfrac{1}{3}(3 \cdot 12)^{2k-2},$$

and

$$|\eta_k| \leqq (\delta_k + \epsilon_k)m_k < 3 \cdot 2^{-7 \cdot 2k-2} \cdot \tfrac{1}{3} \cdot (36)^{2k-2} = \left(\frac{9}{32}\right)^{2k-2}.$$

Consequently $|\eta_k|$ may be as small as we please, and, as it is the absolute value of the difference between y/x and $(c_1a_2 - c_2a_1)/(b_1c_2 - b_2c_1)$, it follows that y/x must lie inside an interval shorter than

$$\theta_k = 2 \cdot \left(\frac{9}{32}\right)^{2k-2}.$$

But every such interval must contain the real number q, the root of the cubic equation $q^3 - q^2 - q - 1 = 0$. Since θ_k tends to zero as $k \to \infty$, the sequence of the intervals of length θ_k cannot contain more than one common number. Thus $y/x \equiv q$, and similarly it is shown that $z/x \equiv q^2$.

This completes the proof of Theorem I.

Note: By further consideration it may now be deduced that, in order to be a non-vanishing cycle, Q_4 must have one of the two forms:

$$(u, u + x, u + x + xq, u + xq^3),$$

or

$$(u + xq^3, u + xq + xq^2, u + xq^2, u),$$

in which u is arbitrary, $x \neq 0$, $q^3 - q^2 - q - 1 = 0$, q real. (Since only then $Q_4' \equiv Q_4^*$.)

THE RATIONAL CANONICAL FORM OF A MATRIX II

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1. Introduction. In a previous paper* we attempted a direct attack on the problem of the reduction of a square matrix A with elements in a field K to rational canonical form. Our result was, in a sense, inconclusive, since we used the theory of invariant factors to complete our reduction. MacDuffee's Carus Monograph, *Vectors and Matrices*† has since appeared. In this book an elegant reduction is obtained. However, the present writer feels that the theory of the derogatory case (VM, pp. 128–136) might well offer difficulty to a beginner. For this reason we shall present in this note a completion of the reduction process of RC which is independent of invariant factors, and which does not involve a special consideration of the derogatory case. In effect, we adapt one of MacDuffee's results to the stage of reduction achieved in RC by means of elementary similarity transformations.

Our notation and terminology will be that of VM with the additional designation of the elementary similarity transformation $p_{ij}(c)$ for $c\epsilon K$ of RC. We shall find it convenient to denote the *companion matrix* (VM, p. 82) of a non-constant monic polynomial $f(x)\epsilon K[x]$ by $C(f)$.

2. Two basic theorems. In this section we shall give the additional theory needed to complete our reduction. The reader will observe that, aside from a bare minimum of the theory of matric polynomials, we employ only the basic Hamilton-Cayley theorem.

THEOREM 1. *If $f(x)$ and $g(x)$ are non-constant monic polynomials of $K[x]$, then the matrix $C(fg)$ is similar in K to*

$$B = \begin{bmatrix} C(f) & 0 \\ U & C(g) \end{bmatrix} \tag{1}$$

where U is a matrix with unity in the upper right-hand corner and zeros elsewhere.

Proof. Let $f(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + x^n$, and let $g(x)$ have degree m. Then the sequence of elementary similarity transformations: $p_{m+n,\, m+n-1}(-a_{n-1})$, $p_{m+n,\, m+n-2}(-a_{n-2}), \cdots, p_{m+n,\, m}(-a_0)$; $p_{m+n-1,\, m+n-2}(-a_{n-1})$, $p_{m+n-1,\, m+n-3}(-a_{n-2}), \cdots, p_{m+n-1,\, m-1}(-a_0); \cdots; p_{n+1,\, n}(-a_{n-1}), \cdots, p_{n+1,\, 1}(-a_0)$, yields B.

Note that if g factors in K, our method applied to $C(g)$ leaves U unaltered.

THEOREM 2. *Let $f(x)$ and $g(x)$ be non-constant monic polynomials of $K[x]$ which satisfy $a(x)f(x)+b(x)g(x)=1$ for polynomials $a(x)$, $b(x)\epsilon K[x]$. Define $P = b(B)g(B)+(0\dotplus I)$, where B is given by (1) and the order of I is the degree of $g(x)$. Then P is non-singular and $P^{-1}BP = C(f)\dotplus C(g)$.*

* The rational canonical form of a matrix, this MONTHLY, vol. 49, 1942, pp. 451–454. We shall refer to this paper as RC.

† We shall refer to this book as VM.

Proof. The Hamilton-Cayley theorem yields the matric equation

$$b(C(f))g(C(f)) = I. \tag{2}$$

By induction we easily see that

$$B^i = \begin{bmatrix} (C(f))^i & 0 \\ X_i & (C(g))^i \end{bmatrix} \qquad (i = 1, 2, \cdots) \tag{3}$$

in which X_i is a matrix which will not concern us. By (2), (3), and the Hamilton-Cayley theorem, we find that

$$b(B)g(B) = \begin{bmatrix} b(C(f)) & 0 \\ Y & b(C(g)) \end{bmatrix} \begin{bmatrix} g(C(f)) & 0 \\ Z & g(C(g)) \end{bmatrix} = \begin{bmatrix} I & 0 \\ W & 0 \end{bmatrix}.$$

The equation $Bb(B)g(B) = b(B)g(B)B$ then gives $WC(f) = U + C(g)W$. A direct matrix calculation now justifies the conclusion of Theorem 2.

Remark. It will be noted that no use was made of the particular form of the matrix U in the proof of Theorem 2.

3. Completion of the reduction process. We now apply the theorems of Section 2 to complete the reduction of the matrix A which was initiated in RC. We may assume (by applying the process of RC to A^T and transposing the result) that A is similar in K to $C(f_1) \dotplus C(f_2) \dotplus \cdots \dotplus C(f_t)$, where $f_i(x) \epsilon K[x]$ for $i = 1, 2, \cdots, t$. We may then factor each f_i into irreducible factors in $K[x]$ and apply Theorem 1 (noting the remark that follows it) and then Theorem 2 to find that $C(f_i)$ is similar in K to $C(g_{i1}) \dotplus \cdots \dotplus C(g_{is_i})$ with each $g_{ij} (j = 1, 2, \cdots, s_i)$ a power of an irreducible polynomial of $K[x]$ and with the g_{ij} $(j = 1, 2, \cdots, s_i;$ i fixed) relatively prime in pairs. Another application of Theorem 1 gives the desired canonical form. We should emphasize that the usual bookkeeping scheme* for recording a sequence of elementary transformations permits a simultaneous calculation of the linear transformation which reduces A.

4. An example. This section is devoted to an illustration of the use of the theorems of Section 2 in a very simple example.

Let us consider the reduction of $C = C(x^3 - x^2 - x + 1)$. Suppose first that the characteristic of K is not two. We then have $x^3 - x^2 - x + 1 = (x+1)(x-1)^2$ and Theorem 1 yields

$$B = Q^{-1}CQ = \begin{bmatrix} C((x-1)^2) & 0 \\ U & C(x+1) \end{bmatrix}$$

where

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* See, for example, G. Birkhoff and S. MacLane, A Survey of Modern Algebra, p. 276, or A. A. Albert, Introduction to Algebraic Theories, p. 42.

$$Q = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

We have $a(x)(x-1)^2+b(x)(x+1)=1$ with $4a(x)=1$ and $4b(x)=3-x$. Compute

$$4b(B)g(B) = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

and

$$4P = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ -1 & 1 & 4 \end{bmatrix}, \qquad 4P^{-1} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & -1 & 4 \end{bmatrix}.$$

Then $P^{-1}BP = C((x-1)^2) \dotplus C(x+1)$. Using Theorem 1 again yields

$$R^{-1}P^{-1}BPR = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{with} \quad R = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If the characteristic of K is two, then $x^3-x^2-x+1=(x-1)^3$ and we obtain our canonical form directly from Theorem 1 by means of

$$S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{for we have} \quad S^{-1}CS = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

5. Concluding remarks. We cannot close this short note without a few brief remarks which our method suggests. It is not our feeling that the general theorems on the relation of a matrix to its invariant spaces and on the characteristic and minimum functions of a matrix should be suppressed. We are as heartily in favor of these "by-products" as is MacDuffee (VM, p. vii). It should be observed, however, that the proofs of these theorems become almost trivial once the rational canonical form is obtained.

MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, University of California, Los Angeles 24, and Institute for Numerical Analysis of the National Bureau of Standards

Material for this department should be sent directly to E. F. Beckenbach.

SELF-DUAL FRAGMENTS OF THE ORDINARY PLANE

KARL MENGER, Illinois Institute of Technology

In the geometry of our ordinary plane, points and lines play different rôles. There exist parallel lines, that is, lines which are not on a common point. In fact, for any given line l, there exists exactly one parallel line on every point which is not on l. But there are no parallel points, in the sense of points which are not on a common line. Every two distinct points are on exactly one common line.

The system of all points (x, y) and all lines $y=mx+c$ (that is, all lines which are not parallel to the Y-axis), however, is perfectly self-dual. That is to say, in this system points and lines do play precisely the same role. We shall call two points which are not on a line belonging to the system, *vertical* points. We call (m, c) the *non-vertical line coordinates* of the line $y=mx+c$, and prove a perfect analogy between m and x, and between c and y. We shall accentuate the analogy by denoting* the slope of a line by X, and its y-intercept by $-Y$. Then the point (x, y) is on the line (X, Y) if and only if $y=Xx-Y$, that is,

$$xX = y + Y.$$

Clearly, we can say:

1. In order that the two lines (X_1, Y_1) and (X_2, Y_2) be parallel it is necessary and sufficient that $X_1=X_2$. And dually: In order that the two points (x_1, y_1) and (x_2, y_2) be vertical it is necessary and sufficient that $x_1=x_2$.

2. If (x, y) is a given point, and (X, Y) is a given line, then there exists one and only one line on (x, y) which is parallel to (X, Y), namely the line $(X, y-xX)$. The dual counterpart of this euclidean parallel postulate reads: If (X, Y) is a given line, and (x, y) is a given point, then there exists one and only one point on (X, Y) which is vertical to (x, y), namely, the point $(x, Y-xX)$.

Starting with the self-dual projective plane we obtain the ordinary plane by deleting a line and all points on it. The result is, of course, not self-dual. We obtain our self-dual system by deleting a line, l_∞, and all points on l_∞, as well as a point, P_∞, and all lines on P_∞—with the proviso that P_∞ is on l_∞. Clearly, the result is again self-dual.

If in the above procedure we delete a point, P_0, which is not on the deleted line, l_∞, then we obtain another self-dual fragment of our ordinary plane. If we

* This notation was suggested by H. S. M. Coxeter.

choose P_0 as the origin (0, 0), then the new system consists of all points $(x, y) \neq (0, 0)$ and all lines $(a, b) \neq (0, 0)$ such that (x, y) is on (a, b) if and only if $ax+by=1$. In this theory, two points are not on a common line or, as we might say, *proportional* if and only if they are collinear with the origin. Again, lines and points satisfy the same postulates. Both, however, satisfy the following non-euclidean parallel postulates:

(a) Given a point P, and a line l, there is at most one line on P which is parallel to l, and at most one point on l, which is proportional to P.

But there is no line parallel to the line (a, b) on a point (x, y) for which $ax+by=0$. Similarly, there is no point proportional to the point (x, y) on the line (a, b) for which $ax+by=0$. All we can say is:

(b) If two points are not on a line parallel to l, then the two points are proportional; and if two lines are not on a point proportional to P, then the two lines are parallel.

We obtain a self-dual fragment of our ordinary space by deleting from the projective space a plane p_∞, and all points and lines on p_∞, as well as a point, P_∞, and all planes and lines on P_∞—with the proviso that P_∞ is on p_∞. We retain the planes $z=ax+by+c$ or $z=Xx+Yy-Z$. The point (x, y, z) and the plane (X, Y, Z) are incident if and only if $xX+yY=z+Z$. This fragment satisfies Euclid's parallel postulate and the dual statement.

Also the theory of order including the Law of Pasch is capable of dualization.

ON THE FEUERBACH POINTS*

VICTOR THÉBAULT, Tennie, Sarthe, France

Let ABC be a triangle with circumcenter O; (I) the incircle with center I and D, E, F as points of contact with the sides BC, CA, AB; G the centroid; D', E', F' the points of intersection of the line $d \equiv OI$ with the sides EF, FD, DE of the triangle DEF; and Z the point of contact of the nine-point circle with the incircle (the *in-Feuerbach point*).

THEOREM 1. *The Miquel circle of the complete quadrilateral $Q \equiv (DEF, d)$ passes through the in-Feuerbach point.*

The lines DD_1, EE_1, FF_1 which join the points D, E, F with the circumcenters D_1, E_1, F_1 of the triangles $DE'F'$, $EF'D'$, $FD'E'$ of the quadrilateral Q, being the isogonal conjugates for the angles D, E, F of the altitudes DD'', EE'', FF'' of the above triangles, are concurrent on the circle (I) at the focus of the parabola inscribed in the triangle DEF and having directrix d, that is to say, at the in-Feuerbach point Z of triangle ABC [1]. On the other hand, since triangle $D_1E_1F_1$ is directly similar to triangle DEF, angles EZD and $E_1F_1D_1$ are

* Translated from the French by Howard Eves.

equal or supplementary, and the circle (I_1) circumscribed about triangle $D_1E_1F_1$ (the *Miquel circle of* Q) passes through Z.

COROLLARY 1.1. *The line* ZM *joining the in-Feuerbach point* Z *to the Miquel point* M *of* Q *coincides with the radical axis of the circles* (I) *and* (I_1).

COROLLARY 1.2. *The points* Z *and* M *are symmetric with respect to the line* IG.

This interesting property is easily established with the aid of complex coordinates.

THEOREM 2. *If* A_1, B_1, C_1 *are the orthogonal projections of the circumcenter on the interior bisectors* AI, BI, CI, *then parallels through* D, E, F *to the lines* B_1C_1, C_1A_1, A_1B_1, *respectively, are concurrent at the in-Feuerbach point.*

The points A_1, B_1, C_1 are located on the circle (I') described on OI as diameter, the lines IA_1, IB_1, IC_1 are perpendicular to the sides EF, FD, DE of triangle DEF, and the lines OA_1, OB_1, OC_1 are parallel to these same sides. As a result we see that

$$\measuredangle\, C_1B_1O = \measuredangle\, C_1IO = \measuredangle\, F'DD'' = \measuredangle\, ZDF,$$

and consequently that lines C_1B_1 and ZD are parallel, and similarly for B_1A_1 and ZF, A_1C_1 and ZE.

We have extended these two constructions of the in-Feuerbach point, which also apply with obvious modification to the three outer-Feuerbach points, to the construction of the orthopole of any diameter of the circumcircle of the fundamental triangle [2].

1. V. Thebault, Journal de Vuibert, 36e année, 1911, p. 2.
2. V. Thebault, Comptes-Rendus de l'Académie des Science (Paris), 1944, vol. 218, p. 434

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

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A NOTE ON A METHOD OF LORD RAYLEIGH

L. M. COURT, Rutgers University

A familiar result in the theory of linear differential equations, associated with Lagrange's name, states that if n linearly independent solutions of the associated homogeneous (reduced) equation are known, a particular integral of any nth order linear differential equation can be found by a simple quadrature.

The integrand in this case is the quotient of two determinants in the n linearly independent solutions (the determinant in the denominator is a Wronskian) multiplied by the function on the right side of the non-homogeneous equation. When $n=2$, all this reduces to the result that if $u_1(t)$ and $u_2(t)$ are two linearly independent solutions of $u''+p(t)u'+q(t)=0$, then a particular integral of the non-homogeneous equation

$$u'' + p(t)u' + q(t) = F(t) \tag{1}$$

is furnished by the expression

$$u(t) = \int_0^t \frac{\begin{vmatrix} u_1(\lambda) & u_1(t) \\ u_2(\lambda) & u_2(t) \end{vmatrix}}{D(\lambda)} F(\lambda)d\lambda, \tag{2}$$

where $D(\lambda)$ is the Wronskian $u_1(\lambda)\, u_2'(\lambda)-u_1'(\lambda)\, u_2(\lambda)$.

The method of obtaining (2) found in most texts, as for example Ince, is purely analytic. But for the special case in which p and q are constants, Lord Rayleigh was able to get (2) by a mechanical "strategem" which, if not nearly as rigorous, is much more intuitive. We shall retrace Rayleigh's reasoning* somewhat more fully than he does himself, and then go on to show that it can be extended to the general case, *i.e.*, equation (2).

When p and q are constants and, in addition, $4q-p^2$ is positive, (2) represents mechanically a particle of unit mass "vibrating" about its position of equilibrium $u=0$ under the influence of the impressed force $F(t)$ and against a frictional resistance proportional to the velocity. When we put $F(t)\equiv 0$, we "disconnect" the impressed force and "free" the system. After solving the homogeneous equation representing the free system, we can express the arbitrary constants in terms of the particle's position $u(\lambda)$ and velocity $u'(\lambda)$ at the time λ:

$$\begin{aligned} u(t) = \frac{e^{p(\lambda-t)/2}}{m}\Big\{ & u'(\lambda)\sin m(t-\lambda) \\ & + u(\lambda)\left[\frac{p}{2}\sin m(t-\lambda) + m\cos m(t-\lambda)\right]\Big\} \end{aligned} \tag{3}$$

$$\left(\text{where } m^2 = q - \frac{p^2}{4} > 0\right).$$

The particle's position at any time t is thus seen to depend parametrically upon its position at the time λ and, what is more important for our purposes, its velocity at the time λ; both of these are, of course, arbitrary.

Now let us "connect" the impressed force with the system at the instant $\lambda-d\lambda$, permitting it to operate for a brief time interval $d\lambda$. The impressed force

* Lord Rayleigh, *The Theory of Sound*, Vol. 1, p. 62.

$F(t)$ generates a velocity increment $F(\lambda)d\lambda$ in our particle of unit mass, so that its velocity at the time λ is no longer $u'(\lambda)$, as before, but $u'(\lambda)+F(\lambda)d\lambda$. Because of this change in its "initial" velocity at λ, the particle's position at any later instant t will differ from what it would have been if the impressed force had never operated; in fact, referring back to (3), we see that this deviation is given by

$$\frac{e^{p(\lambda-t)/2}}{m}[F(\lambda)d\lambda]\sin m(t-\lambda). \tag{4}$$

If now we permit the impressed force to operate continuously from $\lambda=0$ to $\lambda=t$, since displacements along the same straight line are algebraically additive, we see that the total deviation in the particle's position from what it otherwise would have been is given by the integral

$$\frac{1}{m}\int_0^t e^{p(\lambda-t)/2}\sin m(t-\lambda)F(\lambda)d\lambda. \tag{5}$$

But (5) when added to (3) gives the particle's position at any instant t, the mechanical system, of course, having been subjected to the impressed force $F(t)$ all the time. Accordingly, this sum is a solution (the complete solution since it involves two arbitrary constants) of the differential equation representing the "forced" system. That is, (5) is a particular integral of the non-homogeneous equation (1) for p and q constant, with the further proviso that $4q-p^2>0$.

Let us now generalize Rayleigh's reasoning to the case in which p and q are not constants. When p and q are functions of t, (1) no longer represents a "vibrating" system but a general system, *i.e.*, a system consisting of a particle of unit mass moving in a more or less general fashion along a straight line under the influence of the impressed force $F(t)$ and against a frictional resistance. As before, the complete solution of the associated homogeneous equation

$$u(t)=\alpha u_1(t)+\beta u_2(t) \tag{6}$$

represents the motion (position) of the particle when $F(t)$ is "disconnected." By solving a pair of algebraic equations, the arbitrary constants α and β can be expressed in terms of the particle's position $u(\lambda)$ and velocity $u'(\lambda)$ at the time λ, and (6) rewritten as

$$u(t)=\frac{1}{D(\lambda)}\left\{u'(\lambda)\begin{vmatrix} u_1(\lambda) & u_1(t)\\ u_2(\lambda) & u_2(t)\end{vmatrix}-u(\lambda)\begin{vmatrix} u_1'(\lambda) & u_1(t)\\ u_2'(\lambda) & u_2(t)\end{vmatrix}\right\}, \tag{7}$$

where $D(\lambda)$ is once again the Wronskian $u_1(\lambda)\,u_2'(\lambda)-u_1'(\lambda)\,u_2(\lambda)$. If $F(t)$ is brought into play at the instant $\lambda-d\lambda$ and permitted to operate for a time interval $d\lambda$, the particle's velocity at the instant λ will be $F(\lambda)d\lambda$ greater than it would have been otherwise; its position at any subsequent instant t will deviate from what its position would have been otherwise by an amount

$$F(\lambda)d\lambda \frac{1}{D(\lambda)} \begin{vmatrix} u_1(\lambda) & u_1(t) \\ u_2(\lambda) & u_2(t) \end{vmatrix};$$

finally, its total displacement in position if $F(\lambda)$ is permitted to operate continuously from $\lambda = 0$ to $\lambda = t$ will be given by the integral

$$\int_0^t \frac{\begin{vmatrix} u_1(\lambda) & u_1(t) \\ u_2(\lambda) & u_2(t) \end{vmatrix}}{D(\lambda)} F(\lambda)d\lambda, \tag{8}$$

and this, which is identical with (2), must be a particular integral of (1) even when p and q are general functions of the time.

It is fairly obvious that this method can be extended to the general linear differential equation of the nth order. In this case, after solving the associated homogeneous equation, we express the n arbitrary constants of integration in terms of the derivatives $u^{(0)}(t) \equiv u(t)$, $u^{(1)}(t)$, $u^{(2)}(t)$, $\cdots$, $u^{(n-1)}(t)$ evaluated at the time λ. The first three of these have familiar mechanical names; there are no standard designations for the remaining $n-3$ but it is evident that they are replete with mechanical significance, particularly when the acceleration is not constant but varies in a complicated fashion with the time. We might term $u^{(n-1)}(\lambda)$ the particle's *velocity of the* $(n-1)$*th order.* $F(\lambda)$ will be the *impressed force of the* $(n-1)$*th order*, *i.e.*, if it operates for a brief interval $d\lambda$, it generates an increment $F(\lambda)\ d\lambda$ in the velocity of the $(n-1)$th order of a particle of unit mass. The quantity that multiplies $u^{(n-1)}(\lambda)$ after we have expressed the n original arbitrary constants in terms of $u^{(0)}(\lambda)$, $\cdots$, $u^{(n-1)}(\lambda)$ will be the quotient of two determinants of the nth order, the numerator being a function of λ and t and the denominator the Wronskian $D(\lambda)$ of the n linearly independent solutions $u_1(\lambda)$, $\cdots$, $u_n(\lambda)$ of the associated homogeneous equation. If we permit the impressed force of the $(n-1)$th order $F(\lambda)$ to operate continuously from $\lambda=0$ to $\lambda=t$, we see that the deviation in the particle's position will be given by an integral similar to (8)—except that the two determinants are of the nth order—or, in other words, by the known expression for the particular integral in terms of the n linearly independent solutions of the reduced equation.

THE LAWS OF SINES AND COSINES

L. J. BURTON, Bryn Mawr College

Using the distance formula of analytic geometry we may derive the law of cosines in the following simple fashion. For any triangle ABC, choose the coordinate system so that A is at the origin and B is on the positive x-axis. Then B is $(c, 0)$ and C is $(b \cos A, b \sin A)$. From the distance formula:

$$a^2 = (b \cos A - c)^2 + (b \sin A)^2 = b^2 + c^2 - 2bc \cos A.$$

Most textbooks do not point out any connection between the law of cosines

and the law of sines. Assuming the former we can prove the latter algebraically as follows:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad \text{and} \quad \sin^2 A = 1 - \cos^2 A = \frac{4b^2c^2 - (b^2 + c^2 - a^2)^2}{4b^2c^2}$$

or
$$\sin^2 A = \frac{(a + b - c)(a - b + c)(a + b + c)(-a + b + c)}{4b^2c^2}.$$

By symmetry:

$$\sin^2 B = \frac{(a + b - c)(-a + b + c)(a + b + c)(a - b + c)}{4a^2c^2}.$$

Then $b^2 \sin^2 A = a^2 \sin^2 B$; and since $\sin A$, $\sin B \geqq 0$, we have

$$b \sin A = a \sin B.$$

Also, it follows that if $s = \frac{1}{2}(a+b+c)$, then

$$\sin A = \frac{2\sqrt{s(s - a)(s - b)(s - c)}}{bc}$$

and since the area of the triangle ABC is clearly $\frac{1}{2}bc \sin A$, we have a simple proof that the area is $\sqrt{s(s-a)(s-b)(s-c)}$.

The law of cosines can be proved algebraically from the law of sines as follows. Assuming that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k$$

we wish to prove that $a^2 = b^2 + c^2 - 2bc \cos A$, or simply the trigonometric identity

$$\sin^2 A = \sin^2 B + \sin^2 C - 2 \sin B \sin C \cos A$$

where $A = \pi - (B+C)$. Putting $\sin A = \sin B \cos C + \cos B \sin C$ and $\cos A = \sin B \sin C - \cos B \cos C$, the identity is easily verified. In spherical trigonometry, the law of sines follows from the law of cosines by a proof similar to the above, but the law of cosines does not follow from the law of sines.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 881. *Proposed by C. M. Sandwick, Easton High School, Easton, Pa.*

(a) Show that any positive integer can be represented uniquely in the form $\sum_{n=1}^{\infty} a_n n!$, $0 \leqq a_n \leqq n$.

(b) Show that any positive rational number less than unity can be represented uniquely in the form $\sum_{n=1}^{\infty} a_n/(n+1)!$, $0 \leqq a_n \leqq n$.

E 882. *Proposed by C. W. Trigg, Los Angeles City College*

Five regular tetrahedra arranged around a common edge just fail to fill space around that edge. (a) What is the closest approximation to a regular tetrahedron such that five such tetrahedra will fill the space around a common edge to form a decahedron having equilateral faces? (b) Show that the edge of the decahedron is $\sqrt{5}$ times the radius of the sphere touching those edges which radiate from the axis of the decahedron.

E 883. *Proposed by H. T. R. Aude, Colgate University*

Find the smallest integer n which is the sum of the squares of two non-negative integers in exactly (a) three ways, (b) five ways, (c) six ways.

E 884. *Proposed by E. P. Starke, Rutgers University*

Show that there exist infinitely many rational integral equations with integral coefficients and leading coefficient unity, and of degree n, such that $n-1$ of the roots occur within a specified interval, however small.

E 885. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In the tetrahedron $ABCD$ let A_1, B_1, C_1, D_1 divide a set of concurrent cevians AA', BB', CC', DD' in the same ratio, and let A_1B_1, A_1C_1, A_1D_1 pierce BCD in A_2, A_3, A_4. Show that triangles A_2CD, A_3DB, A_4BC have equal areas.

SOLUTIONS

Cevian Triangles of Incenter and Excenters

E 849 [1949, 31]. *Proposed by C. W. Trigg, Los Angeles City College*

The area of a triangle is to the area of the triangle determined by the points of contact of its incircle (or excircle) as its circumdiameter is to its inradius (or

exradius).

Solution by C. S. Ogilvy, Trinity College. Let K and K' denote the areas of the given triangle and the cevian triangle of the incenter. Then, in terms of standard notation,

$$K/K' = r(a+b+c)/r^2(\sin A + \sin B + \sin C) = 2R/r.$$

The same procedure, with due regard to signs, yields the corresponding theorem for an excenter.

Also solved by Roger Lessard, Azriel Rosenfeld, N. C. Scholomiti, W. R. Talbot, Kaidy Tan, P. D. Thomas, Roscoe Woods, and the proposer.

Tan located the problem in Loney, *Plane Trigonometry*, p. 248, and Joseph Langr pointed out that the problem was studied by Neuberg in *Mathesis*, 1910, p. 258. An easy secondary result, noted by the proposer, is that the areas of the cevian triangles of the incenter and the excenters are to each other as the radii of the corresponding circles.

Area of Quadrilateral with Incircle and Circumcircle

E 851 [1949, 104]. *Proposed by Joseph Rosenbaum, Hartford, Connecticut*

The area of a quadrilateral which has both a circumcircle and an incircle is equal to the square root of the product of its sides.

I. *Solution by M. C. Stapp, University of Alabama.* Let $ABCD$ be the quadrilateral, O the incenter, r the inradius, and P, Q, R, S the points of contact of the incircle with the sides AB, BC, CD, DA. Set $t=AS=AP$, $u=BP=BQ$, $v=CQ=CR$, $w=DR=DS$. Since $ABCD$ is concylic, right triangles OQC and APO are similar, whence $r^2=tv$. Similarly $r^2=uw$, and we have $tv=uw$. If K is the sought area we then have

$$\begin{aligned} K^2 &= r^2(t+u+v+w)^2 = uw(t+u+v+w)^2 \\ &= uw(uw/v+u+v+w)^2 = uw(t+u+uw/t+w)^2. \end{aligned}$$

Therefore, taking the product of the last two expressions,

$$\begin{aligned} K^4 &= (uw/tv)^2(uw+vu+v^2+vw)^2(t^2+tu+uw+tw)^2 \\ &= (v+w)^2(v+u)^2(t+u)^2(t+w)^2 \\ &= (AB)^2(BC)^2(CD)^2(DA)^2. \end{aligned}$$

II. *Solution by L. R. Chase, Newport, R. I.* Let a, b, c, d be the sides, in their order. Since the quadrilateral is a circumscribed one, $a+c=b+d=s$. Since the quadrilateral is an inscribed one, the area is

$$K = [(s-a)(s-b)(s-c)(s-d)]^{1/2},$$

which, by appropriate substitutions for s, becomes $(cdab)^{1/2}$.

Also solved by Louis Berkofsky, W. C. Bornmann, W. E. Buker, William Douglas, Ragnar Dybvik, W. O. Pennell, C. M. Sandwick, N. C. Scholomiti,

C. W. Trigg, Lila Walker, Alan Wayne, G. A. Williams, Roscoe Woods, and the proposer.

Trigg pointed out that this problem was solved as problem 1459, *School Science and Mathematics*, vol. 36, p. 1031. It also appeared in the form $r=(abcd)^{1/2}/s$ in No. 472, *National Mathematics Magazine*, vol. 17, p. 182. For other properties of this quadrilateral see *School Science and Mathematics*, vol. 21, p. 280, problem 676 and vol. 31, p. 82, problem 1138. Woods located a trigonometrical solution of the problem in Durell and Robson, *Advanced Trigonometry*, (1930), pp. 27–28. It also appears in Shively, *An Introduction to Modern Geometry* (1939), p. 156, problem 26.

A proof of the famous Brahmagupta formula, $K=[(s-a)\ (s-b)\ (s-c)\ (s-d)]^{1/2}$, for the area of a cyclic quadrilateral may be found in many places. See, for example, Hobson, *Plane Trigonometry* (1928), p. 204 or Johnson, *Modern Geometry* (1929), p. 81.

Extreme Case of Lamé's Theorem

E 852 [1949, 104]. *Proposed by Roy Dubisch, Fresno State College*

A theorem due to Lamé states that the number of divisions D in the Euclidean Algorithm required to find the g.c.d. of two numbers a and b $(a>b)$ is never greater than $5p$, where p is the number of digits in b. While this result has been strengthened for special pairs a, b, show, by a counter-example with b as small as possible and corresponding a as small as possible, that the statement $D<5p$ for all a, b is false.

I. *Solution by Jacob Feldman, University of Pennsylvania.* Let a and b_0, $a>b_0$, be any two numbers such that $D=5p$, and let the D divisions of the Euclidean Algorithm for a and b_0 be

$$\begin{aligned} a &= b_0q_1+b_1, \\ b_0 &= b_1q_2+b_2, \\ &\cdots\cdots\cdots\cdots \\ b_{D-3} &= b_{D-2}q_{D-1}+b_{D-1}, \\ b_{D-2} &= b_{D-1}q_D. \end{aligned}$$

Let u_k be the kth term of the (truncated) Fibonacci sequence defined by $u_1=1$, $u_2=2$, $u_{n+2}=u_{n+1}+u_n$ for $n\geqq 1$. Notice that in the above set of divisions we have

$$b_{D-1}\geqq 1, \qquad b_{D-2}\geqq 2, \qquad b_{D-m}\geqq b_{D-m+1}+b_{D-m+2}.$$

Thus $b_{D-k}\geqq u_k$. Since $D=5p$, we have $D\geqq 5$ and

$$b_0\geqq u_5=8, \qquad b_1\geqq u_4=5, \qquad a\geqq b_0+b_1\geqq 13.$$

But for $a=13$, $b_0=8$, it is easily verified that $D=5p$. This, then, is the required counter-example.

II. *Solution by John Todd, Surrey, England.* The most slowly proceeding Euclidean Algorithms are those in which all the quotients are 1. Of these, that involving smallest numbers has its last remainder 1 and its penultimate remainder 2. Successive remainders in this case can be built up and are, in fact

$$(A) \qquad 1, 2, 3, 5, 8, 13, 21, 34, \cdots .$$

There are five divisions necessary to find the g.c.d. of 13 and 8. This is the counter-example required.

More generally, it is clear that the numbers (A) are those of the Fibonacci sequence, with the initial term $f_1=1$ omitted. This sequence is defined by $f_1=1$, $f_2=1$, and $f_{n+2}=f_{n+1}+f_n$ for $n\geqq 1$. It is known that

$$\sqrt{5}\,f_n = x^n - y^n, \qquad x = -1/y = (\sqrt{5}+1)/2 = 1.6180\cdots .$$

Further, the Euclidean Algorithm when applied to f_{n+2}, f_{n+1} involves n divisions. Now $p-1$ is the integral part of $\log_{10} f_{n+1}$, and this is asymptotically $(n+1)\log_{10} x$, since $|y|<1$. Hence

$$n \sim (1/\log_{10} x)p \doteq (4.785)p,$$

and certainly for sufficiently large n we have $n<5p$. Rough estimations show that this is true for $p\geqq 4$. Reference to a table of Fibonacci numbers shows that this is indeed the best possible result, for the Euclidean Algorithm, when applied to $f_{16}=987$ and $f_{17}=1597$, terminates after 15 divisions.

Also solved by Monte Dernham, Roger Lessard, C. D. Olds, and the proposer.

Lessard claimed that for $p\geqq 4$,

$$D < (4.785)p + 0.673,$$

which gives

$$\begin{array}{ll}
4 \leqq p < 8 & D < 5p \\
8 \leqq p < 13 & D < 5p - 1 \\
13 \leqq p < 18 & D < 5p - 2 \\
18 \leqq p < 22 & D < 5p - 3 \\
22 \leqq p & D < 5p - 4
\end{array}$$

It seems that $(a, b)=(13, 8)$, $(144, 89)$, $(1597, 987)$ are the only counter-examples.

Convergence of an Infinite Involution

E 853 [1949, 104]. *Proposed by C. S. Ogilvy, Trinity College*

If $y_1=x$, $y_2=x^{y_1}$, $\cdots$, $y_n=x^{y_{n-1}}$, what is the maximum x for which $\lim_{n\to\infty} y_n$ exists, and what is this limit?

4361. *Proposed by Melvin Dresher, The Rand Corporation*

If $S_1, S_2, \cdots, S_m$ are m line segments parallel to the y-axis such that through every set of $n+2$ of them the locus of an nth degree polynomial can be passed, then there exists some nth degree polynomial whose locus intersects all m segments.

4362. *Proposed by D. J. Newman, College of the City of New York*

Evaluate

$$\begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n-1} \end{vmatrix}.$$

4363. *Proposed by Paul Erdös, Syracuse University*

Let $a_1 < a_2 < \cdots$ be an infinite sequence of positive upper density (*i.e.*, $\underline{\lim}\ a_k/k < \infty$). Then there exists an infinite subsequence such that no element divides another. In fact there exists an infinite subsequence $a_{i_1}, a_{i_2}, \cdots$ such that $\sum 1/a_{i_k} = \infty$ and no a_{i_k} divides any other.

4364. *Proposed by Joseph Rosenbaum, The Milford School, Connecticut*

On the sides A_iA_{i+1} of an n-gon $A_1A_2 \cdots A_n$ as bases, isosceles triangles $A_iA_{i+1}B_i$ are constructed, either all exteriorly or all interiorly, with the vertex angle $B_i = 360°/n$. Prove

(a) If $A_1A_2 \cdots A_n$ is a projection of a regular n-gon, then $B_1B_2 \cdots B_n$ is regular.

(b) The problem of locating the points A_i when the points B_i are given is a porism.

SOLUTIONS

Monge Point of a Tetrahedron

4234 [1947, 112]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show that if the Monge point of a tetrahedron $ABCD$ lies in the plane of the face BCD, then the altitudes through vertex A, of the triangles ABC, ACD, ADB are coplanar, and conversely. Examine the cases where the Monge point lies on an edge and at a vertex of the face BCD. Show how to construct tetrahedra illustrating each case.

Solution by René Blanchard, Le Havre, France.*

1. *Lemma.* In a tetrahedron:

(a). The distances of the Monge point from the faces are equal to one-half the segments intercepted on the altitudes by the planes of the faces and the circumsphere.

(b). The orthogonal projections of the Monge point on the faces are the midpoints of the line segments joining the orthocenters of the faces to the feet of the altitudes of the tetrahedron.

Let $ABCD$ be the given tetrahedron, (O) its circumsphere, G its centroid, Ω its Monge point, A' the foot of the altitude drawn from A, A'' the point where AA' meets (O) again, O_a' the midpoint of AA'' (orthogonal projection of O on AA'), (O_a), G_a, H_a the circumcircle, the centroid and the orthocenter of triangle BCD, G_a' and Ω_a the orthogonal projections of G and Ω on BCD.

(a). Ω is the symmetric of O with respect to G. We have

$$\overrightarrow{\Omega\Omega_a} = 2\overrightarrow{GG_a'} - \overrightarrow{OO_a}.$$

But $\overrightarrow{GG_a}=\overrightarrow{AA'}/4$ and $\overrightarrow{OO_a}=\overrightarrow{O'A'}-\overrightarrow{AA''}/2-\overrightarrow{A'A''}$. Hence $\overrightarrow{\Omega\Omega_a}=\overrightarrow{AA'}/2-\overrightarrow{AA''}/2+\overrightarrow{A'A''}=\overrightarrow{A'A''}/2$.

(b) Since $(H_aG_a/H_aO_a)(\Omega_aO_a/\Omega_aG_a')(A'G_a'/A'G_a)=1$, H_a, Ω_a, A' are collinear. Furthermore, $H_a\Omega_a/H_aA'=(G_aG_a'/G_aA')(O_a\Omega_a/O_aG_a')=\frac{1}{2}$, and Ω_a is the midpoint of H_aA'.

2 (a). The necessary and sufficient condition that Ω be in the plane BCD is that $A'A''=0$, *i.e.*, that A' be on (O_a). Thus the projections of A' on the sides of the triangle BCD are collinear and the altitude of triangles ACD, ADB, ABC drawn from A are coplanar.

To construct such a tetrahedron it is sufficient to give the triangle BCD, to take an arbitrary point A' on (O_a), and to choose A as an arbitrary point on the perpendicular to the plane BCD erected at A'.

As A' described the circle (O_a), Ω describes the homothetic transform $(H_a, \frac{1}{2})$ of (O_a). The locus of Ω is the nine-point circle of BCD.

(b). If Ω is on CD, it can be either at the midpoint of CD or at the foot of the altitude of BCD from B on CD. To construct the tetrahedron it is sufficient to give the triangle BCD, to take the point A' symmetric to H_a with respect to Ω and to choose A arbitrary on the perpendicular to the plane BCD erected at A'.

(c). If Ω coincides with D, the altitudes of triangle BCD drawn from B and C meet at D. The angle at D of triangle BCD is a right angle, A' coincides with D, and the tetrahedron is trirectangular at D.

A Diophantine System

4284 [1948, 100]. *Proposed by E. T. Bell, California Institute of Technology*

"Diophantus [II, 20 and IV, 45] proposed to find three squares such that

* Translated by W. E. Byrne, Virginia Military Institute.

$$y^2 - x^2 : z^2 - y^2 = a : b,$$

where $a:b$ is a given ratio." (L. E. Dickson, *History of the Theory of Numbers*, v. 2, p. 419.) For each of $a/b=3$, $1/3$ he gave one rational solution. Prove that the complete integer solution is given by

$$2x = k(a_1b_1g^2 + 2agh - a_2b_2h^2),$$
$$2y = k(a_1b_1g^2 + a_2b_2h^2),$$
$$2z = k(a_1b_1g^2 - 2bgh - a_2b_2h^2),$$

where a_1, a_2 are any integers whose product is a, and b_1, b_2 are any integers whose product is b, and k (for any pair of integer values of g, h) is an arbitrary integer multiple of the reciprocal of the G.C.D. of b_1g+a_2h, a_1g-b_2h, $a+b$.

Solution by Roger Lessard, École Polytechnique, Montreal. We have

$$\frac{a}{b} = \frac{(x-y)(x+y)}{(y-z)(y+z)}. \tag{1}$$

Then there exist integers g, g', h, h', such that

$$x + y = gh'a_1, \qquad x - y = g'ha_2, \tag{2}$$
$$y + z = gg'b_1, \qquad y - z = hh'b_2, \tag{3}$$

with $a_1a_2=a$, $b_1b_2=b$. These give

$$2y = gh'a_1 - g'ha_2 = gg'b_1 + hh'b_2,$$

whence

$$g'/h' = (ga_1 - hb_2)/(gb_1 + ha_2). \tag{4}$$

Using (4) it is easy to reduce (2) and (3) to the form $x:y:z=x':y':z'$, where

$$x' = a_1b_1g^2 + 2agh - a_2b_2h^2,$$
$$y' = a_1b_1g^2 + a_2b_2h^2, \qquad z' = a_1b_1g^2 - 2bgh - a_2b_2h^2.$$

Editorial Note. If d is the G. C. D. of x', y', z', we have for some integer s,

$$x = sx'/d, \qquad y = sy'/d, \qquad z = sz'/d. \tag{5}$$

Conversely, if g, h, s are arbitrary integers, then x, y, z as given by (5) will satisfy (1), as may be verified by direct substitution.

The attempt to be more specific than (5) leads to some difficulty. Let f be the G.C.D. cited by the Proposer and m the arbitrary integer multiple, so that $k=m/f$. The Proposer's statement is equivalent to (5) if and only if $d=2f$. We may write

$$x' = a_2hu + a_1gv, \qquad z' = b_1gu - b_2hv, \qquad y' = -a_2hu + a_1gv = b_1gu + b_2hv,$$

where $u=a_1g-b_2h$ and $v=b_1g+a_2h$. It is thus evident that f divides d. (It is easy to show that the G.C.D. of u and v is always a divisor of $a+b$.)

Now unless a_1, a_2, b_1, b_2 are chosen in a special manner, d will not equal $2f$ and the solution in terms of k may be incomplete or may yield fractional values for x, y, z. For example, if $x=1$, $y=19$, $z=29$, we have $a=3$, $b=4$. Of the six ways of selecting the factors, only one way leads to $d=2f$. The others give $d=12f$, $4f$, $3f$, $3f$, f. Thus the Proposer's formulas fail to give the solution 1, 19, 29 at all in four of these cases, while in three of them the solution 3/2, 57/2, 87/2 appears.

Film Wound on a Reel

4285 [1948, 165]. *Proposed by H. N. Davis, Victor Erikson, and Robert Nathans, United States Army*

A full reel of film (thickness τ) of original diameter A is being wound onto an empty spool of original radius a and rotating at a uniform angular velocoity ω. How long does it take to unwind the first spool if its inner diameter is also a?

Solution by W. B. Campbell, Philadelphia Textile Institute. If the film being wound upon the spool is assumed to take the form of a spiral

$$r = a + (\theta/2\pi)\tau^*$$

and if $d\theta/dt=\omega=$constant, we have $dr/dt=\tau\omega/2\pi$. The time for attaining a specific value of r is $\int_a^r(dt/dr)dr$. Since the final value of r is A, the same as on the first spool, the total time is $2\pi(A-a)/\tau\omega$. As a check, the approximate number of turns is $n=(A-a)/\tau$, and the number of revolutions per unit of time is $R=\omega/2\pi$, making the total time $=n/R$. This solution neglects the (unstated) distance between axes.

A matter of possible interest is the length of the film. An obvious estimate for it is obtained from $n=(A-a)/\tau$ and the estimated mean radius $R_m=(A+a)/2$, so that the length $L=2\pi R_m n=\pi(A^2-a^2)/\tau$. The full expression is

$$L = \int [dr^2 + (rd\theta)^2]^{1/2} = \int_a^A [1 + (2\pi r/t)^2]^{1/2}dr.$$

This responds readily to formal integration, but it is interesting to note that if the first term under the radical is suppressed, as negligible in comparison with the second, the result is identical with the estimated value of L.†

The spiral law for the form of the film seems as reasonable as any, since we do not know the shape of the end of the film, nor how it is fastened onto the spool, nor what the winding tension is, nor the nature of the elastic deformation by which the influence of the "hump" becomes negligible as turns accumulate upon the coil.

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* We make the simplifying assumption that a and A in the proposal are meant to represent radii on both spools, a reasonable reading in light of the word "also."

† Assuming the spiral form, the length of one turn at radius r_i, expanded in powers of τ, is $L_i=2\pi r_i+\pi\tau+\cdots$. Hence neglect of the first term under the radical gives the same result as assuming $L_i=2\pi r_i$, which is also equivalent to the assumption of circular layers made by some of the other solvers.

Also solved by F. W. Herlihy, C. S. Ogilvy, Mary H. Payne, Hanan Rubin, and G. W. Walker.

Editorial Note. The Proposers were interested in the acceleration of the first spool. Suppose r', r are the radii of the outer layers on the two spools at time t. Then the lengths of film on the spools are $\pi(r'^2-a^2)/\tau$, $\pi(r^2-a^2)/\tau$, approximately, as in the above solution. But the total length is $\pi(A^2-a^2)/\tau$. Hence $r'^2+r^2=A^2+a^2$. By differentiation and substitution of $dr/dt=\tau\omega/2\pi$ we find for the angular velocity of the first spool $\omega'=r\omega/r'$, and for its angular acceleration

$$d\omega'/dt = \omega^2\tau(A^2 + a^2)/2\pi r'^3.$$

Representation of Integers in the Form: a kth Power Plus a Prime

4287 [1948, 165]. *Proposed by C. R. Phelps, Rutgers University*

Show that for any given integer $k>1$, there are an infinite number of perfect kth powers which cannot be written as the sum of a prime and a kth power. (This disproves a conjecture of Hardy and Wright, *Introduction to the Theory of Numbers*, p. 19.)

Solution by P. A. Clement, University of California at Los Angeles.

(1) The function $(x+a)^k-x^k$, $k>1$, is algebraically factorable and thus can represent a prime for integral x only if $a=1$.

(2) Let $F(x)=(x+1)^k-x^k$ and, for an arbitrary positive integer x_0, let $q=F(x_0)$. Then q is an integer >1. Putting $y_n=x_0+nq$, $n=1, 2, \cdots$, we have

$$\begin{aligned} F(y_n) &= (y_n+1)^k - y_n^k = [(x_0+1+nq)^k - (x_0+nq)^k] \\ &\equiv (x_0+1)^k - x_0^k = q \pmod{q}. \end{aligned}$$

Hence, for all n, $F(y_n)$ is divisible by q, but is not equal to q since $F(x)$ is obviously an increasing function of its argument. Thus, for each choice of x_0, $(y_n+1)^k$ gives an infinite set as required.

Also solved by Robert Breusch, Paul Erdös, William Gustin, W. J. Harrington, Fritz Herzog, Free Jamison, Roger Lessard, Leo Moser, E. Trost, and the Proposer.

Editorial Note. Breusch, Harrington, and Moser cited the set A^k, where $A=n(2^k-1)+2$, $n=1, 2, \cdots$. Several solvers employed the known theorem that any polynomial $F(x)$ with integral coefficients, not a constant, is composite for infinitely many values of x (Hardy and Wright, p. 18). An obvious slight change in the above proof will establish the general theorem.

The Proposer noted that the case $k=2$ was treated by Lischinsky and Webber, *Transactions of the Royal Society of Canada*, v. 27, pp. 71–90 in which Hardy-Littlewood methods are used to show that "almost all" numbers can be represented in the form y^2+p. The Proposer also suggests, in place of the conjecture which lead to the problem, the following: Corresponding to each sufficiently large N there exist integers y, $k(>1)$, and p (prime) such that $N=y^k+p$.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

Unified Calculus. By E. S. Smith, Meyer Salkover, and H. K. Justice. New York, John Wiley and Sons, Inc., 1947. 10+534 pages. $3.50.

The text represents an excellent way to present to the student as early as possible the fundamental notions of differential and integral calculus. After an especially good ten-page chapter on the limit idea the authors introduce differentiation. This is well motivated by the idea of speed and is one of the best presentations the reviewer has seen. With just the necessary formulas for differentiating a polynomial the authors discuss curve tracing as applied to polynomials. Next follows a section on applications of maxima and minima with a very good set of story problems.

After other applications of derivatives the notion of integration is introduced with Chapter 3. Integration as inverse of differentiation and as the limit of a sum is applied to find areas, solids of revolution, *etc.* Thus in the first 73 pages the beginning student is given an excellent introduction to the whole elementary course.

Next comes a chapter on centroid, moment of inertia, work, *etc.* It is not until Chapters 5 and 6 that differentiation formulas for algebraic and transcendental functions are derived. As the differentiation of each new type of function is introduced a list of problems on applications is given. This keeps the student systematically reviewed on maxima, minima, inflection points, *etc.* After two other chapters on applications of derivatives, Chapters 9–12 complete the usual elementary work on integration. The remaining chapters are devoted to the customary topics of series, expansions of fractions, hyperbolic functions, partial differentiation, multiple integrals, and some differential equations. In the back are formulas from previous elementary courses and tables of integrals, trigonometric functions with the argument in degrees and radians, natural logarithms, hyperbolic functions, and powers of e.

The reviewer feels that the definitions of maxima, minima, and inflection points on a curve are poorly done. They lead one to believe these points can occur only when the function and its first two derivatives are continuous.

Otherwise the book seems very good. There is an ample number of well graded exercises in each set to use the book two years without repeating. Answers to the odd numbered exercises are given.

All illustrative examples and exercises are in a slightly smaller type than the body of the text, but the type is clear and the spacing good. The text makes a very pleasing appearance. The reviewer did not find a single typographical error!

An outstanding feature of the book is the set of several hundred diagrams. Those showing the element of integration in solids are especially well done.

J. A. WARD

Mathematics: Our Great Heritage. Essays on the Nature and Cultural Significance of Mathematics. Edited by W. L. Schaaf. New York, Harper and Brothers, 1948. 11+291 pages. $3.50.

"These essays are *about* mathematics. They have been selected for the thoughtful reader who would understand why mathematics means so much to mankind. There is little in them that is technical—not, at least, in the sense that they bristle with esoteric symbols and intricate diagrams" (from the preface). Culled from the expository writings of well-known figures in the mathematical world, the essays focus on five aspects of mathematics:

I. *The Creative Spirit.* "Mathematics as an Art," by J. W. N. Sullivan; "On the Seriousness of Mathematics," by G. H. Hardy; "Mathematics—the Subtle Fine Art," by J. B. Shaw.

II. *Wellsprings.* "On the Development of Mathematics," by E. T. Bell; "On the Genesis of Mathematical Ideas," by George Sarton; "On the Sociology of Mathematics," by D. J. Struik.

III. *The Queen.* "An Introduction to Modern Mathematical Thought," by C. V. Newsom; "On the Nature of Mathematical Truth," by C. G. Hempel; "The Two Realities," by Tobias Dantzig.

IV. *The Handmaiden.* "Mathematics and the Sciences," by Tomlinson Fort; "Mathematics and the Sciences," by J. W. Lasley, Jr.; "On the Relation of Mathematics and Physics," by R. B. Lindsay; "Industrial Mathematics," by T. C. Fry.

V. *Humanistic Bearings.* "On the Nature of Mathematical Knowledge," a report of the Progressive Education Association; "Mathematics and the Humanities," by Archibald Henderson; "Mathematics as a Cultural Bridge," by Arnold Dresden; "The Larger Human Worth of Mathematics," by R. D. Carmichael.

That such a collection of essays should make for splendid reading goes without saying. And they do. The professional mathematician will single out many stimuli for his own ideas, many foci for further development of thought and exposition on his part. Here are said a large number of the things he has always intended to say, often better said than he could say them; the reviewer even finds (to his amusement and chagrin) the likening of mathematics to a juice extractor which he has always considered his very own simile. The student of mathematics, graduate student or undergraduate major, will achieve an orientation and synthesis of hitherto compartmentalized knowledge, a clearing of trees and stumps into greener pastures. He will derive comfort and peace of mind from Carmichael's inspiring summary of the effect of a life-time devotion to mathematics on that life. The beginner in mathematical specialization who won-

ders what to do with his specialty "if he does not want to teach it," will profit from the directive contained in Fry's guide to industrial mathematics and in the essays on the counterpoint of mathematics and the sciences. As collateral reading material the collection will serve as a survey course in general mathematics with its store of topics ripe for further investigation by the thoughtful student or for further amplification by the instructor. As for an intellectually curious lay reader, one with even a modicum of mathematical background should have his eyes opened to many of the meanings and beauties which have escaped him before. And there are many such readers, still insatiably and unsuccessfully questing for knowledge of a field of thought which only their intuition tells them is important.

Every reader will have personalized preferences for some of the essays or parts of essays. Indeed, personal preference on the part of the editor was one of the guides to selection. And any reader, himself his own reviewer, may quarrel with the omission of some favorite essay and the inclusion of another which does not quite catch his fancy. It is therefore not as an index of what the reader should extract from the book that the present reviewer claims the privilege of recording his own preferences and complaints. He is grateful indeed for the beauty of Carmichael's tribute to the human worth of mathematics; for the disarming directness of Hardy's contrast of what is serious in mathematics, because it is forward looking and developable, with what, though fine and interesting, is yet dead end; for the lucidity of Hempel's exposition of the postulational structure of mathematics. He must admire the timely brilliance of Dresden's advocacy of achieving "necessary and sufficient conditions" for an "existence theorem" for world peace, and of securing better intercultural relations by an application of the study of "inverse functions"—"putting one's self in the other man's shoes," the *man muss immer umkehren* of Jacobi in social dress. But he must also confess that, although he ardently contends that mathematics is indeed a subtle fine art, he does not find the artificially artful arguments of Shaw on the subject persuasive to the contention.

Acknowledging the difficulties of selection confronting the editor and his general skill in solving them, the reviewer feels that in some respects the selections were not wisely balanced. Why *two* essays on mathematics and the sciences? Are not the essays of Newsom and Hempel largely co-extensive for this project? Indeed, omission of one of each of these pairs would have opened up space for the further development of Part III, the least convincing section of the book to this reviewer. For, the queenliness of mathematics is attested not alone by its logical formulation and its philosophy of the infinite. What of the unifying concept of function and the invention of techniques for the extraction of information from a functional relationship? What of the central rôle of form and transformation and invariant in algebra and geometry? Is it not a royal thing that a cache of mathematics has been, as mathematics for mathematics' sake, conceived and developed, and stored ready to the hand of the grateful scientist

of a later period? These themes can surely be made vivid to the general reader; and surely there exist articles discussing them which are non-technical and do not "bristle with esoteric symbols." If such are indeed not available (in English or in translation from foreign languages), then it might have been wise of the editor to have commissioned their writing, for without them the beautiful all-pervasiveness of mathematics is not brought home to the reader.

Despite these relatively minor criticisms, the general verdict of this review is that the book is admirable, successfully sustaining a high level of content in pursuit of a worth while purpose.

G. M. MERRIMAN

Differential- und Integralrechnung im Hinblick auf ihre Anwendungen. By Louis Locher-Ernst. Basel, Verlag Birkhäuser, 1948. 594 pages. Fr. 48.

This uncommonly rich text is devoted principally to plane analytic geometry and differential and integral calculus of functions of a single real variable. Although it is the author's aim, on the basis of extended teaching experience, to develop his material ab initio, with emphasis on its applications, he gives relatively few specialized applications. Instead, he has attempted to choose content appropriate for application and to present it in a form technically adapted to application. He anticipates that the "abstract" mathematician may find too much numerical work, and that the technician may find too much genuine mathematics.

The most prominent single feature of the book is the comparatively extensive treatment of numerical and graphical methods. Difference tables are introduced immediately after the calculus of polynomials, and the interpolation formulas of Newton, Bessel, and Stirling are obtained. In the introductory discussion of the definite integral, approximations by the trapezoidal rule and Simpson's rule follow integration of the linear function. Numerical and graphical differentiation, integration, and harmonic analysis are discussed rather fully. The treatment of Taylor series and the expansions of the elementary functions (including the binomial) are based in each case on a careful discussion of the remainder. In practically all numerical work, exact hypotheses and a precise estimate of error are demanded.

Over a fifth (103 pages) of the text is devoted to the conic sections, differential geometry of plane curves, rollers, and kinematic geometry. On the other hand, one finds no general discussion of infinite series of constants (except the geometric series), and the convergence interval of a power series is defined only by uncertain implication (p. 431, italics). Although, in fact, the interval is invariably indicated by an appropriate inequality, the reviewer encountered no mention of the convergence radius.

In accordance with accepted pedagogical practice, the author approaches a fundamental idea first from a descriptive or intuitive point of view, cites illus-

trative examples and results, then gives a rigorous definition, and proceeds to develop, extend, and apply the concept. It seems open to question, however, whether any good purpose is served by an introductory remark on the definite integral, (p. 130) stating that "Das Product $f(x)dx$ bedeutet den Inhalt eines unbegrenzt schmalen rechtecks. . . . " Again, the introductory statement (p. 166) on nullsequences says that the " . . . Glieder mit unbegrenzt wachsender Nummer sich immer weniger von Null unterscheiden . . . , " seeming to imply monotonicity; a rigorous definition follows a few lines later. One reads also (p. 164) that the equation lim $x = x_1$, " . . . soll aber nicht den sinn haben, dass x die Grenze x_1, unbedingt erreichen muss, sondern nur, dass x ihr jedenfalls beliebig nahe kommt." Although cluster points are not used in this text, even a first view of the limiting operation should distinguish unambiguously between convergence and nonconvergence.

Perhaps the most questionable passages refer to differentials and related notions. A differential is defined (p. 163) as follows: "Eine variable grösse, die unbegrenzt dem Werte Null zustrebt, nennen wir eine werdende Null oder ein Differential." Increments Δx and Δy are defined in the usual way and are used in the usual way in treating the derivative of a polynomial. In connection with a more general treatment of differentiation, Δx and Δy are replaced (p. 185) *when approaching zero* by differentials: "$dx = \Delta x \to 0$, $dy = \Delta y \to 0$." The limit of the difference quotient (p. 174) is "kurz dv/du; genauer wäre die Screibewise lim (dv/du)." Finally (p. 191) we learn that $dy = f'(x)dx$ is not an equation in the ordinary sense, but that it must be understood as a "Grenzwertgleichung"; the variable functional increment dy has to the corresponding increment dx a ratio whose limit is $f'(x)$. Comment is offered (pp. 191–192) on the more conventional view, in which, for fixed x, dy and dx are two variables whose ratio is $f'(x)$, and $dy = f'(x)dx$ is the equation of the tangent line at (x, y) in running coördinates (dx, dy) relative to (x, y).

It is stated explicitly (p. 166) that $+\infty = -\infty$, meaning that there is only one special number ∞, so that $\pm\infty$ is more properly $\infty\pm$. One is led to admit (p. 165) that, if $du \to 0\pm$ then $1/du \to \infty\pm$. (Why not $\infty\mp$?)

There are several other less important departures from conventional notation and terminology. For example, for the point P with coördinates (x, y) the notation Px/y is used far more often than the usual $P(x, y)$. Again, the "main theorem of differential and integral calculus" is (p. 251) that, if $|f'(x)| \leqq M$ for $a \leqq x \leqq b$, then $|f(x) - f(a)| \leqq (x-a)M$ for $a \leqq x \leqq b$.

It seems inconsistent with the otherwise highly articulate character of the book that only intuitive evidence is offered (p. 324) for the existence, under suitable conditions, of a solution $y = F(x)$ to the equation $f(x, y) = 0$.

A section of twenty-five pages is devoted to solid analytic geometry and the differential calculus of functions of several variables. There are 53 short biographical sketches, brief tables, over 400 attractive diagrams, and over 1000 problems. One also finds much intuitive and motivational material and other

detail normally reserved for the classroom. Though the print is small and displays are compact, the book is easily read and style and format are quite attractive.

It is the reviewer's opinion that, on the whole, the advantages of this elaborate text outweigh its disadvantages. The latter can be overcome by effective instruction. Aside from the wealth of exercises and much attractive expository detail, the principal advantage consists in a presentation of graphical and numerical methods that is thoroughly integrated with the rest of the text; such a presentation appears not to have been available previously on an elementary level. In any case, this will be a very useful reference book for those teaching the subjects it covers.

F. A. Ficken

NEW BOOKS RECEIVED

College Algebra. Revised Edition. By L. M. Reagan, E. R. Ott, and D. T. Sigley. New York, Rinehart, 1948. 16+447 pp. $4.00.

An Introduction to College Geometry. By E. H. Taylor and G. C. Bartoo. New York, Macmillan, 1949. 8+143 pp. $3.15.

Higher Algebra for the Undergraduate. By M. J. Weiss. New York, Wiley, 1949. 8+165 pp. $3.75.

Probability Theory for Statistical Methods. By F. N. David. Cambridge, University Press, 1949. 10+230 pp. $3.50.

The Mathematics of Circuit Analysis. By E. A. Guillemin. New York, Wiley, 1949. 14+590 pp. $7.50.

Contributions to Applied Mechanics. By H. J. Reissner (Anniversary volume). Ann Arbor, J. W. Edwards, 1949. 8+493 pp. $6.50.

Theory and Application of $\int_0^z e^{-x^2}dx$ *and* $\int_0^z e^{-p^2y^2}dy \int_0^y e^{x^2}dx$. (Part I, Methods of Computation.) By J. B. Rosser. New York, Mapleton House, 1948. 4+192 pp. $8.00.

Mathematics Review Exercises. Revised Edition. By D. P. Smith and L. T. Fagan. New York, Ginn, 1949. 8+280 pp. $2.00.

Modern-school Solid Geometry. New Edition. By R. R. Smith and J. R. Clark. Yonkers-on-Hudson, World Book, 1949. 8+256 pp. $1.76.

Living Mathematics. 2nd Edition. By R. S. Underwood and F. W. Sparks. New York, McGraw-Hill, 1949. 10+374 pp. $3.00.

Sampling Methods for Censuses and Surveys. By F. Yates. London, Griffin, 1949. 14+318 pp. 24s.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

Sigma Mu Pi, Honorary Mathematics Fraternity

Sigma Mu Pi had its origin in 1947 at Asheville-Biltmore College, Asheville, N. C. Its principal activity is to recognize outstanding mathematical scholarship among college students. A miniature slide rule has been adopted as the fraternity key. Further information concerning the fraternity may be obtained by writing to Mr. Thomas Case of Asheville-Biltmore College, the National Secretary.

CLUB REPORTS, 1948–49

Pi Mu Epsilon, St. Louis University

The papers heard by the St. Louis University chapter of *Pi Mu Epsilon* were:

Generating functions, by Rev. Charles Rust, S.J.

Newton's polygon in higher plane curves, by Rev. J. E. Case, S.J.

Special methods of integration, by Charles Koerner

The concept of area in plane geometry, by Prof. C. C. MacDuffee, University of Wisconsin and Director General of Pi Mu Epsilon

The prize winners in the third annual Essay Contest were: Jeanette Maschmann, for her paper *Carl Frederick Gauss, His Life and Work;* Sister Ermelinda Van Domelen, of the Sisters of St. Mary, for the paper *Gauss' fundamental theorem of algebra.*

The officers elected during the year were: Director, Edward Thirkhill; Vice-Director, William Felling; Secretary, Virginia Herre; Faculty Adviser, Dr. Francis Regan.

Mathematics Club, Wellesley College

The Wellesley *Mathematics Club* enjoyed the following talks during 1948–49:

Chinese numerals and their use, by Miss Zung-Nyi Loh

Educational experiences in Vienna and Great Britain, by Miss Ilse Novak

Teaching of algebra and geometry in secondary schools, by Jasper Moulton, Mathematics Department of Wellesley High School

Modern algebra, by Joyce Friedman

Differential equations, by Mary Ann Berry

Boolean algebra, by C. Elizabeth Taylor.

Club members attended the Wellesley Science Conference as part of their activities.

The officers for the year 1948–49 were: President, Carol Rogers; Vice-President, Joyce Friedman; Treasurer, Lindsley Clark; Secretary, Diane Gruhler; Junior Executive, Betsy Martin; Sophomore Executive, Rachel Allen; Faculty Advisor, Miss Marion Stark.

The officers for 1949–50 are: President, Diane Gruhler; Vice-President, Florence Van Dyke; Secretary, Elizabeth Robinson; Treasurer, Elizabeth Weiner; Junior Executive Ursula Loengard; Faculty Advisor, Miss Ilse Novak.

Kappa Mu Epsilon, Pittsburg (Kansas) State Teachers College

The *Kansas Alpha* chapter of *Kappa Mu Epsilon* presented the following topics at six open meetings:

Origins of mathematical symbols, by Prof. J. A. G. Shirk
The scope of mathematics, by Dr. R. G. Smith
Addition-subtraction logarithms, by Prof. F. C. German
The number 2 in mathematics, by James Hudson
Career fields for mathematicians, by Norval Phillips
Mathematics in secondary schools, by Miss Jane Townsend, principal of the Girard High School
Vital statistics, by D. E. Waggoner, Director of the Division of Vital Statistics, Topeka, Kansas.

Joint meetings were held with the Biology Club and with the Physical Science Club. Kappa Mu Epsilon keys were awarded to James Hudson and Frank Slane for the highest scholastic standings in mathematics and general scholarship.

The officers for the past year were: President, James McCollam; Vice-President, James Hudson; Secretary, Betty Multhaup; Treasurer, Norval Phillips; Corresponding Secretary, Prof. J. A. G. Shirk; Sponsor, Dr. R. G. Smith.

Mathematics Club, University of Kansas

The *Mathematics Club* of the University of Kansas presented the following programs during 1947–48:

Careers in mathematics, by Dr. G. B. Price
Thales, by Arnold Wedel
Number systems, by John Michener
Probability, by Bertha Cummins
Cryptography, by Dr. G. W. Smith
Sine curves, by Joseph Hull
Meteorology, by Vernon Benson
Doubling the cube, by Dr. Robert Schatten
Hessian line co-ordinates, by Francis Brooks
Pythagorean triples, by Charles Terry
The boxing-in process, by W. K. Moore

Mr. Francis Brooks was awarded the book *Men of Mathematics* for presenting the best undergraduate talk of the year. The annual picnic was held in May.

Officers elected for the year 1948–49 are: President, Charles M. Terry; Vice-President, Elneta Richmond; Secretary-Treasurer, Christine Mann.

Kappa Mu Epsilon, Mount Mary College

The *Wisconsin Alpha* Chapter of *Kappa Mu Epsilon* held eight regular meetings during 1947–48. The main business of the year was the drawing up of chapter by-laws and the reading of short papers by the members of the chapter. The following papers were read:

Symposium on non-Euclidean geometry, by Eileen Ford, Pat Farrell, Eleanor Grogan, Mildred Oestreich, Mary Lou Marquardt, Janet Kuhn, and June Rose McDonald

Napier, by Audrey Reiff

Logarithms, by Dorothy Karner

Infinity, by Norma Harding.

The social events for the year included a card party and a dinner for new initiates.

Officers for 1948–49 are: President, Norma Harding; Vice-President, Bernadine Spitznogle; Secretary, Mary Alice Gauerke; Treasurer, Marilyn Briggeman; Corresponding Secretary and Faculty Sponsor, Sister Mary Felice.

Pi Mu Epsilon, Michigan State College

Participation in presentation of mathematical topics by both undergraduates and faculty members was the objective of the *Michigan Alpha* Chapter of *Pi Mu Epsilon* during 1947–48. The following programs were offered:

Mathematical problems in geography, by Joyce Clark

An envelope of pedal lines of a triangle, by Dr. B. M. Stewart

A two variable maximum problem, by Joyce Deisch and Eugene Parker

Using IBM cards to find prime numbers, by Dr. J. S. Frame

Radar equations, by Dr. J. H. Bell

A statistics program, by Robert Zavell and Dale Hekhuis

Calculus of variations or history of mathematics, by Dr. J. E. Powell

Is mathematics important?, by Prof. C. C. MacDuffee of the University of Wisconsin and Director General of *Pi Mu Epsilon*.

The L. C. Plant awards based on scholarship, interest in mathematics, and helpfulness to the mathematics department were presented to James Powell (first prize of $50.00) and to Richard Zindler (second prize of $40.00). Fifty-nine new members were admitted during the year.

Officers elected for 1948–49 were: President, James Powell; Vice-President, Robert Houston; Secretary; Dale Hekhuis; Treasurer, Wendell Grove; Faculty Advisors, Dr. J. H. Bell and Mrs. B. B. Houston.

Pi Mu Epsilon, Michigan State College

The *Michigan Alpha* Chapter of *Pi Mu Epsilon* held ten meetings during the year 1948–49 with mathematical papers being presented by both faculty and student members. Picnics were held during the spring and fall terms and the annual banquet was held during the winter term. At the banquet, which was attended by sixty-six members and guests, Dr. R. V. Churchill of the University of Michigan spoke on *Some applications of differential equations.* Two formal initiations were conducted at which the sixty-four initiates gave short biographical talks on famous men in mathematics.

The following papers were presented at the regular meetings:

Calculus of variations, by Dr. J. E. Powell

Solutions of oblique triangles without tables, by Dr. H. E. Stelson

Three solutions of a mathematical problem from the standpoint of calculus, theory of numbers, and probability, by James Collins, Alex Arnot, and Robert Houston

Theory of knots, by Dr. E. A. Nordhaus

What is infinity?, by Philip Hartman and Edward Seligman

Officers for the year 1948–49 were: President, James Powell; Vice-President, Robert Houston; Secretary, Dale Hekhuis; Treasurer, Wendell Grove; Faculty Advisors, Dr. James Bell and Barbara Houston.

Mathematics Club, Boston University

At the first meeting of the *Mathematics Club of* Boston University for the year, Professors Mode, Johanson, Sobczyk, and Syer spoke on *Occupational opportunities in Mathematics.* Other speakers at the bimonthly meetings were:

What are your chances?, by Mr. Arvanitis

Curved and flat spaces, by Prof. Sobczyk

"*Whom the Gods Love*," by Miss Nickerson

Peculiar topological curves, by Dr. Giever

Significant figures, by Mr. Stubbs

Three famous Greek problems, by Mr. Olds

Solutions with solutions, by Mr. Whitcomb

Magic squares, by Mr. Twigg.

Representatives of the club attended two meetings of the Greater Boston Intercollegiate Mathematics Clubs Association. The Harvard Club was host at the first meeting at which Mr. A. M. Gleason of Harvard spoke on *Nim and oriented graphs.* The second assembly was held at Massachusetts Institute of Technology where Prof. Dirk Struik spoke on *Zeno's paradoxes.*

The social activities included a Halloween Party, a supper, a bowling party, a self-prepared Spaghetti Supper and Gym Night, a Theatre Party, and a picnic.

Officers elected for 1949–50 are: President, Irene Calnan; Vice-President, John Swaffield; Secretary, Marjorie Radcliffe; Treasurer, Bernard Olshansky.

Kappa Mu Epsilon, Iowa State Teachers College

The *Iowa Alpha* Chapter of *Kappa Mu Epsilon* at Cedar Falls, Iowa held an Alumni Homecoming Breakfast and two formal initiation dinners in addition to the regular meetings at which the following talks were presented:

Non-cummutative processes, by Loren Sheldahl

Some tricks of paper cutting, by Donna Whiting and Betty Sayre

Postulates of ordinary geometry, by Jim McGrew

Mathematical induction, by Bill Boettcher

When you fly, by Robert Lankton

Finding extremes by algebraic means, by David McClure

Congruencies, by Richmond Trunkey

An alignment chart for the quadratic equation, by Mrs. Robbie Lou Ashworth.

The *Iowa Alpha* Chapter was well represented at the National Convention of *Kappa Mu Epsilon* held at Topeka, Kansas, April 10–12. Eight students and two faculty members attended the sessions at which Dr. H. Van Engen was re-elected National President.

The officers for 1948–49 were: President, George Mach; Vice-President, Orval Knee; Secretary-Treasurer, Lena Abbas; Corresponding Secretary, George Keppers.

Pi Mu Epsilon, Louisiana State University

Louisiana Alpha Chapter of *Pi Mu Epsilon* held its first meeting of the year for the purpose of organizing the calendar for the session. Papers presented at the regular meetings included:

Applications of complex numbers, by Prof. B. B. Townsend

Coördinate systems, by M. C. Wicht

Mathematics of the atomic bomb, by Prof. F. B. Rickey

Straight line construction of conic sections, by Ernest Ikenberry

LaPlace transforms, by Miss Margaret LaSalle

Osculating figures, by Donald Shipp.

The *Pi Mu Epsilon* Lecture Series sponsored by the local chapter was delivered by Prof. Saunders MacLane of the University of Chicago. His topics included *What is topology?* and *Co-homology theory for groups*.

The year was completed with the initiation of sixty-one members. Following the initiation, a banquet was held at which the two annual awards were presented. For making the highest grade on the freshman honors mathematics examination, James Turner, Jr. was recognized; Mrs. Christine Whitman and Marion Smith were honored jointly as outstanding Seniors in mathematics.

Officers for the year 1948–49 were: President, Donald Shipp; Vice-President, Marion Smith; Secretary, Alice Pecot; Treasurer, Frank Woolam; Faculty Advisor and Corresponding Secretary, Prof. H. T. Karnes.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

WAYNE UNIVERSITY ACQUIRES A MATHEMATICAL MACHINE

A complex mathematical machine, designed and built in 1931 by Dr. Vannevar Bush, became the property of Wayne University in September as the result of a gift from the Massachusetts Institute of Technology. The machine is in two units known as differential analyser and cinema integraph. The machine will find the solution to differential equations of considerable complexity. As a result of this gift, the University can be of greater service to local industry in their solution of research problems.

SCIENTIFIC JUBILEE OF PROFESSOR MAURICE FRÉCHET

The following announcement concerning the "Jubilé Scientifique Du Professeur Maurice Fréchet" has been received by Professor T. H. Hildebrandt and is herewith in translation brought to the attention of the members of the Association:

Professor Maurice Fréchet is approaching the age of retirement. His friends colleagues, and pupils desire to express to him on this occasion, their admiration, their affection, and their gratitude by celebrating his Scientific Jubilee.

Professor Fréchet has expressed the wish that this occasion should be characterized by the utmost simplicity. Near the beginning of the year 1950 one or two prizes will be awarded to an author, or authors, of a memoir on general analysis. The prize winners will be selected by a committee appointed by the Council of the Société Mathematique de France and their names will be announced at one of the sessions of this Société. The Bulletin de la Société will publish later the name or names of the prize winners, as well as a list of subscribers.

In keeping with the expressed wish of Maurice Fréchet this letter is signed only by the members of the section in geometry of the Academie des Sciences, the members of the Council of the Société Mathematique and colleagues and former colleagues of Maurice Fréchet on the Faculty of Sciences of the Collège de France and l'École Polytechnique.

E. Borel, E. Cartan, Denjoy, Hadamard, Julia, Montel.

Brard, president; Bayard, Belgodere, Benoit, Boos, Cagnac, H. Cartan, Chatelet, Choquet, Courbon, Courtand, Desforge, Mme Dubreil, Fourès, Janet, Jean, Lamothe, LeLong, Leray, Lichnerowicz, Maillard, Mandelbrojt, Marchand, Schwartz.

Beghin, Borel, Bouligand, Brard, de Broglie, E. Cartan, H. Cartan, Chapelon, Chatelet, Chazy, C. Darmois, Denjoy, Dubreil, Favard, Garnier, Hadamard, Janet, Julia, Leray, P. Levy, Mandelbrojt, Montel, Pérès, Platrier, Poncin, Roy, Thiry, Valiron, Vessiot, Villat.

To these names will be added at the time of publication in the Bulletin those friends and foreign colleagues of Maurice Fréchet who have assisted in obtaining subscriptions. Subscriptions should be sent to M. Daniel Dugué, Professeur a la Faculté des Sciences, 52, rue d'Authie, a Caen, Calvados.

Rules for the competition. Manuscripts in a foreign language should be accompanied by a typewritten summary in French. All manuscripts should be sent to the President of the Société Mathematique, Institut H. Poincaré, 11, rue Pierre Curie, Paris (5^e) before March 1, 1950. They should treat of general analysis (theory of abstract spaces, transformations of abstract elements into abstract elements) or its applications.

Manuscripts should bear the name and address of the author. If preferred, authors who desire to remain anonymous can write an identifying sentence on the manuscript, which is reproduced on a sealed envelope containing the name and address of the author. Envelopes corresponding to memoirs which are not retained will be destroyed without being opened.

PERSONAL ITEMS

Dr. S. A. Schelkunoff of New York City has been awarded the Stuart Ballantine Medal of the Franklin Institute, Philadelphia, for his outstanding contributions to the extension of the electromagnetic wave theory.

Antioch College announces: Professor Max Astrachan, chairman of Department of Mathematics, who is on leave of absence during 1949–50, has been appointed Professor and Head of Department of Statistics, USAF Institute of Technology, Wright-Patterson Air Force Base, Dayton, Ohio; Assistant Professor Parker Hamilton of Boston University has accepted an appointment at Antioch during the current academic year.

At Bowling Green State University: Dean Emeritus J. R. Overman has resumed full time teaching; Mr. H. E. Tinnappel of Ohio State University has been appointed to an assistant professorship; Mr. Irving Gaskill and Mr. Theodore Titgemeyer have resigned to take graduate work.

Case Institute of Technology announces the following: Associate Professor Max Morris has been promoted to a professorship; Mr. F. C. Leone, formerly instructor at Purdue University, has been appointed to an instructorship.

Centenary College makes the following announcements: Mrs. Fariebee P. Self has been promoted to an assistant professorship; Mr. Charles Murrah has been appointed to an instructorship; Associate Professor W. C. Griffith has been elected a vice chairman of the Louisiana-Mississippi Section of the Association.

Central Missouri State College announces the promotion of Assistant Professor L. W. Akers to an associate professorship and Instructor I. A. Gladfelter to an assistant professorship.

At Clark University: Professor C. E. Melville has retired with the title of Professor Emeritus; Mr. R. R. Christian, formerly graduate student at Yale University, has been appointed to an instructorship.

Connecticut College for Women reports: Mr. W. E. Ferguson, graduate as-

sistant at Yale University, has been appointed to an instructorship; Miss Elizabeth Hahnemann, teaching assistant, has returned to the University of Minnesota for further graduate study.

Fenn College announces the following promotions and appointments: Associate Professors K. D. Kelly and W. R. Van Voorhis have been promoted to professorships; Assistant Professor C. W. Topp has been promoted to an associate professorship; Dr. E. E. Haskins who has been located at Wright Field, Dayton, has been appointed to an associate professorship.

Hope College announces the promotions of Instructors J. E. Folkert and C. A. Steketee to assistant professorships.

At Iowa State College: Assistant Professor R. E. Gaskell has been promoted to an associate professorship; Instructors H. D. Block and H. E. Dickey have been promoted to assistant professorships; Dr. Carl Langenhop of Princeton University and Dr. J. E. Foote of Massachusetts Institute of Technology have been appointed to assistant professorships; Mr. W. S. Bicknell, graduate assistant at University of Michigan, and Mr. R. S. Dean, graduate student at North Texas State Teachers College, have been appointed to instructorships; Professor I. F. Neff and Assistant Professor Fay Farnum have retired.

Iowa State Teachers College announces the appointments of Mr. I. H. Brune of Frostburgh State Teachers College to an associate professorship and Mr. F. W. Lott, Jr., formerly student at University of Michigan, to an assistant professorship.

Macalester College reports: Mr. Robert Scherer, graduate assistant at University of Minnesota, has been appointed to an assistant professorship; Assistant Professor M. D. Brown is on leave of absence and is doing graduate work at the University of Minnesota.

North Central College announces the promotion of Assistant Professor Mary A. Seybold to a professorship and the appointment of Mr. H. G. Beck of the Avery Coonley School to an assistant professorship.

At North Texas State College: Assistant Professor J. V. Cooke has been promoted to an associate professorship; Mr. H. C. Parrish of Ohio State University has been appointed to an assistant professorship.

Pomona College makes the following announcements: Professor A. J. Kempner of the University of Colorado has been appointed to a professorship for the year 1949–50; Mr. C. J. A. Halberg has been appointed to an instructorship; Miss Jean B. Walton, who has been appointed Dean of Women, will teach part-time in the Department of Mathematics; Professor H. J. Hamilton is on leave for 1949–50 to teach at Los Angeles City College; Professor Elmer Tolsted is teaching at Exeter College, Devon, England during 1949–50.

Stanford University reports: Professor Kurt Reidemeister, University of Marburg, Germany, and Institute for Advanced Study, and Dr. P. D. Lax of New York University were visiting members of the staff during the Summer Quarter, 1949. Lectures were given by the following people during the summer: Professor Stefan Bergman, Harvard University; Professor K. Mahler, Univer-

sity of Manchester, England; Professor L. V. Ahlfors, Harvard University; Professor Arne Beurling, Sweden; and Professor Witold Hurewicz, Massachusetts Institute of Technology.

Nebraska State Teachers College, Wayne, announces that Mr. M. J. Hassel has been elected President of the Nebraska Section of the National Council of Teachers of Mathematics.

Trinity College announces the following: Mr. A. J. Grace, Jr., has been appointed to an instructorship; Mr. C. S. Ogilvy is on leave of absence and is doing graduate work at Columbia University.

Union College makes the following announcements: Associate Professors O. J. Farrell and A. H. Fox have been promoted to professorships; Mr. M. R. Bates, formerly teaching fellow at Cornell University, has been appointed to an instructorship; Mr. Edward Craig is now a graduate student at Massachusetts Institute of Technology.

At the University of Arizona: Dr. Deonisie Trifan of Case Institute of Technology and Dr. B. C. Meyer of Stanford University have been appointed to assistant professorships.

University of Washington reports the following: Dr. D. G. Chapman of the University of California has been appointed to an assistant professorship; Mr. D. M. Sandelius of the University of Uppsala has been appointed Lecturer; Assistant Professor Edward Paulson is on leave of absence to continue research at Columbia University.

Upsala College announces: Dr. Louis Larriver of the Naval Observatory has been appointed to an associate professorship; Mr. Donald Lindtredt, formerly instructor at the United States Naval Academy, has been appointed to an assistant professorship; Assistant Professor Norma M. Gilbert has resigned to continue graduate work.

Assistant Professor Leonard Bristow of the University of Wyoming has been appointed Head of Department of Mathematics of Wisconsin State Teachers College, Oshkosh.

Professor H. E. Buchanan, Head of the Department of Mathematics of Tulane University, has retired.

Assistant Professor Virginia Carlton of Centenary College has been appointed to an associate professorship at Northwestern Louisiana State College.

Dr. Uttam Chand has been appointed Assistant Professor of Mathematical Statistics at Boston University.

Assistant Professor George Cook of the Colorado School of Mines has been promoted to an associate professorship.

Mr. P. C. Cox, formerly graduate student at the University of Michigan, has been appointed to an assistant professorship at Albion College.

Mr. H. T. Donohoe of Baylor University has been appointed to an assistant professorship at Louisiana College.

Mr. J. D. Edwards has been appointed to an assistant professorship at Howard College.

Dr. Arthur Erdélyi of the University of Edinburgh has been appointed to a professorship at the California Institute of Technology.

Mr. Milford Franks, Dean of Men at Carthage College, is now teaching part-time in the Department of Mathematics.

Assistant Professor W. J. Fuchs of Cornell University has been promoted to an associate professorship.

Mr. G. R. Glabe, formerly instructor in the Galesburg Division of the University of Illinois, has been appointed to an instructorship at Denison University.

Mr. W. W. Gorsline of Wright Junior College has been appointed to an instructorship at Butler University.

Mr. R. T. Gregory, formerly at the United States Naval Proving Ground, Dahlgren, Virginia, has been appointed to an instructorship at Florida State University.

Miss Clara L. Hancock has retired from her position at Virginia Junior College, Minnesota.

Professor Mabel Heren of Knox College has retired with the title of Professor Emerita.

Mr. Edward Hodson of the University of Wisconsin, Milwaukee Branch, has been appointed Instructor of Mathematics and Physics at Cornell College.

Mr. R. T. Hood has been appointed to an instructorship at Beloit College.

Professor J. M. Kindle of the University of Cincinnati has retired with the title of Professor Emeritus.

Miss Elizabeth C. Kleinhans has been appointed to an instructorship at Illinois Institute of Technology.

Mr. D. V. LaFrenz of William Jewell College has been promoted to a professorship.

Mr. J. C. Lanz has been appointed a member of the Department of Mathematics of Hershey Junior College.

Assistant Professor Mary A. Lee of Sweet Briar College has been promoted to an associate professorship.

Professor R. R. McDaniel of Virginia State College has been appointed Acting Dean.

Professor W. R. McEwen of the University of Minnesota, Duluth Branch, has been promoted to an associate professorship.

Mr. V. D. Moore of Indiana State Teachers College, Terre Haute, has been promoted to an assistant professorship.

Assistant Professor Karlem Riess of Tulane University has been promoted to an associate professorship in the Department of Physics.

Mr. Arnold Ritchie has been appointed Supervisor of Mathematics in the Demonstration School of Central State College.

Associate Professor Helen G. Russell of Wellesley College has sabbatic leave for the first semester of 1949–50.

Instructor D. L. Shell of Michigan College of Mining and Technology has been granted a research fellowship at the University of Cincinnati.

Mr. J. L. Slechticky, University of Toledo, has received an appointment at New Mexico Highlands University.

Associate Professor H. E. Stelson is on leave of absence from Michigan State College; he has a position as visiting professor at the University of Hawaii.

Associate Professor Roscoe Stinetorf of Catawba College has been appointed Head of the Department of Physics of Wagner College.

Mr. A. D. Talkington of the University of Missouri has been appointed to an instructorship at DePauw University.

Dr. L. V. Toralballa of Fordham University has been appointed to an associate professorship at Marquette University.

Assistant Professor R. M. Whitmore of Southwestern University has been promoted to an associate professorship.

Professor C. E. Wilder of Dartmouth College has retired.

Assistant Professor Mary E. Williams of Skidmore College has been promoted to an associate professorship.

Miss Zung-nyi Loh, formerly lecturer at Wellesley College, has been appointed Assistant Professor of Physics at Wilson College.

Professor R. E. Gleason of Temple University died on July 7, 1949.

Assistant Professor G. F. Kelsall of Upsala College died on November 30, 1948.

Professor Nelle Miller of the University of Arizona died on June 20, 1949.

Dr. R. G. D. Richardson, dean emeritus of the Graduate School of Brown University, died July 17, 1949. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 80 persons have been elected to membership by the Board of Governors on applications duly certified:

M. R. Bates, A.B. (Cornell) Teaching Fellow, Cornell University, Ithaca, N. Y.

C. M. Bell, B.S. (Franklin & Marshall) Chemist, Naval Ammunition & Net Depot, Seal Beach, Calif.

C. E. Blom, Editor, "Elementa," Stockholm, Sweden.

C. F. Briggs, M.A. (Michigan) Teaching Fellow, University of Michigan, Ann Arbor, Mich.

Bailey Brown, M.A. (Princeton) Professor, Amherst College, Mass.

Elizabeth W. Brownell, B.A. (Vassar) Student, Stanford University, Calif.

G. C. Burns, B.S. (Oklahoma A & M) Grad. Fellow, Oklahoma A & M, Stillwater, Okla.

L. J. BURTON, Ph.D. (Harvard) Asst. Professor, Bryn Mawr College, Pa.
E. F. BUSCH, Student, St. Norbert College, West De Pere, Wis.
L. H. CHAMBERS, Ph.D. (Cornell) Assoc. Professor, U. S. Naval Academy, Annapolis, Md.
W. J. CHERRY, M.A. (Northwestern) Teacher, Morton High School and Junior College, Cicero, Ill.
K. G. CLEMANS, M.A. (Minnesota) Instructor, Willamette Univerity, Salem, Ore.
REGINAL COBB, Student, University of Florida, Gainesville, Fla.
W. W. COLEMAN, B.A. (Cornell) Asst. Vice President, Irving Trust Co., New York, N. Y.
NICOLAS COLMENARES-CARRILLO, Ing. Civ. (Univ. Nac. de Colombia) Engineer, Caracas, Venezuela.
L. E. DAVIS, JR., B.Sc. (Ohio State) Research Asst., Ohio State University, Columbus, Ohio.
MRS. B. PEARSEN DELANY, B.S. (Illinois Institute of Technology) Student, Illinois Institute of Technology, Chicago, Ill.
H. T. DONOHOE, M.A. (Baylor) Asst. Professor, Louisiana College, Pineville, La.
H. L. EMERSON, Jr., B.S. (Beloit) Student, Beloit College, Wis.
PAUL ERDÖS, D.Sc. (Manchester) Visiting Research Professor, Syracuse University, N. Y.
A. R. ERSKINE, M.A. (Michigan) Instructor, Pennsylvania State College, DuBois, Pa.
E. R. FINKBEINER, Student, St. Norbert College, West De Pere, Wis.
MARY J. FLANARY, B.A. (St. Teresa) Teacher, Academy of Holy Angels, Minneapolis, Minn.
CHARLES FRANCIOL, B.S. (Southwestern Lousiana Institute) Grad. Asst., Louisiana State University, Baton Rouge, La.
JACQUELINE GIVEN, M.A. (Colorado) Head of Department, Pueblo Junior College, Colo.
G. R. GLABE, M.A. (Minnesota) Instructor, Denison University, Granville, Ohio.
R. R. GUTZMAN, M.S. (Iowa) Instructor, Fenn College, Cleveland, Ohio.
D. L. GUY, B.A. (Macalester) Wauwatosa, Wis.
C. J. A. HALBERG, JR., B.A. (Pomona) Instructor, Pomona College, Claremont, Calif.
MR. DILLA HALL, M.S. (Chicago) Asst. Professor, Southern Illinois University, Carbondale, Ill.
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ROBERT HOOKE, Ph.D. (Princeton) Assoc. Professor, University of the South, Sewanee, Tenn.
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L. AILEEN HOSTINSKY, Ph.D. (Illinois) Instructor, Temple University, Philadelphia, Pa.
D. W. HULLINGHORST, B.S. (Tulane) New Orleans, La.
C. J. KAUFMAN, Student, New York, N. Y.
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R. E. LEE, M.S. (Missouri) Asst. Professor, Missouri School of Mines and Metallurgy, Rolla, Mo.
W. T. LENSER, Sc.M. (Brown) Instructor, University of Nebraska, Lincoln, Neb.
TADEUSZ LESER, Ph.D. (London) Asst. Professor, University of Kentucky, Lexington, Ky.

ZUNG-NYI LOH, M.A. (Cornell) Asst. Professor, Wilson College, Chambersburg, Pa.

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L. D. POTTS, B.M.E. (Ohio State) Research Engineer, Linde Air Products Co., Tonawanda, N. Y.

H. W. REIMER, Student, Long Island University, Brooklyn, N. Y.

D. D. RIPPE, M.A. (Nebraska) Teaching Fellow, University of Michigan, Ann Arbor, Mich.

ALEX ROSENBERG, M.A. (Toronto) Fellow, University of Chicago, Ill.

L. L. ROSS, M.A. (Ohio State) Instructor, Ohio Northern University, Ada, Ohio.

S. M. SHARTLE, Surveyor, Office of County Surveyor, Danville, Ind.

ELYSE G. SHEPPARD, M.A. (Michigan) Asst. Professor, University of Tampa, Fla.

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MR. DEONISIE TRIFAN, Ph.D. (Brown) Asst. Professor, University of Arizona, Tucson, Ariz.

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MARY E. WILCOX, M.A. (Southwestern) Asst. Professor, Southwestern University, Georgetown, Texas.

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W. H. WITTY, B.A. (Mississippi) Winona, Miss.

ALBERT WOLINSKY, Ph.D. (Vienna) Instructor, New York University, N. Y.

VERBA M. WOOD, B.S. (Roanoke) Instructor, College of William & Mary, Williamsburg, Va.

CHIA-SHUN YIH, Ph.D. (Iowa) Lecturer, University of British Columbia, Vancouver, B. C.

LILLIAN B. ZARLING, M.A. (Minnesota) Instructor, University of Wisconsin, Green Bay, Wis.

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The twenty-ninth regular meeting of the Southern California Section of the Mathematical Association of America was held at John Muir College in Pasadena, California on Saturday, March 12, 1949. Professor E. F. Beckenbach, Chairman of the Section, presided at the morning session and at one of the afternoon sessions; Professor H. R. Pyle, Vice-Chairman of the Section, presided at the other afternoon session.

The attendance was one hundred thirty-five, including the following sixty-five members of the Association: L. J. Adams, H. M. Bacon, E. F. Beckenbach, May M. Beenken, H. F. Bohnenblust, Herbert Busemann, F. A. Butter, Jr.,

W. D. Cairns, B. A. Chiappinelli, P. A. Clement, L. M. Coffin, Myrtie Collier, E. L. Crow, D. R. Curtiss, J. H. Curtiss, P. H. Daus, R. P. Dilworth, W. H. Glenn, Jr., B. K. Gold, Jr., H. J. Hamilton, R. B. Herrera, M. R. Hestenes, E. J. Hills, P. G. Hoel, R. E. Horton, D. H. Hyers, C. G. Jaeger, Glenn James, G. R. Kaelin, Samuel Karlin, P. J. Kelly, Cornelius Lanczos, Ella E. Lausman, L. C. Lay, Margaret B. Lehman, Sophia L. McDonald, G. F. McEwen, W. E. Milne, F. R. Morris, W. H. Myers, J. F. Paris, R. P. Peterson, Jr., H. R. Pyle, E. D. Rainville, L. T. Ratner, L. L. Rauch, E. C. Rex, Wladmir Seidel, G. E. F. Sherwood, Ernst Snapper, R. H. Sorgenfrey, I. S. Sokolnikoff, D. V. Steed, J. D. Swift, T. E. Syndor, A. E. Taylor, W. I. Thompson, C. W. Trigg, S. E. Urner, F. A. Valentine, Morgan Ward, R. L. White, A. L. Whiteman, D. V. Widder, Euphemia R. Worthington.

At the business meeting the following officers were elected for the next academic year: Chairman: H. R. Pyle, Whittier College; Vice-Chairman, Herbert Busemann, University of Southern California; Program Committee, H. F. Bohnenblust (Chairman), Lulu Bechtolsheim, and R. E. Horton; and Secretary, P. H. Daus, ex-officio.

The next meeting was scheduled for March 11, 1950 at Immaculate Heart College, Hollywood, California.

The following papers were presented:

1. *The theory of games*, by Professor H. F. Bohnenblust, California Institute of Technology.

The fundamental notions of the theory of two-person, zero-sum games were reviewed, and the main points illustrated by a simple example. A discussion of the material was led by Mr. L. S. Shapley and Dr. S. Karlin. Mr. Shapley discussed the extension to infinite games, stressing in particular the difficulties which are encountered when the pay-off function is not suitably restricted. Dr. Karlin spoke on the case of continuous games in which the pay-off function is a polynomial. The relationship to the moment problem was discussed, and attention was called to some of the more important unsolved problems.

2. *Divergence in mathematical definitions and concepts*, by Professor C. G. Jaeger, Pomona College.

This speaker pointed out that there is considerable lack of agreement in definition, usage, and interpretation of many mathematical concepts. It was suggested that some national or international mathematical body attempt to bring about a standardization in cases where it is needed; and that the result be used as an authority where differences of opinion occur.

3. *On L-sets*, by Professor F. A. Valentine, University of California at Los Angeles.

A set S is called an L_n set if each pair of points in S can be joined by a polygonal line having at most n segments. Properties of L_2 sets developed by Professor A. Horn and the author were described. The relationship between L_n sets and C_n-convex sets was discussed. A C_n-convex set is one having n components, each of which is convex. The complement of an open, bounded C_n-convex set $(n>1)$ is an L_{n+1} set. To prove this the author uses the notion of a maximal family of n disjoint open convex sets.

4. *Mortality statistics*, by Dr. J. L. Brenner, Santa Barbara College, University of California, introduced by Professor Herbert Busemann.

This paper gives a general description of mortality statistics, and an analysis of the monetary functions based on them. Some of the principal mortality tables are examined; "selection" is mentioned, and the underlying causes of the existence of a "select period" are brought out.

The provisions, and especially the cost, of a retirement plan, will usually change with the mortality of a group of lives covered by the plan. Several published tables purport to predict the mortality of a group of retired lives. The predictions of these tables are mutually inconsistent. This paper mentions a method which is being used for gathering, and analyzing the reliability, of mortality statistics for retired university teachers. The novelty of the formula used to compute the reliability of the statistics lies in the point of view involved in its application.

5. *Local motions and non-euclidean geometry*, by Dr. P. J. Kelly, University of Southern California.

Given a finite portion of a two dimensional metric space, in which geodesics exist uniquely, the extra assumption that neighborhoods possess reflections onto themselves specializes the geometry to being hyperbolic, elliptic, or euclidean.

6. *A miniature theory in illustration of the convolution transform*, by Professor D. V. Widder, Harvard University.

The convolution transform is defined by the equation

$$f(x) = \int_{-\infty}^{\infty} G(x - t)\phi(t)dt.$$

For a large class of "kernels" $G(x)$, the author and I. I. Hirschman, Jr., have discussed elsewhere the inversion of this transform, that is, the determination of $\phi(t)$ from $f(x)$. They have shown in fact that the inversion is accomplished by means of a linear differential operator, of infinite order, with constant coefficients. In the present paper this general theory is illustrated by use of the special kernel $G(x) = e^x$, $(x<0)$; $G(x) = 0$, $(x>0)$. In this case many of the results take the same form as in the general case, but can be established *ab initio*. Since the Laplace and Stieltjes transforms are special cases of the convolution transform, they are also illustrated by the present miniature theory.

7. *Remainder term in linear methods of approximation*, by Professor W. E. Milne, Oregon State College and the Institute for Numerical Analysis.

This paper represents an effort to formulate a systematic treatment of the error for such diverse processes as interpolation, numerical differentiation, numerical integration, harmonic analysis, approximation by least squares, approximation by equating moments, and other allied operations.

First a uniform procedure is exhibited by which a desired method of approximation may be explicitly constructed. Second, by suitable transformation the remainder is put in a form in which it is possible in many cases to estimate its approximate magnitude. And third, the theory is applied to a variety of concrete examples, and bounds are obtained for the magnitude of the error.

8. *Possibility and necessity*, by Dr. J. C. C. McKinsey, Douglas Aircraft Corporation, introduced by Professor E. F. Beckenbach.

The speaker outlined the results in modal logic obtained during the last three decades by C. I. Lewis, W. T. Perry, M. Wajsberg, K. Godel, J. C. C. McKinsey, R. Carnap, and others. Considerable attention was devoted to the formal properties of the five systems of modal logic described by C. I. Lewis.

9. *Report of the state-wide Mathematical Education Committee*, by Professor Frank Morris, Fresno State College, and Professor H. M. Bacon, Stanford University.

P. H. DAUS, *Secretary*

THE APRIL MEETING OF THE MICHIGAN SECTION

The spring meeting of the Michigan Section of the Mathematical Association of America was held in conjunction with the meeting of the Michigan Academy of Science, Arts, and Letters at Wayne University, Detroit, Michigan, April 2, 1949. This meeting also constituted the meeting of the Mathematics Section of the Michigan Academy of Sciences, Arts, and Letters. Morning and afternoon sessions and a luncheon-business meeting were held, at all of which Professor B. M. Stewart, Chairman of the Section, presided.

About seventy-five people attended the meeting including the following fifty-two members of the Association: Bess E. Allen, N. H. Anning, J. W. Baldwin, W. D. Baten, F. A. Beeler, J. H. Bell, C. J. Blackall, W. M. Borgman, J. W. Bradshaw, J. R. Britton, D. M. Brown, R. E. Carr, R. V. Churchill, B. B. Clark, W. H. Clatworthy, P. C. Cox, J. W. Crispin, P. S. Dwyer, P. W. Edmonson, K. W. Folley, J. S. Frame, E. L. Grindall, G. W. Grotts, V. G. Grove, Fritz Herzog, T. H. Hildebrandt, J. D. Hill, E. E. Ingalls, L. G. Johnson, L. S. Johnston, Wilfred Kaplan, H. D. Larsen, L. E. Mehlenbacher, E. D. McCarthy, J. A. McGrail, D. C. Morrow, A. L. Nelson, Mary H. Payne, Gertrude V. Pratt, G. Y. Rainich, P. H. Raker, L. E. Shaefer, Tryphena H. Scibiorski, W. F. Smith, T. H. Southard, R. L. Spencer, H. E. Stelson, B. M. Stewart, P. C. Sweetland, L. O. Thompson, W. R. Utz, J. E. Vollmer.

The following officers were elected for the coming year: Chairman, Professor L. E. Mehlenbacher, University of Detroit; Secretary-Treasurer, Professor P. S. Jones, University of Michigan.

At the morning and afternoon sessions the following program of papers was presented:

1. *Statistics as part of a general education in mathematics*, by Benjamin Epstein, Wayne University.

It is becoming apparent that an understanding of some of the basic concepts in probability and statistics is essential to the solution of many scientific problems. This means, in the opinion of the speaker, that some basic changes may have to be made in the undergraduate mathematics curriculum. These changes may well require a breaking away from traditional concepts of what constitutes elementary mathematics.

2. *A continued fraction formula for the remainder in a certain series*, by Professor Emeritus J. W. Bradshaw, University of Michigan.

The author finds a formula for the remainder in the special case of hypergeometric series $F(p, p; 1; 1)$ after summing a fixed number of terms. It is expressed as the product of the last term taken and a certain continued fraction. It not only provides a means of rapid calculation, comparable to that of the sum in terms of gamma functions $\Gamma(1-2p)/\Gamma^2(1-p)$, but also yields an extension of the series outside its interval of convergence.

3. *Note on the product of power sums*, by Professor J. S. Frame, Michigan State College.

If $S_p(n)$ denotes the sum of the pth powers of the integers from 1 to n, then it is well known that $S_p(n)$ is a polynomial in n of degree $p+1$, whose coefficients are simply expressible in terms of Bernoulli numbers. The relation $S_1^2(n)=S_3(n)$ is also familiar. In this paper relations such as $2S_3^2(n)=S_7(n)+S_5(n)$ are included in a general formula expressing the product $S_p(n)\,S_q(n)$ of two power sums as a linear combination of the power sum polynomials $S_{p+q+1}(n)$, $S_{p+q-1}(n)$, $\cdots$, in which the sum of the coefficients is unity, the subscripts $p+q+1$, $p+q-1$, etc. all have the same parity, and the number of terms is $1+[p/2]$, if $p\geqq q$. The proof is obtained by summing both sides of the identity

$$S_p(m)S_q(m) - S_p(m-1)S_q(m-1) \equiv m^p[S_q(m) - m^{q/2}] + m^q[S_p(m) - m^{p/2}]$$

from $m=1$ to n.

4. *The number systems of algebra and some trouble spots in grade school arithmetic*, by Professor Holmes Boynton, Northern Michigan College of Education, introduced by the Secretary.

The five number systems of algebra (natural, rational or fractional, signed, real, complex) have been shown to be built one on the other in order to satisfy a logical need; the rational number system to create a system closed to division; the signed, closed to subtraction; the real, closed to the process of taking roots of positive numbers; and the complex, closed to the process of taking roots of negative numbers; and to form a system capable of expressing roots of any algebraic equation.

The logical steps involved in creating each system from the previous one, and of showing the previous one isomorphic to a subset of the new, are not easy for a college student. The grade school child, when first confronted with manipulation of fractions, signed numbers, and real numbers (in the form of square roots, and of π) is often similarly confused. The teacher too often progresses from one system to another without knowing what she is doing, or that difficulties are involved which may confuse the gifted child as well as the normal one.

It is, however, possible for a grade school teacher to develop fractions and signed numbers in a manner understandable to her pupils by analyzing the types of situations in which the need for such numbers arises. This will lead the child to associate in his mind the kinds of number with various things he does. But these things are analogous to the reasons which induce the mathematician to develop the systems in a logical manner.

5. *Some remarks concerning pattern integration*, by Professor R. E. Carr, Michigan State College.

When investigating the asymptotic behavior of a number theoretic function such as

$$H(n) = \sum_{i,j}{}_1^m i^2 j^3, \qquad (i + 2j = n)$$

the substitution $x_i=1/n$ leads to

$$n^{-6}H(n) = \sum_{\substack{i=1\\(p)}}^{n-2} x_i^2\left(\frac{1-x_i}{2}\right)^3(x_i - x_{i-1})$$

where $P{:}i\equiv n \bmod 2$ indicates that the summation is to take place over a subset of the set $\{i\}_1^{n-2}$. The result,

$$\lim_{n\to\infty}[n^{-6}H(n)] = \frac{1}{2}\int_0^1 x^2\left(\frac{1-x}{2}\right)^3 dx$$

does not follow directly from the definition of the Riemann integral. If to the Riemann integral,

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}),$$

is added the restriction that the summation is to take place over a certain subset P of the set $\{i\}^n$, the resulting limit, providing it exists, will be called the P-pattern integral of $f(x)$. This paper contains some remarks and a few theorems about pattern integration.

6. *A multiplicative Diophantine equation*, by Gerald Harrison, Wayne University, introduced by the Secretary.

Some consequences, especially in the ring of Gaussian integers, of the general solution of the Diophantine equation $xy = zw$ were discussed.

7. *Theory of tests of statistical hypotheses*, by C. W. Churchman, Wayne University, introduced by the Secretary.

After brief references to the work of Pearson, Neyman, and Wald on the theory of criteria for "best" tests of statistical hypotheses, this speaker considers certain extensions of their work, and outlines the mathematical problems involved if these extensions are to be carried out to completion.

8. *The rate of interest in installment payment plans*, by Professor H. E. Stelson, Michigan State College.

Three formulas (besides the compound interest formula) are commonly used in determining the rate of interest in installment payment plans. These are known as the constant ratio, series of payments, and interest at end formulas. These formulas have been independently developed from certain particular assumptions. It is the purpose of this paper to show that all three of these formulas may be derived as approximations to the compound interest formula (sometimes called the actuarial formula). It is also shown that the rates as determined by the various formulas satisfy definite inequalities. A new and more accurate approximation formula is presented.

9. *Mathematical statistics from the standpoint of elementary algebra*, by Professor H. W. Alexander, Adrian College, introduced by the Secretary.

Most of the available treatments of mathematical statistics presuppose an advanced knowledge of the calculus. This paper will show that the central ideas of mathematical statistics may be developed and illustrated using only algebraic ideas such as are encountered in the typical college algebra course. The central problems which are dealt with are these: (1) Given an infinite population with a discrete distribution, to derive the distributions of the means and variances of small samples; (2) Given a set of observed values, to set up and test a null hypothesis concerning the parent population from which they are assumed to be drawn.

10. *Clarification of the problem of damped free vibrations*, by Professor G. P. Loweke, Wayne University.

The solution of the problem of damped free vibrations is presented incorrectly in one respect or another in a surprising number of texts. No general solution of the problem is presented in any standard text, and speculation arises as to the exact nature of the motion. This paper is intended to clarify the misunderstanding which can arise from these solutions.

11. *The Green's function for the rectangle obtained by the finite Fourier transformation*, by A. W. Jacobson, Wayne University, introduced by the Secretary.

A special function which arises in the solution of boundary value problems is introduced, and

in terms of this function a new formula for Green's function for the rectangle is given. A relation is established between the special function and Weierstrass's sigma function.

L. J. ROUSE, *Secretary*

THE APRIL MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The April meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at the University of Mississippi, Oxford, Mississippi, April 8–9, 1949. Professor W. L. Duren, Chairman of the Section, presided.

Ninety-eight persons attended the meeting, including the following forty-six members of the Association: T. A. Bickerstaff, J. A. Bullard, L. Virginia Carlton, Margaret R. Davis, Mamie M. Davis, W. L. Duren, Virginia I. Felder, L. M. Garrison, M. E. Gillis, F. Monica Goen, W. C. Griffith, A. C. Grimes, R. H. Hopkins, Alfred Hume, W. R. Hutcherson, H. T. Karnes, C. G. Killen, Z. L. Loflin, Saunders MacLane, A. C. Maddox, J. W. McClimans, Dorothy McCoy, Betty McKnight, R. A. Miller, B. E. Mitchell, S. B. Murray, I. C. Nichols, Arthur Ollivier, C. R. Pettis, O. L. Phillips, P. K. Rees, F. A. Rickey, Alta H. Samuels, H. F. Schroeder, H. L. Smith, M. M. Temple, V. B. Temple, W. B. Temple, J. E. Thomson, B. B. Townsend, G. R. Trott, B. A. Tucker, P. M. Tullier, Eleanor B. Walters, K. L. Warren, M. C. Wicht.

The following officers were elected for the coming year: Chairman, G. R. Trott, University of Mississippi; Louisiana Vice-Chairman, W. C. Griffith, Centenary College; Mississippi Vice-Chairman, M. E. Gillis, Blue Mountain College; Secretary-Treasurer, F. A. Rickey, Louisiana State University.

By the invitation of the Executive Committee, Professor Saunders MacLane delivered two addresses, one at the Friday evening informal dinner on the subject *Mathematics in Europe*, the other at the Saturday morning session on *Determinants and Grassman algebras*.

The following short papers were presented:

1. *Third order involution contained on a certain seventh degree surface*, by W. R. Hutcherson, Northwestern State College of Louisiana.

This paper concerns a seventh degree surface which is invariant under the homography

$$T;\ x_1 : x_2' : x_3' = \epsilon x_1 : \epsilon^2 x_2 : x_3 : x_4, \qquad (\epsilon^3 = 1).$$

The equation of the surface is

$$F(x_1, x_2, x_3, x_4) \equiv a x_2^6 x_1 + b x_1 + c x_1^5 x_2 + d x_2^2 + e x_1^2 x_2 + f x_1 x_2^3 = 0,$$

where a, b, c, d, e, and f are homogeneous functions of x_3 and x_4 of degrees 0, 6, 1, 5, 4, and 3, respectively. This third order involution is mapped on a surface in a space of seven dimensions by use of suitable equations. This surface is of order 21, and the five equations defining the surface are

$$X_5^3 = X_1X_2X_3, \qquad \left\| \begin{matrix} X_8X_3X_7X_5 \\ X_4X_7X_8X_6 \end{matrix} \right\| = 0, \qquad \psi(X_1, X_2, X_3, X_4) = 0.$$

The seventh degree surface in three-way space has two isolated invariant points as well as a

line of invariant points. One of these coincident (invariant) points $P_2(0, 1, 0, 0)$, is found to be a perfect point. (A perfect point is one such that a curve through this point, and also its transformed curve, both pass through it in the same direction.)

2. *Undergraduate mathematics in our Louisiana-Mississippi Section*, by Dorothy McCoy, Belhaven College.

This paper is a report on results obtained from a questionnaire sent to the senior colleges of Louisiana and Mississippi. Twenty-one questions covering mathematics curricula, course content, and related topics were included.

3. *Note concerning transforms of Fuchsian groups*, by P. K. Rees, Louisiana State University.

This paper considers a Fuchsian group of transformations $T=(az+\bar{c})/(cz+\bar{a})$, a fixed Fuchsian transformation $G=(\alpha z+\bar{v})/(vz+\bar{\alpha})$, and the transform $S=GTG^{-1}$ of the group. It is proved that if r_s and r_t are the radii of the isometric circles of S and T, and if g_t, g_t', and g are the centers of the isometric circles of T, T^{-1}, and G, and if $r_s\sqrt{k}=r_t$, then there are no relative maximum or minimum values of k, the absolute minimum is zero and is taken on if and only if the mid-points of the segments (g_t, g_t') and $(g, 1/\bar{g})$ coincide; furthermore, this value can be taken if and only if T is an elliptic transformation.

4. *Compound cycloids*, by B. E. Mitchell and K. L. Warren, Millsaps College.

A compound cycloid is defined as a curve resulting from the compounding of two circular motions; i.e., the rotation of a circle called the guide circle, of radius a, about its fixed center and that of the generating circle, of radius b, $b<a$, which rolls on the guide. The locus of any point on the generating circle will be a compound cycloid. Professors Mitchell and Warren selected four loci for investigation, two without and two within the guide, the two points in each case being diametrically opposite. The parametric equations and the properties of these loci were developed. One interesting result is that the lengths of the compound cycloids are all expressible as elliptic integrals. As the ratio a/b increases, the lengths of these curves approach that of the guide circle (which the lengths of the ordinary epi- and hypo-curves do not).

5. *Harmonic measure in one-to-one directly conformal mappings of canonical plane regions, preliminary report*, by R. W. Schmied, Tulane University, introduced by the Secretary.

This paper is a study of a special case. A sequence $\xi_n=f_n(z)$ of functions each of which is analytic in a region $R_n\subset N_1(0)$ and bounded by $C_1(0)$ and three circles, and which maps R_n in a 1–1 directly conformal manner onto a region $\Omega_n\subset N_5(0)$ is described, and the mapping effected by $\xi_0=f_0(z)=\lim_{n\to\infty} f_n(z)$ for each $z\in R=\lim R_n$ of R onto $\Omega=\lim_{n\to\infty} \Omega_n$ is discussed. Harmonic measure plays the dominant role in the analysis of the mappings considered.

6. *On the extensive derivative of absolute extensors*, by B. B. Townsend, Louisiana State University.

The process of extensive differentiation of absolute extensors introduced by Craig in *Mathematics Magazine*, vol. 21 (1947), pp. 21–29, also *Bulletin of American Mathematical Society*, vol. 53 (1947), pp. 332–343, is developed for other types of tensor-extensors. The extensive derivative based on asymmetric connections is considered, and the corresponding extended components of connection are found.

7. (a) *A universal distributive law*, by G. R. Trott, University of Mississippi.

This speaker considers the expression $a\circ(x\Box y)$, where $\circ$ and $\Box$ are certain specified operations,

and α, x, and y (x and y in general contained in the same set) are elements of the same set or different sets. The condition

$$\alpha \circ (x \square y) = \alpha \circ x \square \alpha \circ y \tag{1}$$

is easily recognized as a type of distributive law. By assigning specific operations to $\circ$ and $\square$, such as multiplication, addition, intersection, union, linear transformation, etc., basic requirements are found for homomorphisms, representations, algebras, tensors, linear vector spaces, etc.

(b) Brief reports by three students of the University of Mississippi, introduced by Professor Trott: *A note on DeGua's Theorem*, by Frank Fant, giving a correction to a translation of a theorem of DeGua concerned with the exact number of complex roots of an equation with missing terms; *Some remarks on quasi groups*, by Evelyn Wright, an attempt to simplify the definitions of certain groupoides and domains in which the associative law of multiplication does not hold, using a continuance of the definitions of rings, integral domains, and fields as given by Dubriel wherein he characterizes the definitions by the use of demi-groups and semi-groups; *On moments and semi-invariants*, by S. R. Knox, a paper on some properties of generating functions of moments and semi-invariants.

8. *A note on the natural number system*, by H. L. Smith, Louisiana State University.

Let I be the class of all natural numbers, and, for each natural number n let $s(n)$ denote the "successor of n" as in Peano's foundation. Then without first defining addition, the relation $<$ can be introduced into I as follows. For each n in I let M_n denote the class of all subsets E of I satisfying the following conditions: (1) $1 \in E$, (2) $s(E) \leqq E + [n]$. (Here $s(E)$ denotes the class of all $s(n)$ for n in E). Let I_n be the intersection of the sets of M_n. Then the relation $M < n$ may be defined to mean $m \neq n$, $I_m \subset I_n$.

9. *Report on the testing program*, by H. T. Karnes, Louisiana State University.

The departments of mathematics in the colleges and universities of Louisiana and Mississippi have instituted a testing program for entering Freshmen. This number on the program was a report of the results of the testing for the 1948–49 school year. It was felt that the results were of sufficient use and importance to justify continuation of the program for another year at least.

F. A. RICKEY, *Secretary*

THE APRIL MEETING OF THE TEXAS SECTION

The annual meeting of the Texas Section of the Mathematical Association of America was held in Denton, Texas, April 8–9, 1949. North Texas State Teachers College and Texas State College for Women were co-sponsors.

The attendance was ninety-one, including the following forty-one members of the Association: A. W. Ashburn, A. E. Barksdale, Ina M. Bramblett, H. E. Bray, Myrtle C. Brown, J. E. Burnam, L. A. Colquitt, J. V. Cooke, Don Cude, J. A. Daum, R. E. Greenwood, E. H. Hanson, H. M. Hardy, C. M. Howard, J. M. Hurt, Mrs. Fay Johnson, E. C. Klipple, C. G. Maple, Hazel L. Mason, E. K. McLachlan, W. K. McNabb, Harlan C. Miller, Edith L. Morgan, E. D. Mouson, Jr., C. A. Murray, J. W. Querry, Henry Rainbow, L. W. Ramsey, C. R. Sherer, D. P. Shore, M. M. Slotnick, F. W. Sparks, D. W. Starr, Jennie L. Tate, H. E. Taylor, F. E. Ulrich, Maud Willey, Mabel Williams, H. A. Wood, L. G. Worthington, C. B. Wright.

At the business meeting the following officers were elected for the next academic year: Chairman H. J. Ettlinger, University of Texas; Vice-Chairman C. A Murray, West Texas State Teachers College; Secretary-Treasurer C. R. Scherer, Texas Christian University.

A banquet was given Friday night in the Household Arts Building on the campus of the Texas State College for Women. Dr. Floyd Stovall of North Texas State Teachers College was the principal speaker. A luncheon was given Saturday noon in Marquis Hall on the campus of North Texas State Teachers College.

The first three papers were read Friday afternoon, the next four papers Saturday morning, and the panel discussion was held Saturday afternoon.

1. *Regions of flatness of analytic functions*, by F. E. Ulrich, The Rice Institute.

At the St. Louis meeting of the American Mathematical Society in November 1947, there were given some results of a joint paper by Mandelbrojt and Ulrich concerning regions of flatness of analytic functions. At that time a definition of regions of flatness was stated, and conditions given under which the regions of a set will be regions of flatness for the derivative when they are regions of flatness for the function. For a statement of these results see Abstract No. 58 in the *Bulletin of the American Mathematical Society*, vol. 54 (1948).

In the present paper conditions are stated under which the regions of a set will be regions of flatness for all the derivatives when they are regions of flatness for the function. For a detailed statement of these results see Szolem Mandelbrojt and Floyd Ulrich, *Sur les domaines de comportement uniforme d'une fonction analytique*, Compte Rendus Acad. Sci. Paris, t. 226 (1948), pp. 152–153.

2. *Paths of minimum flight time*, by L. A. Colquitt, Texas Christian University.

The speaker dealt with the problem of minimizing the time of flight, if airspeed is constant and the wind distribution is given. In previous treatments, beginning with that by Zermolo for the plane, the methods were those of the calculus of variations. It was shown that if the formulation is changed (in a way consonant with navigational practice) the problem becomes one in minimizing a function of several variables.

3. *Exponential numerical integration*, by R. E. Greenwood, University of Texas.

A method of numerical integration for functions which can be expanded in powers of e^x has been investigated. Numerical tabulation of the coefficients in the integration formulae include sixth powers in one case, and ranges from negative third to positive third powers of e^x in a second case. Error expressions have also been found. The method has possibilities for use in exponential growth and exponential decay situations.

4. *Concerning certain topics in the calculus*, by H. E. Bray, The Rice Institute.

The speaker discussed the question "How should calculus be taught." He took the view that, on account of its usefulness in science and engineering, calculus should be begun early, that is, in the freshman year; so that by the end of that year the student may be fairly familiar, at least intuitively, with the basic ideas. From that point on calculus should be taught systematically, with emphasis on its logical structure, in order that the student may carry away a thorough under-

standing of the main body of theorems which constitutes the essential structure of the subject. The speaker gave point to his ideas by means of an outline of the central theorems taken from such topics as the theory of limits, continuous functions, derivatives and differentials, anti-derivatives (primitives), definite integrals, and various types of mean value theorems and their applications. It was contended that the more systematic course, of the kind proposed, if skillfullly conducted with an adequate textbook, can be made beneficial and interesting, not only to talented students but to those of average ability.

5. *Polynomials*, by G. R. MacLane, The Rice Institute.

Mr. MacLane discussed the usefulness of the representation

$$f(z) = \frac{1}{2\pi i}\int \frac{g(t)dt}{t - z}$$

where $g(t)$ is not necessarily $f(t)$. His ideas were illustrated by the connection with the following theorem: If $f(z)$ is analytic and non-vanishing in D, the interior of a rectifiable Jordan curve, then there exists a sequence of polynomials $P_n(z)$, all of whose zeros on Γ, such that line $P_n(z) = f(z)$, the convergence being uniform in any closed subset of D.

6. *A Mongean projection board*, by E. K. McLachlan, Baylor University.

The speaker exhibited a Mongean projection board consisting of three mutually perpendicular planes showing each of the eight octants. This projection board, made from plexiglass, was designed primarily for use as a classroom visual aid by teachers of the Mongean method of descriptive geometry to show how the space object corresponds to the representation in the drawing plane. However, it is versatile enough to have uses as a visual aid in other studies of three dimensional space, such as solid geometry, solid analytics, etc.

7. *The kind of mathematics text I would like*, by C. A. Murray, West Texas State Teachers College.

8. Panel Discussion: *What mathematics should be taught during the first four years in college?*

The participants in this discussion were Ina M. Bramblett, Texas Christian University, M. M. Slotnick, Humble Oil and Refining Company, F. E. Ulrich, The Rice Institute, E. C. Klipple, Texas A and M, C. R. Sherer, Texas Christian University.

C. R. SHERER, *Secretary*

THE APRIL MEETING OF THE IOWA SECTION

The thirty-sixth annual meeting of the Iowa Section of the Mathematical Association of America was held at Drake University, in Des Moines, Iowa, on Friday and Saturday, April 15 and 16, 1949, in conjunction with the Iowa Academy of Science. Professor B. E. Gillam, Vice-Chairman of the Section, presided at the Friday afternoon session; Professor W. M. Davis, Chairman of the Section, presided at the Saturday morning session.

Forty-three persons were present, including the following twenty-eight members of the Association: E. W. Anderson, H. D. Block, Marian E. Daniells, W. M. Davis, R. F. Deniston, R. W. Gardner, R. E. Gaskell, B. E. Gillam, R. N. Goss, Cornelius Gouwens, R. A. Griffin, F. S. Harper, Gertrude A. Herr, J. J. L. Hinrichsen, D. L. Holl, O. C. Kreider, R. J. Lambert, Ta Li, F. M. McGaw,

Martha M. McKelvey, I. F. Neff, A. O. Qualley, Fred Robertson, R. M. Robinson, M. F. Smiley, E. R. Smith, F. M. Stein, H. P. Thielman. A charter member of the Association, Professor I. F. Neff, attended both sessions.

At the business meeting held on Saturday morning the Section elected the following officers for next year: Chairman, Professor B. E. Gillam, Drake University; Vice-Chairman, Professor D. L. Holl, Iowa State College; Secretary, Professor Fred Robertson, Iowa State College.

The following papers were presented at the meeting:

1. *Notes on series with holes in them*, by Professor E. R. Smith, Iowa State College.

It may be shown that

$$H_k(x) = 1 + \frac{x^k}{k!} + \frac{x^{2k}}{2k!} + \cdots$$
$$= \frac{1}{k}\left[e^x + e^{wx} + e^{w^2x} + \cdots + e^{w^{k-1}x}\right],$$

where w is a kth root of unity. The set of functions consisting of $H_k(x)$ and its successive derivatives $H_k'(x)$, $H_k''(x)$, $\cdots$, $H_k^{(k-1)}(x)$ may be considered as generalizations of the hyperbolic functions cosh x and sinh x. They have a trigonometry and a calculus which corresponds to those which exist for the trigonometric and the hyperbolic functions. They possess addition and double parameter formulas, differential and integral properties, and satisfy a differential equation of the kth order. Most of the elementary results may be obtained by means of the identities

$$[H_k(x) + H_k'(x) + H_k''(x) + \cdots + H_k^{(k-1)}(x)]^2 = H_k(2x) + H_k'(2x) + H_k''(2x) + \cdots + H_k^{(k-1)}(2x),$$

$$H_k(x+y) + H_k'(x+y) + \cdots + H_k^{(k-1)}(x+y)$$
$$= [H_k(x) + H_k'(x) + \cdots + H_k^{(k-1)}(x)][H_k(y) + H_k'(y) + \cdots + H_k^{(k-1)}(y)].$$

2. *Definitions of limits in abstract sets*, by R. F. Deniston, Iowa State College.

The author discussed briefly developments in the generalized theory of limits which led him to his concept of eventual sets, which is to be discussed elsewhere. Contributions of E. H. Moore, H. L. Smith, A. Weil, Garrett Birkhoff, J. W. Tukey, H. Cartan, and Pierre Samuel were outlined.

3. *On primitive functions*, by Ta Li, Drake University.

If the algebraic differential equation of lowest order having $y(x)$ as an integral is of order n, then y is said to be a primitive function of order n. Theorems concerning the order of primitive functions are proved. For instance if y is of order n, and z is an algebraic function of x and y, then z is of order n. If y is of order p, z is of order q, and u is an algebraic function of x, y and z, then u is either of order $p+q$ or of order $p-q$.

4. *Calendar*, by Professor F. M. McGaw, Cornell College.

The author discussed relations between the rotation and orbital motion of the earth, and efforts made to reconcile these, and obtain therefrom a calendar which would serve both civil and religious purposes. The Julian reformation and the Gregorian adjustment, by dropping days and changing date of year's beginning, were reviewed. The device of "leap year" was described.

The motion of the moon and the device of extra months employed by certain ancient nations, and by some of the Oriental groups in recent years, to better relate this motion to that of the sun were discussed. The meaning of Metonic cycle, golden number and epact was stated. The relation of these quantities to the calculation of religious festivals, particularly Easter, were discussed and illustrated. Formulae for these calculations were illustrated.

The principles involved in the modern proposal of a world calendar, particularly the so-called equal-quarter twelve months calendar of the Calendar Reform Association, were discussed.

5. *Report of the Secretary*, by Fred Robertson, Iowa State College.

The secretary reported on the meetings of the section secretaries at the Madison and Columbus meetings of the Association, and gave some statistics on the sections.

6. *Applications of the Laplace transformations*, by Professor R. E. Gaskell, Iowa State College.

After introductory remarks concerning the Laplace transformation, with applications to systems of ordinary differential equations, an application to a simple boundary value problem was used to illustrate the procedure for solution by transformation. Then applications were made to the problems of temperature in a bar in contact with a finite quantity of stirred liquid, of longitudinal vibration of a bar with attached mass, and of the vibration of a cantilever with a dashpot at the free end. These problems ordinarily require expansions in terms of a set of functions which are not orthogonal, because of the presence of the parameter in the boundary conditions of the characteristic value problem. Finally, an application of the Laplace transformation to a control problem was made. This problem was such that separation of variables would lead to a characteristic value system which is not self-adjoint. The transformation method gave the solution directly.

7. *On generalized means*, by Professor H. P. Thielman, Iowa State College.

The mean $M_f[x_i; c_i]$ of the numbers x_i with respect to a strictly monotone function $f(x)$ and with weights c_i was defined as

$$f^{(-1)}\left[\sum_{i=1}^{n} c_i f(x_i)\right],$$

where $f^{(-1)}[x]$ stands for the inverse function of $f(x)$, and where the c_i are such that $\sum_{i=1}^{n} c_i = 1$. It was shown that if a function $F(x)$ is such that

$$F\{M_f[x_i; c_i]\} = M_g[F(x_i), k_i],$$

then $F(x) = g^{(-1)}[Af(x) + B]$, and $c_i = k_i$, where A and B are constants. A number of examples were given to illustrate this general result. (J. Aczel, *Commentarii Mathematici Helvitici*, vol. 21, 1948, pp. 247–252.) Applications were made to convex solutions of functional equations.

8. *Short proof of the gramian inequality*, by Professor Bernard Vinograde, Iowa State College, introduced by the Secretary.

A short proof is given of the gramian inequality (see Turnbull and Aitken, *Canonical Matrices*, p. 98) based on the triangular factorization of the gramian matrix.

9. *Escalation with multiple roots*, by R. J. Lambert, Iowa State College.

The purpose of this paper is to complete the Morris and Head escalator process for finding the characteristic roots and vectors of a square matrix. The modified process consists of finding a transformation matrix P_0 by using all known vectors and solving for the unknown columns which correspond to the multiple roots. Thus the principal sub-matrix A_p of the matrix A_n is transformed to a matrix A_0 which is diagonal except for a few ones or zeros in the super-diagonal. Therefore the left hand side of the equation

$$\left| (A_{p+1} - I_{p+1}) \right| = \left| \begin{pmatrix} P_0^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_p & D \\ c & a_{p+1,p+1} \end{pmatrix} \begin{pmatrix} P_0 & 0 \\ 0 & 1 \end{pmatrix} - \lambda I_{p+1} \right|$$

$$= \left| \begin{pmatrix} A_0 & -\lambda I_p & P_0^{-1} D \\ c & P_0 & a_{p+1,p+1-\lambda} \end{pmatrix} \right| = 0$$

will be a determinant which can be expanded in terms of the characteristic roots of a previous stage, and which can easily be put into the partial fraction form by dividing by $|(A_0-\lambda I_p)|$. This method reverts to the Morris method if all characteristic roots are distinct at every stage.

10. *A review of certain problems facing the teacher of mathematics and science*, by Professor C. F. Reid, Dubuque, Iowa, introduced by the Secretary.

The speaker discussed the trends in mathematics teaching in Iowa during the years immediately preceding the outbreak of the second world war, during the war, and at present. He emphasized the way in which the new course of study for the state attempted to meet the needs of the 80 per cent of high school pupils who do not plan to attend college, as well as the 20 per cent who do.

11. *The number of optically inactive glycols*, by Professor E. S. Allen, Iowa State College, introduced by the Secretary.

Glycols being organic compounds with two OH radicals in each molecule, the number of optically inactive ones with a given number of carbon atoms is sought. These are the molecules with a plane of symmetry—a plane which may be the perpendicular bisector of the segment joining the two oxygen atoms or which may contain them. The enumeration is made by methods developed by Harvey Diehl and the author, and by the use of data obtained by Russell E. Carr for optically inactive alcohols.

12. *Properties of an integral transformation*, by H. D. Block, Iowa State College.

The author considers the family of integral transformations, of a suitable function $g(y)$, defined by

$$(1) \qquad f_n(k, x) = \int_{-\infty}^{\infty} |x-y|^n e^{-k|x-y|} g(y)dy = L_n\{g(y)\}, \qquad k > 0,\ n = 0, 1, 2, \cdots.$$

The inversion formula

$$\lim_{k\to\infty} k^{n+1} L_n\{g(y)\} = n! [g(x+) + g(x-)]$$

is derived.

Since $f_n(k, x+a) = L_n\{g(y+a\}$, it follows that $L_n\{g'(y)\} = D_x L_n\{g(y)\}$, where $g(y) \in C$ and $\in C_1$ in sections. Corresponding formulas are given when $g(y)$ and its derivatives have finite jumps. The formula for $L_m\{L_n\} = L_n\{L_m\}$ is given as a simple linear combination of $L_0, L_1, \cdots, L_{m+n+1}$. Thus L_i can be written as a linear combination of the iterates of L_0, showing that all the eigenfunctions of L_0 (viewing (1) as an integral equation in which g may involve the parameter k) are also eigenfunctions of L_i, but associated with different eigenvalues. Other properties of the transformation are also studied, and illustrations given of its application to special functional equations.

13. *Rational canonical form of a matrix*, by Professor M. F. Smiley, State University of Iowa, by title.

This paper appears in this issue of the MONTHLY.

14. *Actual construction of fundamental systems of general linear differential equations and the behaviour of such systems*, by Ta Li, Drake University.

Let $f_1, f_2, f_3, \cdots, f_n$ and f be given functions in (a, b). Writing D for d/dx and $K(x, D)$ for $\sum_0 \lambda^{n-1} f_{n-\lambda}(x) D^{\lambda}$, the general linear differential equation takes the form

$$D''y - K(x, D)y = f(x). \tag{1}$$

For any given function $g(x)$ we define:

$$\begin{aligned} F_1(x, g) &= \int_a^x \int_a^{t_1} \cdots \int_a^{t_{n-1}} K(t_n, D)g(t_n)dt_n \cdots dt_1, \\ F_{n+1}(x, g) &= \int_a^x \int_a^{t_1} \cdots \int_a^{t_{n-1}} K(t_n, D)F_n(t_n, g)dt_n \cdots dt_1, \end{aligned} \tag{2}$$

where $n=1, 2, 3, \cdots$, provided the integrals exist. The functions

$$y_{\mu+1}(x) = \frac{(x-a)^\mu}{\mu!} + \sum_1^\infty kF_k\left(x, \frac{(t-c)^\mu}{\mu}\right) \quad \mu = 0, 1, 2, 3, \cdots, n-1 \tag{3}$$

are n linearly independent solutions of the homogeneous equation if the series are uniformly convergent.

The general solution of (1) is given symbolically by

$$Y = \sum_0^{n-1} \mu C_{u+1} Y_{\mu+1} + D^{-n} \frac{1}{1 - K(x, D)D^{-n}} f(x).$$

Different conditions are imposed on $f_1, f_2, \cdots, f_n$ and f to study the behavior of the solutions.

15. *Bernoulli numbers. Their origin and development*, by Professor R. B. McClenon, Grinnell College, by title.

FRED ROBERTSON, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-third Annual Meeting, New York City, December 30, 1949.

International Congress of Mathematicians, Cambridge, Massachusetts, August 30–September 6, 1950.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN

ILLINOIS, Southern Illinois University, Carbondale, May 12–13, 1950.

INDIANA, Wabash College, Crawfordsville, April 29, 1950.

IOWA, State University of Iowa, Iowa City, April 21–22, 1950.

KANSAS, Spring, 1950.

KENTUCKY

LOUISIANA-MISSISSIPPI, Centenary College, Shreveport, Louisiana, Spring, 1950.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Fall, 1949.

METROPOLITAN NEW YORK, Spring, 1950.

MICHIGAN, March, 1950.

MINNESOTA, University of North Dakota, Grand Forks, October 15, 1949.

MISSOURI, Spring, 1950.

NEBRASKA, Nebraska Wesleyan University, Lincoln, May 6, 1950.

NORTHERN CALIFORNIA, Berkeley, January 28, 1950.

OHIO, Denison University, Granville, April 22, 1950.

OKLAHOMA, Oklahoma City, October 14, 1949.

PACIFIC NORTHWEST, University of Washington, Seattle, June, 1950.

PHILADELPHIA, Haverford College, November 26, 1949.

ROCKY MOUNTAIN, University of Denver, April, 1950.

SOUTHEASTERN, University of Florida, Gainesville, April 7–8, 1950.

SOUTHERN CALIFORNIA, Immaculate Heart College, Hollywood, March 11, 1950.

SOUTHWESTERN, Spring, 1950.

TEXAS, Abilene, Spring, 1950.

UPPER NEW YORK STATE, Syracuse University, Spring, 1950.

WISCONSIN, Marquette University, Milwaukee, May, 1950.

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ON ALMOST PERIODIC FUNCTIONS AND THE THEORY OF GROUPS*

HARALD BOHR, University of Copenhagen

1. **Introduction.** The subject which I have chosen for my lecture, the theory of the almost periodic functions, has gradually been increased to a comprehensive and extensive theory by the contributions of numerous mathematicians in various countries. Therefore it would be a rather impossible task to try to give in a single lecture even a very cursory survey of the many different problems which have been taken up for treatment within the scope of this theory and its generalizations. The task, then, which I have set myself today is less comprehensive. First I shall try to describe quite briefly what might be called the main problem of the theory, confining myself however to the consideration of functions of a real variable, and to explain some especially important features of its solution. Subsequently I shall explain to you in a few words how this main problem could later on be fitted into a much wider class of problems than was originally the case, that is to say, that it could be considered as a problem in the so-called theory of groups. The points of view which led to this extension were first emphasized by Weyl, whereas the accomplishment of the group theoretical treatment is due to von Neumann.

2. **Periodic functions.** Before I begin to speak about the almost periodic functions it will be natural and convenient to say first a few words about the theory of the purely periodic functions. We shall start with a very simple, but at the same time very general notion, namely, a quite arbitrary periodic continuous motion in the plane. Let t denote the time, and let us use complex numbers $w=u+iv$ to characterize the points of the plane. Then this motion can be represented by an equation

$$w = w(t) = u(t) + iv(t),$$

where $w(t)$ is a continuous complex function for $-\infty < t < \infty$, periodic with a period p. Such a motion can of course be extremely complicated. Among these motions the most primitive one is certainly a uniform motion on a circle, for instance with its centre at the origin. Such a motion may be represented by the equation

$$w = ae^{i\lambda t}$$

where a is a complex constant and λ a real number. Let $a=re^{i\theta}$. Then r indicates the radius of the circle, and λ is the angular velocity, so that the period is $2\pi/|\lambda|$, whereas θ indicates the phase, *i.e.*, determines where on the circle the point is to be found at the time $t=0$. Such a uniform circular motion, represented by $w=ae^{i\lambda t}$, will be called a pure oscillation. For the present, we shall consider only those pure oscillations which have a given number p as one

* Rouse Ball Lecture, delivered in Cambridge, England, in May, 1946.

of their periods; let us call them mutually harmonic. If for the sake of convenience we choose $p = 2\pi$, the circular motions selected in this manner are just the ones represented by $a_n e^{int}$, where n is an arbitrary integer and a_n is a complex constant. Combining an arbitrary finite number of such simple circular motions with the period 2π, *i.e.*, considering an exponential polynomial $s(t)$ of the form

$$s(t) = a_0 + a_1 e^{it} + a_{-1} e^{-it} + \cdots + a_n e^{int} + a_{-n} e^{-int},$$

we get of course again a continuous periodic motion of period 2π, which may, however, look very complicated. Such motions, produced by so-called superposition of mutually harmonic oscillations were, as is well known, not unfamiliar even to ancient Greek astronomers. Of every motion produced in this way we shall say that it can be decomposed into mutually harmonic pure oscillations. But we shall extend the meaning of this notion somewhat further, it being convenient to operate not only with finite sums, but also with infinite sums, that is to say, also to involve a limit transition. We shall here consider only a limit process uniform for all t, as the simplest possible limit transition. Thus more generally we shall say about a function $w(t)$ that it can be decomposed into mutually harmonic oscillations, if the function can be represented as the result of a uniform limit transition on finite sums of the kind in question. From the point of view of pure mathematics as well as of the applications, it is evidently a problem of decisive importance to find out which continuous periodic motions with the period 2π may in this way be composed of uniform circular motions. As is well known, this problem was solved by Weierstrass in his famous theorem that every continuous motion which is periodic with the period 2π allows such a decomposition. In other words any quite arbitrary continuous periodic motion with the period 2π can be approximated for all t by one of our polynomials $s(t)$ with an arbitrarily given degree of approximation. This being the case, the question naturally arises as to the intensity and phase with which a definite one of our oscillations e^{int} occurs in the arbitrary motion $w = w(t)$ under consideration. That is, what value does the coefficient of e^{int} assume? Evidently the coefficients of the single terms of the approximating polynomial $s(t)$ cannot be fixed exactly before we have gone to the limit, *i.e.*, as long as we only consider a polynomial $s(t)$ approximating $w(t)$ with a certain degree. However, $s_v(t)$ being a sequence of polynomials which for $v \to \infty$ converges uniformly to the given function $w(t)$, the coefficient $a_n^{(v)}$ with which the oscillation e^{int} occurs in the polynomial $s_v(t)$ will for every fixed n converge to a definite value A_n. This number A_n is called the nth Fourier coefficient of the function, and it may be said to indicate the intensity and phase with which the corresponding oscillation e^{int} occurs in the given motion $w(t)$. This nth Fourier coefficient A_n is determined by

$$A_n = \mathfrak{M}\{w(t)e^{-int}\},$$

where $\mathfrak{M}\{w(t)e^{-int}\}$ denotes the mean value $(1/2\pi)\int_0^{2\pi} w(t)e^{-int}dt$. There is

another way, namely, by a more formal consideration, by which we may immediately arrive at these expressions for the Fourier coefficients. Let us write formally $w(t)$ as an infinite series

$$w(t) = \sum_{-\infty}^{\infty} A_n e^{int}$$

and make use of the fact that the system of functions e^{int} forms a normalized orthogonal system in the sense that for any two arbitrary functions $\phi(t) = e^{in_1t}$ and $\psi(t) = e^{in_2t}$ of the system

$$\mathfrak{M}\{\phi(t)\bar{\psi}(t)\} = \mathfrak{M}\{e^{in_1t}e^{-in_2t}\} = \begin{cases} 0 & \text{for} \quad n_1 \neq n_2 \\ 1 & \text{for} \quad n_1 = n_2. \end{cases}$$

Then, by multiplying our infinite series by e^{-int} and integrating term by term we get just the expression given above for the coefficient A_n of e^{int}. We have indicated above how, starting from Weierstrass' approximation theorem and by performing the limit transition, we were led to the Fourier coefficients and thus to the Fourier series of the function $w(t)$. Conversely, however, we can prove Weierstrass' approximation theorem starting from the formally formed Fourier series of the function $w(t)$. Indeed, the exponential polynomials which are directly determined by the partial sums $\sum_{-n}^{n} A_\nu e^{i\nu t}$ of the Fourier series cannot always be used as uniform approximation sums; these partial sums do, it is true, in a certain sense approximate the function best, namely in the so-called mean, but not uniformly, as we claim here; however, starting from the Fourier series we may in different ways by various so-called summation methods form finite sums $S_N(t)$ which for $N \to \infty$ converge uniformly to the given function $w(t)$. From the mere fact that it is possible from the Fourier series of the function $w(t)$ to determine, *i.e.*, to come back to the function $w(t)$, we see in particular that the function $w(t)$ is uniquely determined by its Fourier series, *i.e.*, that two different continuous periodic functions cannot have one and the same Fourier series. This fundamental theorem, the so-called uniqueness theorem, can also be formulated in the following way: The function $w(t)$ identically 0 is the only function the Fourier constants of which vanish altogether; consequently there does not exist any function $w(t)$ which may be added to the system e^{int} so that the extended system again becomes a normalized orthogonal system. This is expressed by saying that the system e^{int} is a complete normalized orthogonal system.

3. Almost periodic functions. I have dwelt at comparative length on the theory of the continuous purely periodic functions and the theory of their Fourier series, and adapted my remarks about them, in order to be able to speak all the more briefly of the corresponding more general theory of the almost periodic functions. Just as before, we start our discussion of the general

theory by considering the simple pure oscillations

$$w = ae^{i\lambda t},$$

but now we include all of them in our considerations, *i.e.*, we do not select a simple mutually harmonic system by considering only those oscillations which have a given period. This gives from the very beginning an essentially different situation in view of the fact that the total number of pure oscillations has the power of the continuum whereas there exists only a denumerable number of pure oscillations with a given period. As before, so also here, we consider all finite sums of our pure oscillations, *i.e.*, all exponential polynomials of the form

$$s(t) = \sum a_n e^{i\lambda_n t}$$

where now, however, the exponents λ_n may be quite arbitrary real numbers and not all of them multiples of one and the same number as before. As formerly, we are interested in the continuous motion which is determined by the function $w = s(t)$. A principal difference, though, is that in this case the composed motion is no longer periodic; certainly the single components $a_n e^{i\lambda_n t}$ are still periodic, but in general they will have no common period, since the exponents may be incommensurable. However, as pointed out by Bohl, who in some very interesting papers studied some classes of continuous functions which include the periodic functions and are contained in the more general class of the almost periodic functions, the motion described by $w = s(t)$ must at any rate present certain periodic-like features, namely, for every $\epsilon > 0$ there is an infinite number of so-called almost periods or translations numbers $\tau(\epsilon)$. By a translation number $\tau(\epsilon)$ we understand a number τ which for all t satisfies the inequality

$$|w(t+\tau) - w(t)| \leqq \epsilon.$$

In the study of the general class of motions which may be decomposed into a finite or infinite number of pure oscillations the first important problem is, of course, to find the theorem analogous to Weierstrass' theorem concerning the special case of mutually harmonic oscillations e^{int}. To put it more exactly, we ask in analogy with the former case: Which continuous motions $w = w(t)$ may be represented either by finite sums of pure oscillations or may be uniformly approached by such sums. Evidently these motions need not be periodic, but they are far from being quite arbitrary, since they must certainly present some periodic-like features. The exact solution of the problem is that the function $w(t)$ should be what I have called an almost periodic function. The definition of such a function reads: A function $f(t)$ continuous for all t is called almost periodic if, firstly, for any $\epsilon > 0$ it possesses translation numbers $\tau(\epsilon)$ in the sense defined above and if furthermore, the set of translation numbers belonging to a given $\epsilon > 0$ is relatively dense, which means that there do not exist arbitrarily great intervals which are free from such translation numbers $\tau(\epsilon)$.

In view of the subsequent group theoretical investigations I should like to insert a remark about another way of characterizing the almost periodic func-

tions, a way which proves to be closely associated with the original definition given above. Already in my early investigations of almost periodic functions I had occasion to use the following theorem: Let $f(x)$ be an almost periodic function and $h_1, h_2, \cdots$ an arbitrary sequence of real numbers. Consider the sequence of functions $f(x+h_1), f(x+h_2), \cdots$ whose elements originate from the given almost periodic function $f(x)$ by the corresponding translations of the independent variable. Then we can always select a subsequence $h_{n_1}, h_{n_2}, \cdots$ so that the new sequence of functions $f(x+h_{n_1}), f(x+h_{n_2}), \cdots$ converges uniformly on the whole x-axis. Later on Bochner found the interesting result that this theorem can be converted so that we have really a new characterization of the very notion of almost periodicity. This new definition reads in exact formulation: A function $f(x)$ continuous for all x is called almost periodic, if from each sequence of functions $f(x+h_1), f(x+h_2), \cdots$ formed from $f(x)$ by translations of the x-axis a sub-sequence may be selected which converges uniformly for all x. This may also be expressed by saying that the set of functions $\{f(x+h)\}$, $-\infty < h < \infty$, formed from $f(x)$ by all possible translations is compact.

It is not very difficult to prove that every function $w(t)$ which can be decomposed into pure oscillations, *i.e.*, can be approximated uniformly by exponential polynomials $s(t)$, is an almost periodic function. The essential difficulty lies in the proof of the converse, *i.e.*, that every almost periodic motion $w(t)$ can be approximated uniformly by sums of pure oscillations. Here let me stress that the oscillations $a_n e^{i\lambda_n t}$ which occur with no quite negligible coefficients in an exponential sum $s(t)$ which approximates the given function $w(t)$ sufficiently closely must have exponents λ_n, which, in contrast to the coefficients a_n, have exactly determined values characteristic of the function $w(t)$ in question. This is intimately associated with the fact that while an oscillation $ae^{i\lambda t}$ is only slightly changed if the coefficient a is changed a little, the least change of the exponent λ means an essential change of the course of the oscillation, since we are interested in this course for all times, *i.e.*, for $-\infty < t < \infty$. In proving that an arbitrary almost periodic function $f(t)$ can really be approximated by finite sums of pure oscillations, we must therefore begin by finding a way to make the given almost periodic function $f(t)$, so to speak, deliver as its oscillation exponents certain numbers Λ_n characteristic of that function. This is obtained by connecting a Fourier series with the function, just as in the case of the periodic functions. This Fourier series, however, has here the general form

$$\sum A_n e^{i\Lambda_n t},$$

where the set of the exponents Λ_n, characteristic of the function, may be any enumerable set of real numbers and not just of integers. Thus, since the oscillation exponents Λ_n of an almost periodic function are first disclosed by its Fourier series, the Fourier series assumes, in a sense, a still more central position in the theory of the almost periodic functions than it does in the more restricted class of the purely periodic functions, where the exponents are given beforehand. In this lecture I am, of course, not able to go further into the structure of the theory,

but shall only say a few words about it. The starting point is that the total non-enumerable system of all pure oscillations $e^{i\lambda t}$, where λ runs over all real numbers, forms a normalized orthogonal system, but now, of course, only if we consider the system on the whole t-axis; for, denoting by $\mathfrak{M}\{f\}$ the mean value over the infinite t-interval, namely,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(t)dt,$$

we have

$$\mathfrak{M}\{e^{i\Lambda_1 t}e^{-i\Lambda_2 t}\} = \begin{cases}0 & \text{for } \Lambda_1 \neq \Lambda_2,\\ 1 & \text{for } \Lambda_1 = \Lambda_2.\end{cases}$$

Therefore, writing formally our given almost periodic function $f(t)$ as an infinite sum of pure oscillations, that is,

$$f(t) = \sum_{1}^{\infty} A_n e^{i\Lambda_n t},$$

we can determine the coefficient A_n of the oscillation $e^{i\Lambda_n t}$ by a formal calculation, quite analogous to that in the case of the purely periodic functions. Multiplying the equation by $e^{-i\Lambda_n t}$,and taking the mean value on both sides, we get

$$A_n = \mathfrak{M}\{f(t)e^{-i\Lambda_n t}\}.$$

However, this formal starting point must be seen from a point of view essentially different from that of the purely periodic case, where the exponents Λ_n were numbers given beforehand and where the relation was only to serve to determine the corresponding coefficients A_n. In our case where we do not know the exponents of the function but have to determine them, we proceed in the following way. For an arbitrary real λ we form the mean value

$$\mathfrak{M}\{f(t)e^{-i\lambda t}\} = a(\lambda).$$

The forming of this mean value may be interpreted as a question put to the function: Do you for the given λ contain the oscillation $e^{i\lambda t}$? The result $a(\lambda)=0$ means that the function $f(t)$ answers the question in the negative, whereas the answer $a(\lambda)\neq 0$ means that the function admits that it contains the oscillation $e^{i\lambda t}$ in question, and with a coefficient the value of which is given by the number $a(\lambda)$. Now it is relatively easy to show, and this result is a decisive point in the development of the theory, that the answer is negative for almost all values of λ, *i.e.*, that $a(\lambda)=0$ for all λ apart from an enumerable set, say $\Lambda_1, \Lambda_2, \cdots$. These values of λ are called the Fourier exponents of the function, and the corresponding mean values $A_n = a(\Lambda_n)\neq 0$ are called the Fourier coefficients of the function. With these pairs of numbers Λ_n, A_n we form the infinite series $\Sigma A_n e^{i\Lambda_n t}$ called the Fourier series of the almost periodic function $f(t)$ in question. Thus

we have obtained a first, important, though for the present, formal, starting point of the theory.

About the further development, leading up to the final result that $f(t)$ can be approximated uniformly by finite exponential sums $s(t)$, I shall say only a few words. We have seen how to every almost periodic function $f(t)$ is attached an infinite series $\Sigma A_n e^{i\Lambda_n t}$ with real exponents Λ_n and complex coefficients A_n, the Fourier series of the function. It now becomes a question of decisive importance to ascertain whether conversely the function $f(t)$ is uniquely determined by its Fourier series, or in other words, whether two different almost periodic functions always have two different Fourier series. We may also formulate this question in another way, and ask whether our normalized orthogonal system $e^{i\lambda t}$ is complete in the set of the almost periodic functions, *i.e.*, whether we may add a further almost periodic function to this system so that the extended system becomes again a normalized orthogonal system. In a more vague formulation we may interpret the question in the following way: Can an almost periodic function really be entirely decomposed into a denumerable number of pure oscillations or does there remain an undecomposable remainder of the function after the pure oscillations given by the Fourier series have been removed? Fortunately the uniqueness theorem saying that every almost periodic function is entirely characterized by its Fourier series is valid. To prove this theorem or other theorems equivalent to it is the central, but also the most difficult point, of the theory. Several proofs exist, varying as to starting point and method. The most simple, and in a certain sense the most elementary one, is de la Vallée Poussin's proof; it may be characterized as a considerable simplification of the original and rather complicated proof of the lecturer. Other proofs were given by Norbert Wiener in connection with his interesting general spectral theory and by Hermann Weyl. Weyl's proof, based on an analogy with the theory of integral equations, has proved to be especially significant for the group-theoretical generalization of the theory about which I shall speak in a moment.

With the uniqueness theorem at our disposal, we may advance in various ways to obtain the main theorem of our theory, the theorem of the uniform approximation by exponential polynomials which is the counterpart and the generalization of Weierstrass' approximation theorem for purely periodic functions. The original proof, given by the lecturer, and generalizing a method of Bohl which was developed for an essentially more restricted class of functions, was based on a theory of Fourier series for the so-called limit-periodic functions of an infinite number of variables. Later on, Bochner succeeded in showing that the approximating exponential sums $s(t)$ in question could also be obtained directly from the Fourier series of the function $f(t)$ without the transition to functions of an infinite number of variables. Other proofs of the approximation theorem were given by Weyl and Wiener among others. Among these proofs Wiener's proof has turned out to be especially suitable for generalization. In one sense or the other all the different proofs may be regarded as summation methods, the application of which to the Fourier series of an almost periodic

function produces finite sums $s(t)$ which converge uniformly to the function.

Before I proceed to place the theory within the scope of the theory of groups, I want briefly to mention that, like the classical theory of Fourier series of purely periodic functions, the theory of the Fourier series of the almost periodic functions has also been generalized in various ways by numerous mathematicians. Of special interest is a generalization due to Besicovitch, who succeeded in obtaining a class of functions almost periodic in a generalized sense, the Fourier series of which may be characterized in an especially simple way, being just the series $\Sigma A_n e^{i\Lambda_n t}$ for which $\Sigma|A_n|^2$ is convergent, while the Λ_n may be quite arbitrary real numbers.

4. The theory of almost periodic functions as a part of the theory of groups. In the remaining part of my lecture I shall try to describe briefly the points of view which made Weyl and von Neumann see the theory of the almost periodic functions (including in particular the classical theory of the purely periodic functions) in a far more general light, *i.e.*, to see it as belonging within the general theory of groups.

We may start our considerations by observing what in the present connection may be said to be the essential properties of the normalized orthogonal system in question, *i.e.*, the system of the pure oscillations $e^{i\lambda x}$. These pure oscillations may, of course, be looked upon in different ways, but their main characteristics may be said to be that they satisfy the simple functional equation

$$\phi(x + y) = \phi(x)\cdot\phi(y).$$

As is well known, this functional equation has an infinite number of solutions, both continuous and discontinuous, the latter being of a rather unpleasant or abnormal character, in fact, not even measurable in the general sense of Lebesgue. As regards the continuous solutions, they simply consist of all functions $e^{\alpha x}$, where α is an arbitrary complex constant. Selecting from among them only those for which the exponent α is a purely imaginary number $i\lambda$, we get only our pure oscillations $e^{i\lambda x}$, which thus may be characterized as the bounded continuous solutions of the functional equation in question. However, a functional equation of the type

$$\phi(x + y) = \phi(x)\cdot\phi(y)$$

is also encountered in quite another discipline, namely, in the general theory of groups, where the independent variable x, however, need not be a number, but may be a symbol of quite a different kind, whereas the values of the function $\phi(x)$ are still ordinary complex numbers. Before going into details I must first say a few words about the general notion "abstract group" which plays such a fundamental part in the mathematics of our day; this importance follows from the fact that in the theory of groups many apparently quite different investigations taken from numerous mathematical disciplines may be comprised. For the sake of simplicity I shall confine myself in what follows to the consideration of

the so-called commutative or Abelian groups. Such a group is a set of a finite or an infinite set of elements $x, y, \cdots, A, B, \cdots$ (the term "element" is used if we have to deal with abstract investigations where we do not want to commit ourselves beforehand to the considered objects being of any definite kind). For these elements a single so-called composition rule is given which is called multiplication or addition, and is denoted by the multiplication sign $\cdot$ or the addition sign $+$, respectively, but which need not have anything to do with ordinary multiplication or addition, for the simple reason that the elements need not be numbers.

If we use the multiplication sign for the composition of the group, the formulas

$$A \cdot B = B \cdot A$$
$$A \cdot (B \cdot C) = (A \cdot B) \cdot C$$

are valid, and, furthermore, we claim that the equation

$$A \cdot X = B,$$

where A and B are two arbitrary given elements, is always satisfied by one and only one element X. If we use the addition sign, these rules read that

$$A + B = B + A$$
$$A + (B + C) = (A + B) + C$$

and that the equation

$$A + X = B$$

has one and only one solution.

Let me give you one or two examples of such Abelian groups, chosen in close connection with our subject.

First, let us consider the set of all rotations of a circle, for instance, the unit circle, about its center. Let the single rotation be characterized by its rotation angle x, where the real number x is determined modulo 2π. Then the composition rule, *i.e.*, the composition of two rotations x and y, is simply expressed by the sum $x+y$, this number being of course only determined modulo 2π. On the other hand, if we characterize the rotation by the point or complex number $X=e^{ix}$ on the unit circle, into which the point 1 is transformed by the rotation, then the composition of two rotations, determined by $X=e^{ix}$ and $Y=e^{iy}$, respectively, is expressed by ordinary multiplication $X \cdot Y=e^{i(x+y)}$.

As the second example I choose the set of the translations of a straight line into itself. Let a single translation be characterized by the (positive or negative) number x, indicating the length of the translation or, what comes to the same thing, by the point (number) x into which the origin is transformed by the translation. Then the composition of two translations given by x and y, respectively, is, of course, expressed by the ordinary sum $x+y$. Both the rotation group and the translation group are so-called topological Abelian groups, which means

that, besides the structure fixed by the given composition, they have also another so-called topological structure, in consequence of which we may, for example, speak about two elements of the group lying near each other or far from each other. We observe that the first group, the rotation group, may in a certain sense be placed under the latter group, the translation group, by identifying those points on the line which differ by a multiple of 2π, or more geometrically expressed, by imagining the straight line twisted around the unit circle.

Besides these two examples of infinite groups, *i.e.*, groups containing an infinite number of elements, I shall mention a classical example of a finite Abelian group, the group of the classes of residues modulo n, where n is a positive integer. As is well known, this group is a dominating factor in an essential part of the elementary theory of numbers. Let us consider, for instance $n=3$; then the group contains only two elements which we may denote by a_1, a_2 corresponding to the two classes of integers which are relatively prime to 3. These two classes consist of the numbers of the form $3n+1$ and $3n+2$, respectively. The composition of the group is given by the following scheme

$$a_1 \cdot a_1 = a_1, \qquad a_1 \cdot a_2 = a_2 \cdot a_1 = a_2, \qquad a_2 \cdot a_2 = a_1.$$

This scheme states that the product of two numbers, both of the form $3n+1$ or both of the form $3n+2$, is a number of the form $3n+1$, whereas the product of two numbers, one of the form $3n+1$ and the other of the form $3n+2$, is of the form $3n+2$.

While generally the elements of an Abelian group are themselves not numbers, but symbols of one kind or another, they can in a natural way be connected with numbers, generally complex numbers, so that the composition rule for two arbitrary elements of the group is reproduced by ordinary multiplication of the numbers attached to these elements. This is obtained by means of the so-called group characters. A character belonging to an Abelian group is a function $\chi(X)$, where the independent variable X ranges over the elements of the group, while the values of the function are ordinary complex numbers, satisfying, for any two elements X and Y of the group, the equation

$$\chi(X \cdot Y) = \chi(X) \cdot \chi(Y)$$

where $X \cdot Y$ determines the element resulting from X and Y by the composition of the group. If the composition of the group is expressed by $X+Y$ instead of by $X \cdot Y$, the equation characteristic of a character reads

$$\chi(X + Y) = \chi(X) \cdot \chi(Y).$$

Thus we see an evident association with the functional equation valid for the exponential function, *i.e.*,

$$\phi(x + y) = \phi(x) \cdot \phi(y)$$

where x and y denote ordinary real numbers. It may be noted that it is not

demanded that a character $\chi(X)$ should take two different values for two different elements X_1 and X_2; thus for any group we have the trivial so-called main character which assumes the value $\chi(X)=1$ for every X. Generally there exist both real characters for which $\chi(X)$ is a real number for every X and complex characters $\chi(X)$ which assume complex values for certain elements X. If $\chi(X)$ is a complex character, the conjugate function $\bar{\chi}(X)$ is obviously again a complex character. Further, as the functional equation provides immediately, the product $\chi_1(X)\chi_2(X)$ of two characters $\chi_1(X)$ and $\chi_2(X)$ is again a character of the group.

I shall speak briefly about the characters of the three particular Abelian groups mentioned above. As regards the group of the classes of residues modulo n with $h=\phi(n)$ elements $X_1, X_2, \cdots, X_h$, where $\phi(n)$ is the Euler function indicating the number of elements among $1, 2, \cdots, n$ which are relatively prime to n, we have at the same time, $h=\phi(n)$ different characters $\chi_1(X), \chi_2(X), \cdots, \chi_h(X)$. For these characters we have the important relations

$$\frac{1}{h}\sum_{m=1}^{h}\chi_\nu(X_m)\chi_\mu(X_n) = \begin{cases} 0 & \text{for} \quad \nu \neq \mu, \\ 1 & \text{for} \quad \nu = \mu \end{cases}$$

which may be said to express the fact that the characters form a normalized orthogonal system by a simple formation of mean value. It was the study of the characters of this group of the residues modulo n which was the starting point of Dirichlet's famous proof that every arithmetical progression contains an infinite number of primes.

Concerning the two other groups, the rotation group and the translation group, the elements of which we will denote by x modulo 2π and by x, respectively, where x ranges over the real numbers, we will call attention only to some of their characters, namely, the bounded characters; the unbounded characters are of no importance for our purpose. For a bounded character it is seen readily that $|\chi(x)|=1$ for all x. If furthermore we require that our bounded characters should be continuous functions on the group, (considering the group as a topological group, *i.e.*, continuous functions of the variable x), then, in the case of the rotation group (where a character must be periodic with the period 2π) these characters are only the functions e^{inx}, $n=0, \pm 1, \pm 2, \cdots$, whereas for the translation group we get the much more comprehensive set of characters $e^{i\lambda x}$, where λ is an arbitrary real number. Thus we realize that the mutually harmonic pure oscillations forming the basis of the theory of Fourier series of the purely periodic functions may be characterized from a group theoretical point of view as the bounded continuous characters of the rotation group, while the set of all pure oscillations $e^{i\lambda x}$ forming the basis of the theory of the almost periodic functions may be characterized as the bounded continuous characters of the translation group. Now we understand how the main problem solved in the theory of the purely periodic and of the almost periodic functions may, according to Weyl, be generalized to the following general problem concerning a quite arbitrary Abelian group: Which function $f(X)$ defined on the group, *i.e.*,

the independent variable of which ranges over the elements of the group, can be represented by a linear composition of the bounded characters of the group? Precisely speaking, which functions defined on the given Abelian group can either be represented as a finite sum $s(X)=\sum a_\chi \chi(X)$, where the coefficients a_χ are complex constants, or can be uniformly approached by such sums?

Passing on to an outline of the solution of this general problem I start with the following remark, where for the sake of connection with the first part of my lecture it will be convenient to use an additive and not multiplicative notation for the composition of the group. We consider first a single, arbitrarily chosen bounded character $\chi(X)$ of the group. Let H be a parameter which ranges over the whole group. Then the set of all the functions $\{\chi(X+H)\}$ will, as is seen easily from the functional equation $\chi(X+H)=\chi(X)\chi(H)$, be a compact set, in the sense that from any sequence of functions $\chi(X+H_1)$, $\chi(X+H_2)$, $\cdots$ taken from the set, we can choose a subsequence of functions converging uniformly on the group. Furthermore from this property of any single bounded character we may without difficulty conclude that every function $f(X)$ which can be composed linearly of bounded characters of the group will have the same quality, *i.e.*, for every such function $f(X)$ the set of functions $\{f(X+H)\}$ will be compact. Now, guided by Bochner's formulation of the definition of the notion "almost periodicity" for the functions of a real variable, von Neumann set up the following general definition: A complex function $f(X)$ defined on an arbitrary Abelian group is called almost periodic on the group if the set of functions $\{f(X+H)\}$ is compact in the above sense.

As just mentioned, it is easily seen that every function on the group which can be composed linearly of bounded characters is almost periodic on the group. Von Neumann has shown that the converse theorem is valid for a quite arbitrary Abelian group (and not only for the translation group and the rotation group), so that the functions on an arbitrary Abelian group which can be linearly composed of the bounded characters of the group are exactly the almost periodic functions on the group. In its main ideas, von Neumann's proof of this fundamental theorem follows previous proofs given in the theory of the ordinary almost periodic functions. Corresponding to the theory of the ordinary almost periodic functions, where a Fourier series of the form $\sum A_n e^{i\Lambda_n x}$ was attached to every almost periodic function, there is also, in the general case of an arbitrary Abelian group, attached to any function $f(X)$ almost periodic on the group, a Fourier series, here of the form $\sum A_n \chi_n(X)$, where $\chi_1(X)$, $\chi_2(X)$, $\cdots$ is a denumerable set of bounded group characters characteristic of the function $f(X)$ under consideration. By a generalization of the method which Weyl developed to prove the uniqueness theorem for the ordinary almost periodic functions, it is shown that also in the general group theoretical case the Fourier series determines the function $f(X)$ uniquely. Thus, to two different almost periodic functions belong two different Fourier series. A generalization of the method applied by Wiener in proving the approximation theorem for the ordinary almost periodic functions is further used to accomplish the proof of the main theorem.

This theorem states that also in the general group theoretical case, starting from the Fourier series $\sum A_n\chi_n(X)$, we may form finite sums $s(X)=\sum a_n\chi_n(X)$ which approximate the given function $f(X)$ uniformly on the whole group.

So far, the theory of the almost periodic functions on arbitrary Abelian groups is quite parallel to the theory concerning the special case of the translation group. But before the theory could get started at all and be developed on the lines indicated above, there was a fundamental difficulty to be overcome which would seem to make the whole problem quite unapproachable. In fact the very basis of the formation of the Fourier series of an ordinary almost periodic function was the consideration of the mean value of the function $f(x)$ defined by

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x)dx.$$

In the general case, however, when we consider an arbitrary Abelian group without any topological structure, it might seem at first that there is no possibility of defining the notion of a mean value of a function over the group. I have no time to explain the simple and ingenious way in which von Neumann succeeded in defining this notion of the mean value $\mathfrak{M}\{f(X)\}$ of a function $f(X)$ almost periodic on the group. I shall only mention that, as soon as the mean value had been defined, the way was open to the further development of the theory in analogy with the theory of the ordinary almost periodic functions. In particular, as might be expected, the set of the bounded group characters proved to be a normalized orthogonal system, that is,

$$\mathfrak{M}\{\chi_1(X)\bar{\chi}_2(X)\} = \begin{cases} 0 & \text{for} \quad \chi_1 \neq \chi_2, \\ 1 & \text{for} \quad \chi_1 = \chi_2. \end{cases}$$

This being the case we naturally get the Fourier series $\sum A_n\chi_n(X)$ of an almost periodic function $f(X)$ by forming the mean value

$$a(\chi) = \mathfrak{M}\{f(X)\bar{\chi}(X)\}$$

for an arbitrary character $\chi(X)$ and by showing that this mean value is equal to 0 for all χ apart from a denumerable number of characters $\chi_n(X)$ characteristic for the function in question. These $\chi_n(X)$ are of course just the characters which occur in the composition of the almost periodic function $f(X)$ under consideration.

So far I have spoken as if the general theory of the almost periodic functions on an arbitrary Abelian group included in particular the ordinary theory of the purely periodic and of the almost periodic functions of a real variable, if we simply specialize our group to be the rotation group or the translation group, respectively. But as you may have observed, this is not the case. Indeed the pure oscillations forming the basis of the ordinary theory of the purely periodic or the almost periodic functions are not all of the bounded characters of the rota-

tion group or the translation group, but only those bounded characters which are continuous on these particular groups with their natural topology. In the general case of an arbitrary Abelian group, however, we must necessarily treat all the bounded characters of the group on the same footing, as a restriction concerning continuity cannot be formulated at all on a non-topological group. Seen from a general group theoretical point of view, the different bounded characters are therefore all equally simple. In the special case of the translation group, the class of all von Neumann almost periodic functions on this group, which functions were investigated by Ursell prior to and independently of the general theory, is essentially more comprehensive than the class of the ordinary continuous almost periodic functions, as von Neumann does not claim continuity for his class. Fortunately, and this is another beautiful chapter of von Neumann's theory, the theory of the ordinary, *i.e.*, of the continuous almost periodic functions, proves to fit quite naturally into his general theory. I must confine myself to a few words about this point. Let us consider an arbitrary Abelian group, and let it be possible to introduce into this group (as it is possible in the case of the translation group and the rotation group) a topology of some kind or other in order to be able to ascribe any sense at all to the notion of continuous function on the group. Then the main theorem will remain valid if in the whole of the theorem we restrict the functions under consideration by demanding that they shall be continuous on the group considered as a topological group. More precisely, it holds for any topological group that the continuous functions $f(X)$ almost periodic on the group are just those functions which can be composed linearly of the continuous characters of the group.

5. Concluding remarks. In conclusion, I should like to make two remarks in order to emphasize what has been gained by von Neumann's general theory, of which I have given you only a rough outline. In the first place, and this must be said to be a characteristic feature of the mathematics of our day, we have achieved the combination and harmonization of investigations hitherto quite unrelated into one single theory of a general abstract character. Thus, in our case, we have learned about an intimate, hitherto unobserved, connection between the theory of Fourier series of purely periodic and almost periodic functions not only of one variable but of several variables, indeed of an infinite number of variables, and the theory of the group characters of the finite Abelian groups, in particular the group of the classes of residues, which forms the basis of the investigations of the distribution of the prime numbers in the different arithmetical progressions.

As for the other remark, it concerns in particular the translation group and the difference emphasized above between the theory of the almost periodic functions of this group considered as a group without any structure (apart of course from the structure given by the composition itself) and considered as a group topologized by means of the ordinary metric of the straight line. Von Neumann's theory has enabled us to fit into our considerations the total set of all the

bounded characters of this group, *i.e.*, to operate not only with the continuous, but also with the discontinuous solutions of the classical functional equation

$$\phi(x + y) = \phi(x)\cdot\phi(y),$$

and to treat these discontinuous solutions, hitherto disdained, on exactly the same footing as the continuous solutions, *i.e.*, the pure oscillations. Indeed we have seen that in order to obtain the right systematization, it is even necessary to include these discontinuous solutions.

Here, as so often before in the history of mathematics, phenomena which appeared at first to be, so to speak, of a pathological nature, and which therefore from the start had to be excluded by means of protecting definitions, were later recognized, from a more general point of view, to be pertinent, even indispensable, to the subject under consideration.

ARE VARIABLES NECESSARY IN CALCULUS?*

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1. The definite integral. We begin with a case in which the variables are superfluous beyond any doubt. By virtue of their definitions, the numbers

$$\int_0^\pi \sin x dx, \qquad \int_0^\pi \sin z dz, \qquad \int_0^\pi \sin \gamma d\gamma$$

are identical. Hence it does not make any difference which letter we use for the variable. But then why write the dummy part at all? We shall simply write

$$\int_0^\pi \sin.$$

A geometric consideration confirms this view. In a cartesian coördinate system, the sine function represents a curve, the sine curve

$$y = \sin x, \quad w = \sin z, \quad \text{or} \quad \delta = \sin \gamma,$$

according to whether the points are denoted by (x, y), (z, w) or (γ, δ). The numbers 0 and π determine an arc on the curve. The $\int$ sign indicates that we form the area under this arc. How we denote the points has no bearing on the area. We have

$$\int_0^\pi \sin = 2$$

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and similarly

$$\int_{-1}^{1} \sinh = 0, \qquad \int_{1}^{2} \log = 2 \log 2 - 1.$$

If we wish to denote $\int_0^1 e^x dx$ without dummy variables we experience the first difficulty. For e^x denotes the value which the exponential function associates with the number x rather than the exponential function itself. But we remember that if in e^x we have to replace x by a long expression such as

$$-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_1^2} - \frac{2\rho xy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2}\right),$$

then typographical difficulties force us to print

$$\exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_1^2} - \frac{2\rho xy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2}\right)\right]$$

where the symbol, exp, is used for the exponential function in the same way that the symbol, log, is used for the logarithmic function or the symbol, sin, for the sine function. In this notation we can write

$$\int_0^1 \exp = e - 1.$$

In eliminating the dummy part of $\int_0^1 x^n dx$ we are confronted with the complete lack of a symbol for the nth power function which associates the number x^n with x. If we feel that this important function deserves a symbol and we denote it by ${}^n\Pi$, then we can write

$$\int_0^1 {}^n\Pi = \frac{1}{n+1} \cdot$$

We do have a symbol for the less important nth root function, namely, $\sqrt[n]{}$. The typical polynomial has neither a name nor a symbol. The polynomial whose value for x is $a_0 + a_1 \cdot x + \cdots + a_n \cdot x^n$ might be denoted by $p_{a_0, a_1, \cdots, a_n}$. Polynomials of special importance may, of course, be denoted by special symbols. For instance, $p_{-1/2, 0, 3/2}$ is the second Legendre polynomial and is usually denoted by P_2. The frequent occurrence of the function associating with x the number $+\sqrt{1-k^2x^2}$ might warrant the introduction of a symbol. The fact that the graph of the function is the upper half of an ellipse of eccentricity $\sqrt{1-k^2}$ suggests the symbol $\mathrm{ell}_{\sqrt{1-k^2}}$ for this function. In particular, $y = \sqrt{1-x^2}$ is the equation of a semi-circle, and we might write cir instead of ell_0. We should then have

$$\int_0^1 \mathrm{cir} = \pi/4.$$

Instead of the constant polynomial p_c, which associates c with every x, we shall simply write c. For instance,

$$\int_0^\pi 2 = 2\cdot\pi, \qquad \int_0^\pi \sin 2 = \sin 2\cdot\pi.$$

All in all, what we need in the calculus of definite integrals are symbols for the most important functions rather than variables.

2. Substitution and the identity function. Instead of

$$\int_0^\pi \sin 2x dx = 0$$

we might write

$$\int_0^\pi \sin p_{0,2} = 0$$

where sin $p_{0,2}$ denotes the function obtained by substituting the polynomial $p_{0,2}$ into the sine function. This notation for substitution is modelled after the symbol, log sin, in the classical theory where log sin x denotes the value for x of the function obtained by substituting the sine into the logarithmic function. Denoting substitution by juxtaposition is unambiguous if we consistently use a dot in denoting a product. For instance, while sin $p_{0,2}$ denotes the function whose value for x is sin $2x$ we write $\sin\cdot p_{0,2}$ for the function whose value for x is sin $x\cdot 2x$. Similarly we distinguish between the functions log sin and log·sin assuming the values log sin x and log $x\cdot\sin x$, respectively.

But polynomials of the form $p_{0,c}$ are so frequently studied and the function $1/c\cdot p_{0,c}$, that is, the polynomial $p_{0,1} = {}^1\Pi$, is of such paramount importance that it seems to deserve a short symbol of its own. For $p_{0,1}$ is the function which associates x with x; the function which may be substituted into any function f without changing f; the function into which any function may be substituted without being changed. It is, in other words, the identity function, and the lack of a current symbol for this function strikingly illustrates how little heed we give to the algebraic aspects of calculus. We shall denote the identity function by j. The above integral reads

$$\int_0^\pi \sin\,(2\cdot j) = 0.$$

Denoting substitution by juxtaposition we can express the properties of j as follows

$$fj = jf = f \text{ for every } f, \text{ in particular, } jj = j.$$

Incidentally, these equalities show that it would be incorrect to describe the introduction of j for the identity function as "just writing the letter j instead of

the letter x." For could we in the classical notation write

$$fx = xf = f \quad \text{or} \quad f(x) = x(f) = f?$$

In particular, could we write

$$xx = x \quad \text{or} \quad x(x) = x?$$

As a matter of fact, the only way of transcribing the simple formula $fj=jf=f$ into the classical notation is by way of an implication of the following form: if $j(x)=x$ for every x, then $f(j(x))=j(f(x))=f(x)$ and, in particular, $j(j(x))=j(x)$.

The function j also enables us to define pairs of inverse functions, such as sin and arcsin, exp and log. We call g and g^* inverse functions if $gg^*=j$. (The functions g and g' are reciprocal if $g\cdot g'=1$.)

As long as we refrain from introducing a special symbol for the identity function, we are comparable to virtuosos in multiplication without a symbol for the number 1.

3. Differential calculus. It is obvious also that the formulae of differential calculus can be written without variables. If $\mathcal{D}f$ denotes the derivative of f, then the basic formulae read as follows:

$$\text{I.} \quad \mathcal{D}(f+g) = \mathcal{D}f + \mathcal{D}g;$$

$$\text{II.} \quad \mathcal{D}(f\cdot g) = f\cdot\mathcal{D}g + g\cdot\mathcal{D}f;$$

$$\text{III.} \quad \mathcal{D}(fg) = (\mathcal{D}f)g\cdot\mathcal{D}g.$$

In III, the term $\mathcal{D}(fg)$ denotes the derivative of the function obtained by substituting g into f, the term $(\mathcal{D}f)g$, the function obtained by substituting g into the derivative of f. By conventions about the scope of the symbol $\mathcal{D}$, we could dispense with parentheses in either one of the two expressions.

If we set

$$f = \text{logabs}, \quad \text{and} \quad g = \sin$$

where logabs is the function assuming the value $\log|x|$ for x, then we have

$$\mathcal{D}f = \text{rec} \quad \text{and} \quad \mathcal{D}g = \cos$$

where rec is the function ${}^{-1}\Pi$ associating with x the reciprocal number $1/x$, and we obtain from formula III

$$\mathcal{D}(\text{logabs}\sin) = \text{rec}\sin\cdot\cos = \cot.$$

4. The calculus of antiderivatives. We shall denote by $\mathcal{S}f$ any antiderivative of f, that is, any function having the derivative f. Hence we have

$$\textbf{A.} \quad \mathcal{D}(\mathcal{S}f) = f.$$

Moreover we shall write

$$\textbf{B.}\quad f \sim g \text{ if and only if } \mathcal{D}f = \mathcal{D}g.$$

Obviously, the relation $\sim$ is reflexive, symmetrical, and transitive. We further readily prove

$$\textbf{C.}\quad \mathcal{S}f \sim g \text{ if and only if } f = \mathcal{D}g,$$

$$\textbf{D.}\quad \mathcal{S}(\mathcal{D}f) \sim f.$$

In this notation, the classical results concerning antiderivatives can be written without variables. For instance,

$$\mathcal{S}\cos \sim \sin, \qquad \mathcal{S}\exp \sim \exp, \qquad \mathcal{S}\operatorname{rec} \sim \operatorname{logabs}, \qquad \mathcal{S}\log \sim (j-1)\cdot\log.$$

5. Changing variables. The crucial test for a notation without variables is integration by substitution. For, traditionally, this method is treated as a change of variables. In our notation we have, first of all,

$$(\mathcal{S}h)g \sim \mathcal{S}[hg\cdot\mathcal{D}g]. \tag{1}$$

For, by **B** of Section 4, this formula is equivalent to

$$\mathcal{D}\{(\mathcal{S}h)g\} = \mathcal{D}\{\mathcal{S}[hg\cdot\mathcal{D}g]\}$$

and this last equality is true since both expressions are equal to $hg\cdot\mathcal{D}g$; the expression on the right side by virtue of **A** of Section 4; the expression on the left side since by virtue of Rules III of Section 3 and **A** of Section 4 we have

$$\mathcal{D}\{(\mathcal{S}h)g\} = [\mathcal{D}(\mathcal{S}h)]g\cdot\mathcal{D}g = hg\cdot\mathcal{D}g.$$

This completes the proof of (1). If on both sides of (1) we substitute an inverse of g, that is, a function g^* such $gg^*=j$, then, in view of $(\mathcal{S}h)j=\mathcal{S}h$, we obtain

$$\mathcal{S}h \sim [\mathcal{S}(hg\cdot\mathcal{D}g)]g^*$$

which is the formula for integration by substitution. The formula clearly indicates the four steps that we have to take in integrating h by the substitution of g, namely,

1) the substitution of g into h;
2) the multiplication by $\mathcal{D}g$;
3) the integration of the product;
4) the substitution of g^* into the antiderivative.

For instance, let h be rec cir, that is, the function associating the number $1/\sqrt{1-x^2}$ with x. If we wish to find an antiderivative of h by the substitution of the function $g=\sin$ for which $\mathcal{D}g=\cos$ and $g^*=\arcsin$, then, noting that cir sin $=\cos$ and $\mathcal{S}1\sim j$, we obtain

$$\mathcal{S}\operatorname{rec}\operatorname{cir} \sim [\mathcal{S}(\operatorname{rec}\cos\cdot\cos)]\arcsin \sim (\mathcal{S}1)\arcsin \sim j\arcsin = \arcsin.$$

Thus we do not need variables in order to "change variables."

It is important to realize that we can apply the method described above even if we refrain from introducing new symbols for special functions beyond j for the identity function. For instance, if we denote the function associating $1/\sqrt{1-x^2}$ with x by f and note that $1/\sqrt{1-\sin^2 t}=\sec t$ or, without variables, that $f \sin = \sec$, then we still have

$$\mathbb{S} f \sim [\mathbb{S} f \sin \cdot \mathcal{D} \sin] \arcsin \sim [\mathbb{S} \sec \cdot \cos] \arcsin$$
$$\sim [\mathbb{S}\, 1] \arcsin \sim j \arcsin = \arcsin.$$

6. Functions of more variables. In what follows we shall confine ourselves to the study of functions of two variables and shall mainly stress a few points which are of importance for the calculus concerning functions of one variable. For more general considerations the reader is referred to other publications [1].

The relevance of functions of two variables for the theory of functions of one variable is illustrated by the following examples.

An implicitly defined function f of one variable is a function which, for some function F of two variables, satisfies the condition $F(x, f(x))=0$ for every x. Functions of one variable which are solutions of a first order differential equation present similar problems of notation.

Some important functions of two variables are defined as integrals with regard to a parameter, that is, of functions of two variables with regard to one of them. For instance, the value of the Gamma function for x is defined as

$$\Gamma(x) = \int_0^\infty e^{-y} \cdot y^{x-1} dy.$$

Laplace transforms and integral equations lead to similar expressions.

The difference quotients of a function f of one variable are the values of a function of two variables, *viz.*, the function

$$\frac{f(y) - f(x)}{y - x} \cdot$$

Similarly, the definite integral $\int_x^y f(t)dt$ of a function of one variable between arbitrary limits x and y is a function of two variables which, without dummy variables, can be written in the form $\int_x^y f$.

Functional equalities for functions of one variable are connected with functions of more variables. For instance, the equality for the exponential function,

$$e^x \cdot e^y = e^{x+y} \quad \text{or} \quad \exp x \cdot \exp y = \exp (x + y),$$

is related to a function of two variables. The function f is homogeneous of degree n if

$$f(x \cdot y) = f(x) \cdot y^n.$$

Finally, even the basic operations of adding and multiplying two functions, f and g, can be interpreted as the result of substituting f and g into a sum-function S and a product-function P, respectively. Here S and P are functions of two variables for which

$$S(x, y) = x + y, \qquad P(x, y) = x \cdot y.$$

7. Substitution in the realm of functions of two variables. Besides the functions f, g, sin, exp, $\cdots$ of one variable we shall now also admit functions of two variables which we shall denote by capital letters F, G, S, P, $\cdots$. With regard to the substitution of i functions of k variables into a function of i variables we shall make the following conventions (ik):

(11). By substituting g into f we obtain a function fg of one variable;

(12). By substituting G into f we obtain a function fG of two variables; if $f=\sin$ and $G=S$, then $fG=\sin S$ is the function associating the number $\sin(x+y)$ with (x, y);

(21). By substituting (g, h) into F we obtain a function $F(g, h)$ of one variable; if $F=P$, $g=\sin$, $h=\exp$, then $F(g, h)=P(\sin, \exp)$ is the function associating the number $\sin t \cdot \exp t$ with t.

(22). By substituting (G, H) into F we obtain a function $F(G, H)$ of two variables. If $F=P$, $G=S$, $H=\sin P$, then $F(G, H)=P(S, \sin P)$ is the function associating the number $(x+y)\cdot\sin(x\cdot y)$ with (x, y).

Each of these substitutions satisfies important one-side distributive laws in conjunction with addition and multiplication, namely,

$$(f_1 + f_2)\gamma = f_1\gamma + f_2\gamma \quad \text{and} \quad (f_1 \cdot f_2)\gamma = f_1\gamma \cdot f_2\gamma$$

where γ may be either a function g of one variable or a function G of two variables, and

$$(F_1 + F_2)(\gamma, \delta) = F_1(\gamma, \delta) + F_2(\gamma, \delta) \quad \text{and} \quad (F_1 \cdot F_2)(\gamma, \delta) = F_1(\gamma, \delta) \cdot F_2(\gamma, \delta)$$

where (γ, δ) may be either a pair (g, h) or a pair (G, H).

Interpreting F as a surface $z=F(x, y)$ in the same way as we have interpreted f as a curve $y=f(x)$ we see that

$$z = \sin x \cdot \exp x \quad \text{and} \quad z = \sin y \cdot \exp y$$

are different surfaces. Both surfaces are cylindrical, their cross-sections are congruent, but their generating lines are perpendicular. Does the variable matter, after all?

The explanation lies in the generalization of the identity function j of one variable, which associates x with x. In the realm of functions of two variables, there are two functions, J_1 and J_2, such that

$$J_1(x, y) = x \quad \text{and} \quad J_2(x, y) = y \qquad \text{for every } (x, y).$$

We have

$$F(J_1, J_2) = F, \qquad J_1(\gamma, \delta) = \gamma, \qquad J_2(\gamma, \delta) = \delta$$

for every function F and every pair (g, h) and (G, H). If a function F of two variables has the property that $F(j, h) = F$ for every function h, then, in the classical terminology, F is a function of x and y depending on x only. The condition $F = F(j, 0)$ is sufficient for F to have the above property. An example of a function of this kind is J_1 and, in fact, fJ_1 for every function f of one variable. Similarly, $J_2 = J_2(0, j)$ is an example of a function of two variables depending upon the second variable only.

The two different cylindrical surfaces mentioned before correspond to two different functions of two variables, namely,

$$(\sin \cdot \exp)J_1 = \sin J_1 \cdot \exp J_1$$

and

$$(\sin \cdot \exp)J_2 = \sin J_2 \cdot \exp J_2,$$

respectively. After what was said in Section 3 about j it should not be necessary to emphasize that the notation without variables for the two functions does not amount to just replacing the letters x and y by the letters J_1 and J_2. We also see that

$$S = J_1 + J_2 \quad \text{and} \quad P = J_1 \cdot J_2.$$

The difference quotients of a function f of one variable now appear to be the values of the following function of two variables

$$\frac{fJ_2 - fJ_1}{J_2 - J_1}.$$

The functional equality of the exponential function reads

$$\exp J_1 \cdot \exp J_2 = \exp (J_1 + J_2) \quad \text{or} \quad P(\exp J_1, \exp J_2) = \exp S.$$

The function f is homogeneous of degree n if, and only if,

$$f(J_1 \cdot J_2) = fJ_1 \cdot {}^n\Pi J_2 \quad \text{or} \quad fP = P(fJ_1, {}^n\Pi J_2).$$

As examples of the role of j in the realm of functions of two variables, we mention the implicit definition of a function f of one variable by the equality $F(j, f) = 0$ for some function F and the first order differential equation $\mathcal{D}f = F(j, f)$ for a given function F.

The differential equation which is traditionally written in the form

$$(\alpha) \qquad M(x, y)dx + N(x, y)dy = 0$$

where M and N are given functions, can be interpreted in one of the following three ways:

(1) as the condition $M(j, y) + N(j, y) \cdot \mathcal{D}y = 0$ for a function y of one variable;
(2) as the condition $M(x, j) \cdot \mathcal{D}x + N(x, j) = 0$ for a function x of one variable;

(3) as the condition

$$(\beta) \qquad M(x, y)\cdot\mathcal{D}x + N(x, y)\cdot\mathcal{D}y = 0$$

for a pair of functions x, y of one variable which are traditionally written in the form $x(t)$ and $y(t)$. We have here used the letters x and y rather than f and g as symbols for functions, in order to avoid the impression that we attribute to the first letters of the alphabet properties not shared by the last letters. Yet, the resemblance of (β) with the classical form (α) is somewhat illusive. For in (β), x and y do not mean variables, and $\mathcal{D}x$ and $\mathcal{D}y$ do not mean differentials.

8. Partial derivatives. There are two operators of differentiation in the realm of functions of two variables, $\mathcal{D}_1$ and $\mathcal{D}_2$. In the classical theory, $\mathcal{D}_1F$ and $\mathcal{D}_2F$ are called the two partial derivatives of F with regard to the first and the second variable, respectively. As before, the derivative of the function f of one variable will be denoted by Df. In particular, we have

$$(2) \qquad \mathcal{D}_1J_1 = \mathcal{D}_2J_2 = 1 \quad \text{and} \quad \mathcal{D}_2J_1 = \mathcal{D}_1J_2 = 0.$$

Now let $\int f$ be the function of two variables associating with (x, y) the definite integral $\int_x^y f$. (There is no danger of confusing $\int f$ with an antiderivative of f since the latter function has been denoted by $\mathsf{S}f$.) The definite integral $\int_0^\pi \sin$, studied in Section 1, is the value of the function $\int \sin$ for $(0, \pi)$. Hence, in a systematic notation we should write

$$\left(\int \sin\right)(0, \pi) = 0, \qquad \left(\int \log\right)(1, 2) = 2\cdot\log 2 - 1, \text{ etc.}$$

The computation of a definite integral by substitution, written without variables, reads

$$\int f = \left(\int fg\cdot\mathcal{D}g\right)(g^*J_2, g^*J_1).$$

The fundamental laws concerning the reciprocity of differentiation and definite integration, traditionally written in the form

$$\int_a^b f'(t)dt = f(b) - f(a) \quad \text{or} \quad \int_a^b f(t)dt = g(b) - g(a) \quad \text{where} \quad g'(t) = f(t)$$

and

$$\frac{d}{dy}\int_a^y f(t)dt = f(y) \quad \text{and} \quad \frac{d}{dx}\int_x^b f(t)dt = -f(x),$$

can now be written without variables

$$\int (\mathcal{D}f) = fJ_2 - fJ_1 \quad \text{or} \quad \int \mathcal{D}f = (\mathsf{S}f)J_2 - (\mathsf{S}f)J_1$$

where $\mathcal{S}f$ is any antiderivative of f, but the same in both places, and

$$\mathcal{D}_1 \int f = -fJ_1, \qquad \mathcal{D}_2 \int f = fJ_2.$$

Partial differentiation is connected with addition and multiplication by the analogues of formulae I and II of Section 3,

$\mathbf{I}_i$ $\mathcal{D}_i(F+G) = \mathcal{D}_iF + \mathcal{D}_iG$,

$\mathbf{II}_i$ $\mathcal{D}_i(F\cdot G) = F\cdot\mathcal{D}_iG + G\cdot\mathcal{D}_iF$.

Differentiation is connected with substitution by the following generalizations of Formula III of Section 3.

$\mathbf{III}_{12}$. $\mathcal{D}_i(fG) = (Df)G\cdot D_iG \qquad (i = 1, 2).$

Example. If $G = P = J_1\cdot J_2$, then, in view of formulae (2),

$$\mathcal{D}_2(fP) = J_1\cdot(\mathcal{D}f)P.$$

If f is homogeneous of degree n (*cf.* Section 6), then

$$fP = fJ_1\cdot {}^n\Pi J_2,$$

and thus

$$\mathcal{D}_2(fP) = fJ_1\cdot n\cdot {}^{n-1}\Pi J_2.$$

Hence

$$fJ_1\cdot n\cdot {}^{n-1}\Pi J_2 = J_1\cdot(\mathcal{D}f)P$$

and, noting that

$$J_1(j, 1) = j, \qquad J_2(j, 1) = 1, \qquad P(j, 1) = j, \qquad fj = f, \qquad (\mathcal{D}f)j = \mathcal{D}f,$$

we obtain Euler's relation

$$n\cdot f = j\cdot\mathcal{D}f.$$

$\mathbf{III}_{21}$. $\mathcal{D}[F(g, h)] = (\mathcal{D}_1F)(g, h)\cdot\mathcal{D}g + (\mathcal{D}_2F)(g, h)\cdot\mathcal{D}h.$

Example. If $g = j$ and thus $\mathcal{D}g = \mathcal{D}j = 1$, then

$$\mathcal{D}[F(j, h)] = (\mathcal{D}_1F)(j, h) + (\mathcal{D}_2F)(j, h)\cdot\mathcal{D}h.$$

If h is implicitly defined by $F(j, h) = 0$, then $\mathcal{D}[F(j, h)] = \mathcal{D}0 = 0$ and we obtain

$$\mathcal{D}h = -(\mathcal{D}_1F)(j, h)/(\mathcal{D}_2F)(j, h)$$

for the derivative of the implicit function h.

$\mathbf{III}_{22}$. $\mathcal{D}_i[F(G, H)] = (\mathcal{D}_1F)(G, H)\cdot\mathcal{D}_iG + (\mathcal{D}_2F)(G, H)\cdot\mathcal{D}_iH \qquad (i = 1, 2).$

Example. If we define a function H implicitly by $F(x, H(x, y)) = y$, then, without variables $F(J_1, H) = J_2$. Using formulae (2) from III_{22}, we obtain the expres-

sions for $\mathcal{D}_1 H$ and $\mathcal{D}_2 H$.

As has been shown, *l.c.* [1], Formula III and its extensions to functions of any number of variables can be synthesized.

9. Partial integration. Corresponding to the two operators $\mathcal{D}_1$ and $\mathcal{D}_2$, there are two operators $\mathcal{S}_1$ and $\mathcal{S}_2$ of antiderivation and two partial definite integrals $\int_1 F$ and $\int_2 F$. For instance, the Laplace transform of a function f

$$\int_0^\infty \exp(-x\cdot y)\cdot f(y)dy$$

is a definite integral with regard to the second variable of a function of two variables which, without variables, can be written in the form

$$\exp(-J_1\cdot J_2)\cdot fJ_2 \quad \text{or} \quad \exp(-P)\cdot fJ_2.$$

The integral is a function of two variables depending only upon the first. If, in this function, we substitute $(j, 0)$ and observe that $gj = g$ we obtain a function of one variable which we may denote by

$$\mathcal{L}f = \left[\int_{0^2}^{\infty} \exp(-P)\cdot fJ_2\right](j, 0).$$

Integral equations and the Gamma function can be written in a similar way.

10. Conclusions. While variables are not necessary for the presentation of fundamental results of calculus, there remain two questions. To what extent are variables necessary in proving these results? And, are variables not desirable even in formulating the theorems?

Since most students learn calculus as a tool, and since books on physics, engineering, statistics, mathematical economics, etc., are written in the classical notation, it is clear that, in initiating students into calculus, we have to use the classical notation. Yet I feel that the possibility of a consistent notation without variables should influence our teaching, namely, in the direction of reducing the use of variables. I further think that, at least in a few cases, we should mention the alternative form and, in particular, make the student aware of the possibility of a consistent notation which dispenses with dummy variables. I even believe that the ability to grasp, say, integration by substitution without variables (Sections 5 and 8) is a gauge for a student's real understanding of calculus.

In proving formulae, we shall make use of variables although perhaps again at a diminishing rate. In proving, for instance, Formula III of Section 3 we show that if for a number x_0 the three numbers

$$\mathcal{D}g, \qquad (\mathcal{D}f)g, \qquad \mathcal{D}(fg)$$

are meaningful, then the third is the product of the first two. (In fact, we prove even more.) This result may be interpreted in the following form: At a place

where the three terms of formula III are meaningful, the formula is true. Many formulae can be interpreted in the sense that they are true provided that every term is meaningful. For elementary functions, one could even develop an algebra of their domains of definition accompanying the algebras of their substitution and differentiation.

Another point brought out by these developments is that the application of the limit concept can be confined to the proof of very few basic formulae from which all the other formulae can be obtained by some algebra.

In concluding we mention the existence of finite models for the algebra of analysis [2] comparable to finite rings or fields enjoying the essential properties of the system of all real numbers, and to finite planes which have the essential properties of our affine or projective plane.

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A GENERALIZATION OF FEUERBACH'S THEOREM

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1. Introduction. The theorem that a circle touching the sides of a triangle touches the nine-point circle is due to Feuerbach. A generalization due to Hart which seems to complete the theorem in one direction is that four circles touching four given circles touch a fifth. In the geometry of the triangle many generalizations have been given, but one which is not a statement about conics is McCay's [1], that the pedal circle of two isogonal conjugates which are collinear with the circumcenter touches the nine-point circle.

The theorem of this paper is a generalization of the last, and may be stated in two ways:

1. *The angle between the nine-point circle of a triangle and the pedal circle of two isogonal conjugates is equal to the angle which the line joining the points subtends at the inverse of either in the circumcircle.*

2. *The angle between the nine-point circle of a triangle ABC and the pedal circle of a point P is equal to*

$$\sphericalangle PBC + \sphericalangle PCA + \sphericalangle PAB \pm \pi/2.$$

The proof is a deduction from three formulas found in *Inversive Geometry* by

Morley and Morley.

2. Angle between two circles. Since the proof is by complex numbers it is convenient to regard the angle between two lines, denoted by $\sphericalangle AOB$, as a measure of the anticlockwise rotation which brings OB into coincidence with OA [2], and thus to have the convention

$$\sphericalangle AOB = - \sphericalangle BOA.$$

If two intersecting circles have radii ρ_1, ρ_2 and the distance between their centers is δ_{12}, and if θ is the angle, positive or negative, subtended by the line joining the centers at either point of intersection, then

$$2\rho_1\rho_2 \cos \theta = \rho_1^2 + \rho_2^2 - \delta_{12}^2.$$

It is convenient to take this as the definition of the angle between two circles [2], and thus to make no distinction between $\pm\theta \pm 2n\pi$.

3. Angle between pedal circle and nine-point circle. Taking the circumcenter as origin and the circumradius as unity, if the vertices of a triangle are denoted by the complex numbers t_1, t_2, t_3, then

$$t_1\bar{t}_1 = t_2\bar{t}_2 = t_3\bar{t}_3 = 1.$$

When the Euler line is taken as the axis of real numbers

$$t_1 + t_2 + t_3 = \bar{t}_1 + \bar{t}_2 + \bar{t}_3.$$

If x, y are isogonal conjugates and ρ is the radius of their pedal circle, the three formulas [3] mentioned are

$$x + y + \bar{x}\bar{y}s_3 = s_1 \tag{1}$$

$$\bar{x} + \bar{y} + xy/s_3 = s_1 \tag{2}$$

$$(1 - x\bar{y})(1 - \bar{x}y) = 4\rho^2 \tag{3}$$

where

$$s_1 = t_1 + t_2 + t_3, \qquad s_2 = t_2t_3 + t_3t_1 + t_1t_2, \qquad s_3 = t_1t_2t_3.$$

Since $(x+y)/2$ is the center of the pedal circle and $s_1/2$ is the center of the nine-point circle, and its radius is $1/2$, the angle between the pedal circle and the nine-point circle is given by

$$\tfrac{1}{2}\left| 1 - x\bar{y} \right| \cos \theta = \tfrac{1}{4} + \tfrac{1}{4}(1 - x\bar{y})(1 - \bar{x}y) - \tfrac{1}{4}(x + y - s_1)(\bar{x} + \bar{y} - s_1).$$

Substitution from (1) and (2) gives

$$\begin{aligned} \tfrac{1}{2}\left| 1 - x\bar{y} \right| \cos \theta &= \tfrac{1}{4} + \tfrac{1}{4}(1 - x\bar{y})(1 - \bar{x}y) - \tfrac{1}{4}xy\bar{x}\bar{y} \\ &= \tfrac{1}{4}(1 - x\bar{y} + 1 - \bar{x}y). \end{aligned}$$

Thus θ is equal to the amplitude of $1 - x\bar{y}$. This is the basis of the discussion.

4. First statement of theorem. Since the equation of the circumcircle is

$$t\bar{t} = 1,$$

then, if y' denote the inverse of y,

$$y'\bar{y} = 1.$$

Hence the angle between the nine-point circle and the pedal circle of x, y has been proved equal to

$$\text{amp } (y' - x)/y' = \text{amp } (y' - x)/(y' - y).$$

This is the first statement of the theorem.

5. Second statement of theorem. Elimination of y from (1) and (2) gives

$$\bar{y} = (x^2/s_3 - xs_1/s_3 + s_1 - \bar{x})/(1 - x\bar{x}),$$

so that

$$\begin{aligned} 1 - x\bar{y} &= (s_3 - s_2x + s_1x^2 - x^3)/s_3(1 - x\bar{x}) \\ &= (t_1 - x)(t_2 - x)(t_3 - x)/t_1t_2t_3(1 - x\bar{x}). \end{aligned}$$

The amplitude of

$$(t_1 - x)(t_2 - x)(t_3 - x)/t_1t_2t_3$$

is the (algebraic) sum of the angles the line joining x and the circumcenter subtends at the vertices, and is equal to

$$\measuredangle\ xt_2t_3 + \measuredangle\ xt_3t_1 + \measuredangle\ xt_1t_2 - \pi/2.$$

The expression

$$1 - x\bar{x}$$

is of course real, and is negative if x is outside the circumcircle, and positive if x is inside. Thus

$$\theta = \measuredangle\ xt_2t_3 + \measuredangle\ xt_3t_1 + \measuredangle\ xt_1t_2 \pm \pi/2,$$

where the sign is plus if x is outside the circumcircle and minus if x is inside.

This is the second statement of the theorem.

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MATHEMATICAL NOTES

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ON THE CONVERSE OF FERMAT'S THEOREM

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Following Lehmer we shall call an integer n a *pseudoprime* if $2^n \equiv 2 \pmod n$ and n is not a prime. The smallest pseudoprime is $341 = 11.31$. Recently Sierpinski[1] gave a very simple proof that there are infinitely many pseudoprimes, by proving that if n is a pseudoprime then $2^n - 1$ is also a pseudoprime. Lehmer[2] proved that there exist infinitely many pseudoprimes n with $v(n) = 3$, where $v(n)$ denotes the number of different prime factors of n. In the present note we prove the following theorem.

THEOREM. *For every k there exist infinitely many squarefree pseudoprimes with $v(n) = k$.*

First we repeat Lehmer's proof[3] that there are infinitely many pseudoprimes n with $v(n) = 2$. It is well known[4] that for every $m > 6$ both $2^m - 1$ and $2^m + 1$ have a primitive prime factor; that is, there exist primes p and q such that

$$2^m - 1 \equiv 0 \pmod p, \qquad 2^l - 1 \not\equiv 0 \pmod p, \qquad \text{for } 1 \leqq l < m;$$
$$2^m + 1 \equiv 0 \pmod q, \qquad 2^l + 1 \not\equiv 0 \pmod q \qquad \text{for } 1 \leqq l < m.$$

It is easy to see that $p \cdot q$ is a pseudoprime. In fact we have $p \equiv q \equiv 1 \pmod{2m}$, $2^{2m} \equiv 1 \pmod{p \cdot q}$, thus

$$2^{pq-1} \equiv 2^{(p-1)(q-1)} \cdot 2^{p-1} \cdot 2^{q-1} \equiv 1 \pmod{pq}.$$

Also it is immediate that to different values of m correspond different values of $p \cdot q$, which proves the theorem for $k = 2$.

The proof of the general case will be very similar to that of Lehmer. We use induction on k. Let $n_1 < n_2 < \cdots$ be an infinite sequence of pseudoprimes with $v(n_i) = k - 1$. Let p_i be one of the primitive prime factors of $2^{n_i - 1} - 1$. We claim that $p_i \cdot n_i$ is a pseudoprime. In fact, by definition, $2^{n_i - 1} \equiv 1 \pmod{p_i \cdot n_i}$, also $2^{p_i - 1} \equiv 1 \pmod{p_i}$. Further, since $p_i - 1 \equiv 0 \pmod{(n_i - 1)}$, we have $2^{p_i - 1} \equiv 1 \pmod{n_i}$ and finally $2^{n_i - 1} \equiv 1 \pmod{n_i}$. Thus

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[1] Colloquium Math., vol. 1 (1947), p. 9.

[2] This MONTHLY, vol. 56 (1949) p. 306.

[3] This MONTHLY, vol. 43 (1936), pp. 347–356.

[4] Bang, Tiddsskrift for Mat. 1886, pp. 130–137. See Also Birkhoff-Vandiver, Annals of Math., 1904.

$$2^{n_ip_i-1} = 2^{(n_i-1)(p_i-1)}\cdot 2^{n_i-1}\cdot 2^{p_i-1} \equiv 1 \pmod{p_in_i}.$$

Also $p_i > n_i$, since $p_i \equiv 1 \pmod{(n_i-1)}$ and n_i is not a prime. Thus $p_i\cdot n_i$ is squarefree, and $v(p_i\cdot n_i)=k$, and all the integers $p_i\cdot n_i$ are different; this completes the proof of the theorem.

Following Lehmer we call n an absolute pseudoprime if $a^n\equiv a \pmod n$ for every a prime to n. The smallest absolute pseudoprime is 561. It seems very difficult to determine whether there are infinitely many absolute pseudoprimes. A similar question is whether there exist any composite numbers n with $n-1\equiv 0 \pmod{\phi(n)}$.

Two further questions are: Are there integers n so that 2^n-1 has more than k primitive prime factors? Are there infinitely many primes p for which 2^p-1 is composite? The smallest such prime is 11.

Denote by $f(x)$ the number of pseudoprimes not exceeding c. I can prove that

$$c_1\cdot\log x < f(x) < c_2\frac{x}{(\log x)^k}, \tag{1}$$

for every k if x is sufficiently large. In other words, the number of pseudoprimes is considerably smaller than the number of primes. The proof of (1) (second inequality) is complicated and we do not discuss it here.

A THEOREM ON THE DISTRIBUTION OF PRIMES

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Using elementary properties of integers only, we shall establish the following result.

THEOREM: *For every positive integer r, there exists a prime p, with $3.2^{2r-1} < p < 3.2^{2r}$.*

This theorem is almost as strong as Bertrand's postulate, which states that for every real value of $x\geqq 1$ there is a prime p with $x < p \leqq 2x$.

The following four lemmas follow easily from Legendre's expression for $n!$, namely,

$$n! = \prod_p p^{\sum_{i=1}^{\infty}[n/p^i]}$$

They are proved in [1], which contains an exposition of Erdös' proof of Bertrand's postulate.

(1) *If $n < p \leqq 2n$, then p occurs exactly once in $\binom{2n}{n}$.*

(2) *If $n \geqq 3$ and $2n/3 < p < n$, then p does not occur in $\binom{2n}{n}$.*

(3) *If* $p^2 > 2n$, *then* p *occurs at most once in* $\binom{2n}{n}$.

(4) *If* $2^a \leqq 2n < 2^{a+1}$, *then* p *occurs at most* a *times in* $\binom{2n}{n}$.

Suppose there is no prime p with

$$3.2^{2r-1} < p < 3.2^{2r}; \tag{5}$$

then

$$\binom{3.2^{2r}}{3.2^{2r-1}} \leqq \binom{2^{2r}}{2^{2r-1}}\binom{2^{2r-1}}{2^{2r-2}} \cdots \binom{2}{1}\left\{\binom{2^{r+1}}{2^r}\binom{2^r}{2^{r-1}} \cdots \binom{2}{1}\right\}^{2r}. \tag{6}$$

By (1), (2), (3), and (5), every prime which appears on the left-hand side of (6) appears also on the right; and those primes which appear with multiplicity greater than 1 on the left appear on the right with multiplicity at least $2r+1$, which by (4) is at least equal to the multiplicity with which they appear on the left.

On the other hand, a simple computation shows that for $r \geqq 6$, we have

$$3.2^{2r} > (2^{2r} + 2^{2r-1} + \cdots + 2) + 2r(2^{r+1} + 2^r + \cdots + 2). \tag{7}$$

If we now interpret $\binom{2n}{n}$ as the number of ways of choosing n objects from $2n$, it follows from (7) that the inequality in (6) should be reversed. Hence for $r \geqq 6$ we have a contradiction which proves the theorem for these values of r. The primes 7, 29, 97, 389, and 1543 show that the theorem is also true for $r < 6$.

Our method may also be used to prove Bertrand's Postulate. For this purpose, if $2^\alpha < 2n \leqq 2^{\alpha+1}$, we replace (6) by

$$\binom{2n}{n} \leqq \binom{2a_1}{a_1}\binom{2a_2}{a_2} \cdots \binom{2}{1}\left\{\binom{2a_k}{a_k}\binom{2a_{k+1}}{a_{k+1}} \cdots \binom{2}{1}\right\}^{\alpha},$$

where

$$a_1 = \left[\frac{n+1}{3}\right], \qquad a_i = \left[\frac{a_{i-1}+1}{2}\right] \qquad \text{for } i > 1, \text{ and } k = \left[\frac{\alpha+1}{2}\right].$$

The details of the proof in this case can be filled in by the reader. Although the resulting proof is not quite as neat as the proof of the somewhat weaker result, it is still, we believe, simpler than the known proofs of Bertrand's Postulate [**1, 2, 3**].

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FERMAT'S EQUATION AND TSHEBYSHEFF'S POLYNOMIALS

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There is an interesting relation between Fermat's equation and the Tshebysheff polynomials which seems, so far, to have gone unstated.

It will be recalled that Fermat's equation (also known, inappropriately, as Pell's equation) is

$$v^2 - aw^2 = 1, \tag{1}$$

where v and w are integers, and a is a positive constant which is not the square of a rational number. The two smallest positive integers v_1 and w_1 which satisfy (1) constitute its *fundamental solution*. It is known that such a fundamental solution always exists, for if $\sqrt{a}$ be expanded into a simple continued fraction, there exist positive convergents v_m/w_m such that v_m and w_m satisfy (1), and v_1/w_1 is the least of these convergents. (See pages 36–38 of Harry N. Wright, *First Course in the Theory of Numbers*, John Wiley and Sons, New York, 1939.)

It is known also that all solutions v_n and w_n of (1), in zero or integers, without any exception, are obtained by equating the rational and irrational parts in the relation

$$v_n + w_n\sqrt{a} = \pm (v_1 + w_1\sqrt{a})^n, \tag{2}$$

where n is any integer or zero. (See page 351 of Uspensky and Heaslet, *Elementary Number Theory*, McGraw-Hill, New York, 1939.)

Now let us put $v_1 = \cos\theta$. Then, from (1), $w_1 = -i \sin\theta/\sqrt{a}$. Substituting these results in (2), and making use of De Moivre's Theorem, we find that $v_n + w_n\sqrt{a} = \pm(\cos n\theta - i \sin n\theta)$.

Let $v_n = \pm\cos n\theta$. Then $w_n = \mp i \sin n\theta/\sqrt{a}$, and again making use of (1), we have $w_n = (\pm \sin n\theta/\sin\theta)w_i$. Thus $v_n = \pm T_n(v_1)$, and $w_n = \mp w_1 U_{n-1}(v_1)$, where $T_n(x)$ and $U_n(x)$ are the Tshebysheff polynomials of the first kind and second kind, respectively, defined by $x = \cos\theta$, $T_n(x) = \cos n\theta$, and $U_n(x) = (\sin (n+1)\theta)/\sin\theta$, for integer or zero n. (See pages 55–56 of Madelung and Erwin, *Die Mathematischen Hilfsmittel des Physikers*, Dover Publications, New York, 1943.)

A NOTE ON FERMAT'S CONGRUENCE

V. L. KLEE, JR.,* University of Virginia

Carmichael [1, p. 53] defines a function λ as follows (ϕ being Euler's function): $\lambda(2^e) = \phi(2^e)$ if $0 \leqq e \leqq 2$; $\lambda(2^e) = \frac{1}{2}\phi(2^e)$ if $e \geqq 3$; $\lambda(p^e) = \phi(p^e)$ if p is an odd prime; $\lambda(2^e p_1^{e_1} \cdots p_n^{e_n})$ is the least common multiple of $\lambda(2^e)$, $\lambda(p_1^{e_1})$, $\cdots$, $\lambda(p_n^{e_n})$, where $2, p_1, \cdots, p_n$ are distinct primes. He proves (p. 54) that if a is prime to m, then $a^{\lambda(m)} \equiv 1 \pmod{m}$; and later (pp. 72–74) he shows that each integer m has a primitive λ-root—that is, an integer which belongs modulo m to

* I am indebted to the referee for helpful suggestions regarding the arrangement of the proof.

the exponent $\lambda(m)$. Now denote by $L(m)$ the set of all integers k such that $a^k \equiv 1 \pmod{m}$ whenever a is prime to m. Then from Carmichael's results it follows that $L(m)$ consists of all multiples of $\lambda(m)$.

Now let us, in contrast, fix our attention on a definite exponent n, and consider the congruence, $a^n \equiv 1 \pmod{h}$, for various values of h. Denote by $C(n)$ the set of all integers h such that this congruence holds whenever a is prime to h, and let $\gamma(n)$ denote the greatest integer in $C(n)$. (We shall show that $C(n)$ is bounded.) As will be indicated below, γ is closely related to another function considered by Carmichael **[2]**.

In this note we establish some "dual" relations between L and λ on the one hand and C and γ on the other. The first such relation, which is an immediate consequence of the definitions, is:

(1) $n \in L(m)$ if and only if $m \in C(n)$.

We next establish the following result:

(2) $L(m)$ consists of all multiples of $\lambda(m)$, $C(n)$ of all divisors of $\lambda(n)$.

The proof of (2) for L and λ was indicated above. To prove the statement for C and γ we note first that $\phi(x) \to \infty$ as $x \to \infty$, whence $\lambda(x) \to \infty$ as $x \to \infty$. Now let $\gamma^*(m)$ denote the greatest integer x for which $\lambda(x) \mid n$. (By the preceding remark, $\gamma^*(n)$ is defined for each n.) Consider an arbitrary $h \in C(n)$ and let r be a primitive λ-root of h. From the congruence, $r^n \equiv 1 \pmod{h}$, we have $\lambda(h) \mid n$, whence $h \leqq \gamma^*(n)$ and $C(n)$ is bounded. Now if $d \mid \gamma^*(n)$, we have $\lambda(d) \mid \lambda(\gamma^*(n))$, whence $\lambda(d) \mid n$ and $d \in C(n)$. Hence $\gamma(n) = \gamma^*(n)$ and $C(n)$ contains all divisors of $\gamma(n)$.

It remains to show that $C(n)$ contains only divisors of $\gamma(n)$. Suppose that $h \in C(n)$, $h \nmid \gamma(n)$, and let b denote the least common multiple of h and $\gamma(n)$. Since $b > \gamma(n)$, there is an integer a prime to b such that $a^n \not\equiv 1 \pmod{b}$. But we also have $a^n \equiv 1 \pmod{h}$ and $a^n \equiv 1 \pmod{\gamma(n)}$, whence $a^n \equiv 1 \pmod{b}$, which is a contradiction. Thus the proof of (2) is complete.

Immediately from (1) and (2) we have the following results.

(3) $m \mid \gamma(n)$ if and only if $\lambda(m) \mid n$.

(4) $L(m)$ may also be characterized as the set of all integers k for which $m \mid \gamma(k)$.

(5) $C(n)$ may also be characterized as the set of all integers h for which $\lambda(h) \mid n$.

And from (3), first with $m = \gamma(s)$ and $n = s$, then with $m = s$ and $n = \lambda(s)$, we have:

(6) For each integer s, $\lambda(\gamma(s)) \mid s$ and $s \mid \gamma(\lambda(s))$.

From (3) and the above-stated relation between λ and ϕ we see:

(7) If n is odd, $\gamma(n) = 2$; if n is even, $\lambda(n)$ is twice the least common multiple

$M(n)$ of all prime powers p^e for which $p^{e-1}(p-1)|n$. Hence if n is even, $24|n$.

The function M was introduced by Carmichael in [2], where there is a table of values of $M(n)$ for $n \leqq 150$ and such that the equation $\phi(n)=n$ has a solution. To complete this table up to $n=150$, we record the following values of $M(n)$: $M(n)=12$ if n is 14, 26, 34, 38, 62, 74, 86, 94, 98, 118, 122, 134, 142, or 146; $M(n)=120$ if n is 68, 76, or 124; $M(50)=132$; $M(114)=252$; $M(90)=4898124$.

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ON THE LEAST POSSIBLE ODD PERFECT NUMBER

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In 1908 A. Turčaninov proved that no odd number less than two million can be perfect [3c], a figure which is generally accepted as the minimum in standard texts [for example, see 1, 4]. However, it is easy to show by means of well-known proofs that the least possible odd perfect number is greater than ten billion.

Let the prime factorization of any odd perfect number N_0 be denoted by

$$N_0 = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m} q_1^{b_1} q_2^{b_2} \cdots q_n^{b_n}, \tag{1}$$

where $p_1, p_2, \cdots, p_m, q_1, q_2, \cdots, q_n$ are odd primes, $a_1, a_2, \cdots, a_m$ are odd integers, and $b_1, b_2, \cdots, b_n$ are even integers. It has been known for centuries [3a] that

$$m = 1, \tag{1a}$$

$$p_1 \equiv 1 \pmod 4, \tag{1b}$$

$$a_1 \equiv 1 \pmod 4. \tag{1c}$$

In addition, Sylvester [3b] has demonstrated that

$$n \geqq 4. \tag{1d}$$

Certain other conditions have been established. Steuerwald has shown

$$b_1 = b_2 = \cdots = b_n = 2 \text{ is impossible.} \tag{1e}$$

More recently, Brauer [2] extended this theme to prove that

$$b_i \neq 4, \text{ when } b_1 = b_2 = \cdots = b_{i-1} = b_{i+1} = \cdots = b_n = 2. \tag{1f}$$

Another useful fact has been demonstrated by Sylvester [5], namely, that

$$N_0 \not\equiv 0 \pmod{3 \cdot 5 \cdot 7}. \tag{1g}$$

The only numbers less than ten billion which satisfy conditions (1*a*), (1*b*), $\cdots$, (1*g*) are $3^6\cdot 5^2\cdot 11^2\cdot 13^2\cdot 17$ and $3^6\cdot 5^2\cdot 11^2\cdot 13\cdot 17^2$. Each is found to be abundant by directly computing the sum of its divisors.

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CLASSROOM NOTES

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COORDINATE SYSTEMS PROJECTED ON BLACKBOARDS

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It was suggested by R. M. Sutton (this MONTHLY, vol. 54, p. 276) that effective teaching use could be made of coordinate systems projected on classroom blackboards from slides or film strips. Although it may seem unlikely at first thought, the projection of white coordinate grids with black backgrounds onto blackboards gives satisfactory results even in undarkened rooms. The use of this method enables the teacher to undertake accurate plotting on a great variety of different types of coordinate systems.

For most teachers the practical difficulty of using this technique lies in the preparation of suitable slides or film strips. Through the courtesy and cooperation of a local concern a variety of such slides was prepared for use at Haverford College. In the light of our experience with them, a film strip of 36 frames was prepared and copies can be obtained from the manufacturer.

This film strip contains several varieties of rectangular cross sections, polar coordinates, logarithmic and semi-logarithmic systems, and probability and log-probability grids. The abscissa of one of the rectangular grids is calibrated in terms of fractions and multiples of π for use in plotting the graphs of trigonometric

functions. Other frames contain a single straight line, parallel straight lines, and a circle. These are projected on string models of cones and other ruled surfaces to illustrate conic sections and simple space curves.

Although this film is still regarded as experimental, limited commercial production is available at reasonable cost. Inquiries should be addressed to: Engineers Publishing Company, 401 N. Broad Street, Philadelphia 8, Pa.

SUMS OF SINES CONVERTED INTO NUMERICAL SUMS

H. H. Downing, University of Kentucky

Interesting results are obtained from equations involving sums, both finite and infinite, of sines of multiples of x. Both members of the equation are divided by x, x is allowed to approach zero, and limits are then found.

For the first illustration let us take the trigonometric formula (Rothrock, *Elements of Plane and Spherical Trigonometry*, 1914, Example 3, page 109).

$$\sin x + \sin 2x + \cdots + \sin nx = \sin\frac{(n+1)x}{2}\cdot\sin\frac{nx}{2}\Big/\sin\frac{x}{2}\cdot \tag{1}$$

Divide both members by x and write as follows:

$$\frac{\sin x}{x} + 2\,\frac{\sin 2x}{2x} + \cdots + n\,\frac{\sin nx}{nx}$$

$$= \frac{n+1}{2}\cdot\frac{\sin\dfrac{(n+1)x}{2}}{\dfrac{n+1}{2}\,x}\cdot\frac{n}{2}\cdot\frac{\sin\dfrac{nx}{2}}{\dfrac{n}{2}\,x}\Bigg/\frac{1}{2}\cdot\frac{\sin\dfrac{x}{2}}{\dfrac{x}{2}}\cdot$$

Now allow x to approach zero. The resulting limit is

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2},$$

the formula for the sum of the first n consecutive integers.

As a second example consider the trigonometric formula (Rothrock, loc cit., Example 5, page 109).

$$\sin x + \sin 3x + \cdots + \sin (2n-1)x = \sin^2 nx/\sin x. \tag{2}$$

Divide by x and write in the form

$$\frac{\sin x}{x} + 3\,\frac{\sin 3x}{3x} + \cdots + (2n-1)\,\frac{\sin (2n-1)x}{(2n-1)x} = n^2\,\frac{\sin^2 nx}{n^2x^2}\Big/\frac{\sin x}{x}\cdot$$

Pass to limits and find

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2,$$

the sum of the first n odd integers.

Next, consider the Fourier expansion (Widder, *Advanced Calculus*, 1947, Example C, page 327)

$$x = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots + (-1)^{n-1}\frac{\sin n\pi}{n} + \cdots\right), \qquad -\pi \leqq x \leqq \pi. \tag{3}$$

Divide by x and obtain

$$1 = 2\left(\frac{\sin x}{x} - \frac{\sin 2x}{2x} + \frac{\sin 3x}{3x} - \cdots + (-1)^{n-1}\frac{\sin nx}{nx} + \cdots\right).$$

Let x approach zero and divide by 2. We obtain

$$1/2 = 1 - 1 + 1 - 1 + \cdots .$$

This result is found by the Cesàro summability method (Widder, loc. cit., Example A, page 263).

The next example is interesting in that it relates the preceding oscillating series and an infinite series giving $\pi/4$. Consider the Fourier expansion (Miller, *Partial Differential Equations*, 1941, Exercise 8, page 150)

$$\begin{aligned} x = {} & \left(1 + \frac{2}{\pi}\right)\sin x - \frac{1}{2}\sin 2x + \left(\frac{1}{3} - \frac{2}{3^2\pi}\right)\sin 3x - \frac{1}{4}\sin 4x \\ & + \left(\frac{1}{5} + \frac{2}{5^2\pi}\right)\sin 5x - \cdots, \qquad 0 < x \leqq \pi/2. \end{aligned} \tag{4}$$

Divide by x, pass to limits, and obtain

$$1 = \left(1 + \frac{2}{\pi}\right) - 1 + \left(1 - \frac{2}{3\pi}\right) - 1 + \left(1 + \frac{2}{5\pi}\right) + \cdots,$$

or, on rearranging,

$$1 = (1 - 1 + 1 - 1 + \cdots) + \frac{2}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right). \tag{5}$$

The series in the second parenthesis is the Leibnitz or Gregory series for $\pi/4$. Thus, if either the first or the second parenthesis is taken as known, the other can be determined.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 886. *Proposed by P. L. Chessin and R. Gilman, New York City*

Let M be a fixed point on the radius OR of a circle of center O and set $OR/OM=k$. If AB is a variable chord such that $\measuredangle AMR=\measuredangle OMB$, then (1) AB passes through a fixed point on line OR, (2) $AB=k(MB-MA)$, (3) A, B, O, M are concyclic.

E 887. *Proposed by C. W. Trigg, Los Angeles City College*

Let N be an n-digit integer and M the kn-digit integer formed by writing N down k times. Show that there is no scale of notation in which $N^k=M$ if $k>1$.

E 888. *Proposed by H. D. Grossman, New York City*

Show how to cut a hole in a cube through which another cube of equal size can pass.

E 889. *Proposed by Michael Golomb, Purdue University*

Let k be a circle with its interior, F_1F_2 one of its diameters, P a point on the axis of the circle, F_1PF_2 a circular arc, and A_1PA_2 a semiellipse with foci at F_1, F_2. Show that the solid angles subtended by k at the points of arc F_1PF_2 are no smaller, and at the points of arc A_1PA_2 are no greater, than the solid angle subtended by k at P.

E 890. *Proposed by Leo Moser, University of Manitoba*

Let $a_1, \cdots, a_n$ be n integers with $a_1=1$ and $a_i \leqq a_{i+1} \leqq 2a_i$, $i=1, \cdots, n-1$. Show that there exists a sequence $\{\epsilon_i\}$ of plus and minus ones such that $\sum_{i=1}^{n}\epsilon_i a_i=0$ or 1.

SOLUTIONS

Pythagorean Triangles with Equal Perimeters

E 812 [1948, 248 and 1949, 32]. *Proposed by Monte Dernham, San Francisco, California*

Find the shortest perimeter common to two different primitive Pythagorean triangles.

Comment by C. W. Trigg, Los Angeles City College. There are shorter perimeters common to two and to three non-primitive Pythagorean triangles than those reported by Ogilvy and Buker in this MONTHLY [1949, 33]. The shortest perim-

eters common to x non-primitive Pythagorean triangles, together with the corresponding sets of triangles, are given below.

$x=2$. $p=60$; (15, 20, 25), (10, 24, 26).
$x=3$. $p=120$; (30, 40, 50), (20, 48, 52), (45, 24, 51).
$x=4$. $p=360$; (90, 120, 150), (60, 144, 156), (135, 72, 153), (36, 160, 164).
$x=5$. $p=660$; (165, 220, 275), (110, 264, 286), (55, 300, 305), (297, 60, 303), (210, 176, 274).
$x=6$. $p=720$; (180, 240, 300), (270, 144, 306), (315, 80, 325), (45, 336, 339), (72, 320, 328), (120, 288, 312).

Some sets of triangles, containing one primitive member, have a common perimeter shorter than those listed above. That is,

$x=4$. $p=240$; (60, 80, 100), (40, 96, 104), (90, 48, 102), (15, 112, 113).
$x=5$. $p=420$; (105, 140, 175), (70, 168, 182), (175, 60, 185), (126, 120, 174), (195, 28, 197).
$x=7$. $p=1320$; (330, 440, 550), (220, 528, 572), (495, 264, 561), (110, 600, 610), (594, 120, 606), (430, 352, 548), (231, 520, 569).
$x=8$. $p=840$; (210, 280, 350), (140, 336, 364), (315, 168, 357), (105, 360, 375), (252, 240, 348), (350, 120, 370), (390, 56, 394), (399, 40, 401).
$x=10$. $p=1680$; (420, 560, 700), (280, 672, 728), (630, 336, 714), (210, 720, 750), (504, 480, 696), (700, 240, 740), (105, 784, 791), (780, 112, 788), (798, 80, 802), (455, 528, 697).

In addition, we find that the pair of Pythagorean triangles having the smallest common perimeter and a common side is (21, 28, 35) and (35, 12, 37), $p=84$.

A Series for π

E 854 [1949, 104]. *Proposed by Jerome C. R. Li, Oregon State College*

Show that $\pi=\sum_{n=0}^{\infty}(n!)^2 2^{n+1}/(2n+1)!$.

I. *Solution by Ragnar Dybvik, Levanger, Norway.* (Using the beta function.) We have

$$\begin{aligned}\sum_{n=0}^{\infty}(n!)^2 2^{n+1}/(2n+1)! &= \sum_{n=0}^{\infty}2^{n+1}\Gamma(n+1)\Gamma(n+1)/\Gamma(2n+2)\\ &= \sum_{n=0}^{\infty}2^{n+1}B(n+1,n+1) = \sum_{n=0}^{\infty}2^{n+1}\int_0^1 x^n(1-x)^n dx\\ &= 2\int_0^1\sum_{n=0}^{\infty}(2x)^n(1-x)^n dx = 2\int_0^1[1-2x(1-x)]^{-1}dx\\ &= \int_0^1[(x-1/2)^2+(1/2)^2]^{-1}dx = 2\arctan 2(x-1/2)\Big|_0^1 = \pi.\end{aligned}$$

desired result.

V. *Solution by D. H. Browne, Buffalo, N. Y.* (Using Euler's transformation of series.) If we apply Euler's transformation

$$\sum_{n=0}^{\infty}(-1)^n v_n = \sum_{n=0}^{\infty}(\Delta^n v_0)/2^{n+1}$$

to the Leibnitz series

$$\pi/4 = \sum_{n=0}^{\infty}(-1)^n/(2n+1)$$

we find

$$\Delta^n v_0 = (n!)^2 2^{2n}/(2n+1)!,$$

whence

$$\pi = \sum_{n=0}^{\infty}(n!)^2 2^{n+1}/(2n+1)!.$$

Also solved by Louis Berkofsky, W. G. Brady, Roger Lessard, and E. D. Rainville.

Solution V may be found in Bromwich, *Theory of Infinite Series*, 2nd ed., p. 62; Knopp, *Theory and Application of Infinite Series*, p. 244 and ex. 2, p. 246; Chrystal, *Text Book of Algebra*, part II, pp. 408–409.

Four Cospherical Circles

E 855 [1949, 104]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Planes through the orthocenter of an orthocentric tetrahedron perpendicular to four concurrent cevians cut the spheres described on these cevians in four cospherical circles.

Solution by the Proposer. Let $ABCD$ be the tetrahedron, AP_a, AP_b, AP_c, AP_d the four cevians concurrent in P, and H the orthocenter. Let us designate the four circles by (C_a), (C_b), (C_c), (C_d), and let A', B', C', D' be the feet of the altitudes of the tetrahedron. The sphere on AP_a as diameter passes through A'. Therefore the power of H with respect to circle (C_a) is given by $(HA)(HA')$. Now

$$\begin{aligned}(HA)(HA') &= (HB)(HB') = (HC)(HC') = (HD)(HD') \\ &= (HO^2 - R^2)/3 = \rho^2,\end{aligned} \tag{1}$$

where O is the circumcenter, and R and ρ are the radii of the circumsphere and the conjugate sphere of $ABCD$. Therefore H has the same power with respect to each of the four circles (C_a), (C_b), (C_c), (C_d). Since the normals to the planes of these circles at their centers are concurrent at P, this is the center of a sphere

(S_1) passing through the four circles. The square of the radius of this sphere is, then,

$$(2) \qquad R_1^2 = HP^2 - \rho^2 = HP^2 - R^2 + (a^2 + a'^2)/4,$$

where a and a' are any pair of opposite edges of $ABCD$.

Editorial Note. For relations (1) and (2) see N. A. Court, *Modern Pure Solid Geometry*, sections 218, 795, 823, 825.

The analogous theorem in the plane is also true, and reads as follows: Lines through the orthocenter of a triangle perpendicular to three concurrent cevians cut the circles described on these cevians as diameters in six concyclic points.

A Cyclic Number

E 856 [1949, 179]. *Proposed by J. T. Hurt, Texas Agricultural and Mechanical College*

Let N be an integer of p digits. If the last digit is removed and placed before the remaining $p-1$ digits, a new number of p digits is formed which is $(1/n)$th of the original number. Find the most general such number N.

Solution by N. D. Lane, St. Andrew's College, Ontario. Let

$$N = a_{p-1}10^{p-1} + \cdots + a_1 10 + a_0, \qquad a_{p-1} \neq 0,$$

and

$$M = a_0 10^{p-1} + a_{p-1}10^{p-2} + \cdots + a_1, \qquad a_0 \neq 0,$$

such that $M = N/n$. Since $10M = N + a_0(10^p - 1)$, we find

$$N = n(10^p - 1)a_0/(10 - n).$$

We see that $n < 10$, and since $10^p - 1$ is the largest p-digit number, $na_0/(10-n) \leqq 1$, or $n(a_0+1) \leqq 10$, and $n \leqq 5$. The requirement that M be integral eliminates $n = 5, 4, 2$. If $n = 1$ we get nine solutions for each value of p. If $n = 3$, a_0 must be 1 or 2, and there exist solutions to the problem if p is a multiple of 6. In this case

$$N = 3(10^{6k} - 1)/7 \quad \text{or} \quad N = 6(10^{6k} - 1)/7,$$

the two smallest such numbers being 428571 and 857142.

Also solved by Monte Dernham, B. B. Dressler, R. T. Hood, M. S. Klamkin, Roger Lessard, Azriel Rosenfeld, C. M. Sandwick, and the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4365. *Proposed by Paul Erdös, Syracuse University*

Let $a_1 < a_2 < \cdots < a_k \leqq n$ be such that the least common multiple of any two a's exceeds n. Prove that

$$\sum_{i=1}^{k} \frac{1}{a_i} < 2.$$

4366. *Proposed by J. Rosenbaum, the Milford School, Connecticut*

Determine the condition which two concentric spheres must satisfy in order that a tetrahedron can be simultaneously inscribed in one and circumscribed about the other. Give a construction for the tetrahedron.

4367. *Proposed by F. E. Wood, University of Oregon*

Evaluate the determinant

$$\begin{vmatrix} {}_nP_n & {}_nC_1 & {}_nC_2 \cdots & {}_nC_n \\ {}_{n-1}P_{n-1} & {}_{n-1}C_0 & {}_{n-1}C_1 \cdots & {}_{n-1}C_{n-1} \\ {}_{n-2}P_{n-2} & 0 & {}_{n-2}C_0 \cdots & {}_{n-2}C_{n-2} \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ {}_0P_0 & 0 & \cdots\ 0 & {}_0C_0 \end{vmatrix}$$

of the $(n+1)$th order where ${}_nP_r$ and ${}_nC_r$ are the usual permutation and combination symbols.

4368. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a tetrahedron $T \equiv ABCD$, the planes tangent at A, B, C, D to the circumsphere of T cut the planes of the opposite faces in four lines. A necessary and sufficient condition for these four lines to be rulings of a hyperbolic paraboloid is that T be orthocentric, and a necessary and sufficient condition for these four lines to be coplanar is that T be isodynamic.

4369. *Proposed by Orrin Frink, Pennsylvania State College*

Show that in every neighborhood of a point on a surface $z = f(x, y)$, with

$f(x, y)$ a continuous function, there exist four points on the surface which are the corners of a square.

SOLUTIONS

Metaharmonic Tetrahedrons

4228 [1946, 594]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a tetrahedron $ABCD$ the straight lines joining each vertex to the points of intersection V and W of the six spheres of similitude of four given spheres with centers A, B, C, D, meet the circumsphere in the vertices of two equal tetrahedrons $A'B'C'D'$ and $A''B''C''D''$.

*Solution by the Proposer.** In a tetrahedron $T \equiv ABCD$, whose circumsphere is (O, R), we designate the lengths of the edges BC, DA, CA, DB, AB, DC by a, a', b, b', c, c'. Let V and W be the points common to the six spheres of similitude of the four given spheres (A, l), (B, m), (C, n), (D, p). Then the specified points A', B', C', D' and A'', B'', C'', D'' are transforms of the vertices A, B, C, D of T by inversions of poles V and W and of powers

$$\lambda^2 = \overline{VO}^2 - R^2 \quad \text{and} \quad \mu^2 = \overline{WO}^2 - R^2.$$

The theory of inversion shows that

$$B'C' = \lambda^2 a/VB \cdot VC, \qquad D'A' = \lambda^2 a'/VD.VA, \cdots,$$
$$B''C'' = \mu^2 a/WB \cdot WC, \qquad D''A'' = \mu^2 a'/WD \cdot WA, \cdots.$$

Hence

$$B'C'/D'A' = a \cdot VD \cdot VA/a' \cdot VB \cdot VC = a \cdot WD \cdot WA/a' \cdot WB \cdot WC, \cdots.$$

But $VA : VB : VC : VD = l : m : n : p = WA : WB : WC : WD$, so that

$$B'C'/D'A' = alp/a'mn = B''C''/D''A'',$$
$$C'A'/D'B' = bmp/b'ln = C''A''/D''B'',$$
$$A'B'/D'C' = cnp/c'lm = A''B''/D''C''.$$

The metaharmonic tetrahedrons $A'B'C'D'$ and $A''B''C''D''$ of T, associated with the points V and W, are similar since corresponding edges are proportional, As they are inscribed in the same sphere (O, R) they are equal.

Note. If the squares l^2, m^2, n^2, p^2 of the radii of the spheres of centers A, B, C, D are proportional to the products $a'bc$, $ab'c$, abc', $a'b'c'$ of the edges of T, we have shown that the corresponding metaharmonic tetrahedrons are isosceles. (*Annales de la Société Scientifique de Bruxelles*, 1947, p. 15.)

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* Translated by W. E. Byrne, Virginia Military Institute.

Lower Bound to a Trigonometric Sum

4290 [1948, 253]. *Proposed by P. T. Bateman, Yale University*

The function $-\log|2 \sin \frac{1}{2}x|$ has the Fourier series

$$\cos x + \tfrac{1}{2} \cos 2x + \tfrac{1}{3} \cos 3x + \cdots .$$

Prove that no partial sum of the series is ever less than -1.

Solution by Otto Szasz, University of Cincinniat. Let

$$S_n(x) = \sum_{v=1}^{n} \frac{\cos vx}{v} \geqq -1, \qquad n = 1, 2, \cdots ; 0 \leqq x \leqq \pi.$$

We have

$$-S_n'(x) = \sum_{v=1}^{n} \sin vx = \left\{\cos \tfrac{1}{2}x - \cos (n + \tfrac{1}{2})x\right\}/2 \sin \tfrac{1}{2}x.$$

Hence

$$-S_{n-1}'(x) - S_n'(x) = (1 - \cos nx)/\tan \tfrac{1}{2}x \geqq 0, \qquad n \geqq 2; 0 \leqq x \leqq \pi.$$

It follows that $-S_{n-1}(x)-S_n(x)$ increases as x increases to π, whence

$$-S_{n-1}(x) - S_n(x) \leqq -S_{n-1}(\pi) - S_n(\pi) = 2\sum_{v=1}^{n-1} (-1)^{v-1}/v + (-1)^{n-1}/n.$$

Let $n=2k$, $k\geqq 1$. Then

$$\begin{aligned} -S_{2k-1}(x) - S_{2k}(x) &= -2S_{2k}(x) + \frac{\cos 2kx}{2k} = -2S_{2k-1}(x) - \frac{\cos 2kx}{2k} \\ &\leqq 2\sum_{v=1}^{2k-1} (-1)^{v-1}/v - 1/2k = 2t_k - 1/2k, \quad \text{say:} \end{aligned}$$

hence

$$-2S_{2k}(x) \leqq 2t_k - (1 + \cos 2kx)/2k \leqq 2t_k,$$

and

$$-2S_{2k-1}(x) \leqq 2t_k - (1 - \cos 2kx)/2k \leqq 2t_k.$$

Obviously $t_1=1$, and $t_k<1$ for $k>1$; it follows that

$$S_n(x) \geqq -t_k \geqq -1, \qquad n \geqq 1.$$

It is easily seen that equality here holds only for $n=1$ and $x=\pi$. Furthermore t_k decreases to $\log 2$, as $n\to\infty$, and

$$\liminf_{n\to\infty} \min_{0\leqq x\leqq\pi} S_n(x) = -\log 2.$$

Also solved by R. P. Boas, Jr., Fritz Herzog, E. Lukacs, C. D. Olds, Norman Miller, and the Proposer.

Editorial Note. The present result is due to W. H. Young (*Proceedings of the London Mathematical Society*; v. 11, 1913, p. 359) and has been here reproposed because of an error in Young's proof. The problem is also found in Polya-Szegö, *Aufgaben und Lehrsätze aus der Analysis*, v. 2, p. 79, problem 28, the solution being given on p. 271. In this solution Young's error is corrected.

An Enumeration Problem

4291 [1948, 253]. *Proposed by J. P. Ballantine, University of Washington, Seattle*

$$\begin{array}{cccccc} 1 & 2 & 3 & \cdots & & n \\ 0 & 1 & 2 & \cdots & & n-1 \\ 0 & 0 & 1 & \cdots & & n-2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & & 1 \end{array}$$

Consider the square array with 1's down the principal diagonal, increasing consecutive integers above the diagonal, and zeros below. (a) How many different paths of $i+j-2$ steps are possible from the 1 in the upper left corner to the element in the ith row and jth column, $j \geqq i$, without passing through any zero element?

(b) Define the value of each path as the product of the elements through which the path passes, not counting the terminal elements. Show that the sum of the values of all the paths in (a) is the coefficient of $(\tan x)^{j-i+1}$ in the $(j+i-2)$nd derivative of $\tan x$ with respect to x, expressed in terms of $\tan x$.

Solution by Fritz Herzog, Michigan State College. Let $i \leqq j$ and denote by (i, j) the element $j-i+1$ in the ith row and jth column of the given array. We call a path from $(1, 1)$ to (i, j) a forbidden path if it passes through at least one zero element, otherwise it is a proper path. Let the number of proper paths required in (a) be represented by $N(i, j)$. Evidently $N(i, j) = N(i, j-1) + N(i-1, j)$, a relation which leads by an easy induction to the result below. The following alternate derivation has some interest.

Let P be any forbidden path and let $(k, k-1)$ be the first zero element reached by P. $(2 \leqq k \leqq i.)$ Let $P = P_1 + P_2$, where P_1 is the part of P from $(1, 1)$ to $(k, k-1)$ and P_2 the part of P from $(k, k-1)$ to (i, j). Let Q_1 be the reflection of P_1 about the principal diagonal, so that Q_1 is a path from $(1, 1)$ to $(k-1, k)$, and let Q_2 be obtained by translating P_2 so as to begin at $(k-1, k)$ and thus end at $(i-1, j+1)$. $Q = Q_1 + Q_2$ is then a path (not necessarily a proper path) from $(1, 1)$ to $(i-1, j+1)$. If P' and P'' are two different forbidden paths from $(1, 1)$ to (i, j) then the first unit segment of P' (P'') which is not common to P' and P'' will belong either to P_1' (P_1'') or to P_2' (P_2''). This shows that the corresponding paths

Q' and Q'' will also be different from one another. Finally, every path Q from $(1, 1)$ to $(i-1, j+1)$ is actually obtained from a forbidden path P; in order to construct this path P, we need only define $(k-1, k)$ as the first element to the right of the principal diagonal reached by Q and then reverse the process described above. Thus a one-to-one correspondence has been established between the forbidden paths from $(1, 1)$ to (i, j) and all paths from $(1, 1)$ to $(i-1, j+1)$. The number of the latter is known to equal the binomial coefficient ${}_{j+i-2}C_{i-2}$, while the number of all paths from $(1, 1)$ to (i, j) equals ${}_{j+i-2}C_{i-1}$. The answer to (a) is therefore

$$N(i, j) = {}_{j+i-2}C_{i-1} - {}_{j+i-2}C_{i-2} = \frac{(j - i + 1)(j + i - 2)!}{(i - 1)!j!}.$$

Let the sum of all the values of all the paths (not necessarily proper) from $(1, 1)$ to (i, j) be denoted by $S(i, j)$. Any path ending at (i, j), $i>1, j>1$, must pass through either $(i-1, j)$ or $(i, j-1)$. Consequently, we have

$$S(i, j) = (j - i + 2)S(i - 1, j) + (j - i)S(i, j - 1), \tag{1}$$

and this relation will hold for $1 \leqq i \leqq j+1$, $(i, j) \neq (1, 1)$, if we define $S(0, r) = S(r, r-2) = 0$ for $r \geqq 2$ and $S(1, 1) = 1$. (As a consequence $S(1, 2) = S(2, 1) = 1$. These values, although not defined by the proposal, are acceptable in the same sense as $0! = 1$.)

We shall show that

$$D_x^n(\tan x) = \sum_{k=0}^{[(n+1)/2]} S(k + 1, n - k + 1)(\tan x)^{n-2k+1}, \quad n = 0, 1, 2, \cdots. \tag{2}$$

Because of $S(1, 1) = 1$, (2) is true for $n = 0$. Assume the equation

$$D_x^{n-1}(\tan x) = \sum_{k=0}^{[n/2]} S(k + 1, n - k)(\tan x)^{n-2k}.$$

has been proved for a positive integer n. Differentiating and making use of $S(0, r) = S(r, r-2) = 0$, $r \geqq 2$, we obtain

$$\begin{aligned} D_x^n(\tan x) &= \sum_{k=0}^{[(n-1)/2]} (n - 2k)S(k + 1, n - k)(\tan x)^{n-2k-1}(1 + \tan^2 x) \\ &= \sum_{k=0}^{[(n+1)/2]} \{(n - 2k + 2)S(k, n - k + 1) \\ &\qquad + (n - 2k)S(k + 1, n - k)\}(\tan x)^{n-2k+1}. \end{aligned}$$

Applying (1) to the expression in braces, we obtain (2), which is thus established by mathematical induction.

Substituting $k+1 = i$, $n-k+1 = j$ in (2), we conclude that $S(i, j)$ is the coefficient of $(\tan x)^{j-i+1}$ in the $(j+i-2)$nd derivative of $\tan x$.

Also solved by Y. S. Luan, and the Proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

Number Theory and Its History. By Oystein Ore. New York, McGraw-Hill Book Co., Inc., 1948. x+370 pages. $4.50.

This book gives an interesting account of many topics of elementary number theory, interwoven with considerable historical material. The theory is available to readers with a limited mathematical knowledge, since the mathematical requirements are reduced to a minimum, and only "topics of systematic and historical importance capable of a simple presentation" are chosen. The author has succeeded in making the book attractive to amateurs, to people interested in number theory as a pastime; and it is well suited as a text for a first course in number theory especially for students who may not wish to study the subject further.

For students looking to further use of the subject, the desirability of a mixed mathematical-historical development may well be debated. While the present treatment is interesting, and hence probably inspiring, it has the disadvantage that the most significant ideas and methods do not stand out sufficiently. A teacher well acquainted with the subject could make up for this deficiency.

The historical material seems to be largely up to date. The history of number symbols associated with various cultures is given at length in Chapter 1, with plates of finger symbols, tallies, Egyptian, Greek, Chinese, and other numerals. Reproductions are given in Chapter 6 of part of the Papyrus Rhind, and in Chapter 8 of two Babylonian tablets, giving tables of reciprocals and of right triangles with integral sides. Early mathematical writers are frequently quoted; problems proposed by them are given, and their methods of solution are contrasted with modern methods. Most of the mathematical sections are accompanied by remarks on recent developments on those topics.

Among the mathematical sections may be listed divisibility of numbers (including binary number systems), Euclid's algorithm, prime numbers (including a discussion of their distribution), the simplest arithmetical functions, an extensive discussion of linear indeterminate equations, Diophantine problems, congruences, Wilson's and Euler's theorems and their consequences, theory of decimal expansions, classical construction problems. It is regrettable that no mention is made of quadratic residues (or hence of Gauss's "gem of the higher arithmetic," the quadratic reciprocity law). Among topics not usually found in textbooks there may be noted Thue's theorem that if a is prime to p, $ax \equiv \pm y \pmod{p}$ is solvable with x and y between 1 and $\sqrt{p}$, this being used to prove that a prime p of the form $4n+1$ is a sum of two squares. The function $\lambda(n)$ which is

the least exponent such that $a^{\lambda(n)} \equiv 1 \pmod{n}$ for all a prime to n, is studied and applied to a problem in telephone splicing. The author takes the opportunity afforded by the properties of greatest common divisors, least common multiples, and of congruences, to discuss concepts of importance in mathematics generally, such as lattices, rings, and moduls.

Numerous numerical examples and historical remarks make the progress of the mathematics somewhat slow, but this may be an advantage for some readers.

The following corrections should be made. It would be desirable, on page 29, to define "a is a divisor of c" even when $c=0$, since such cases are used later. On pages 30–32, b is treated several times as though it were positive, although it was introduced as merely not zero. The greatest common divisor (p. 40) should be defined for integers not all zero. The word "number" is overworked; perhaps, integer, positive integer, nonzero integer, and so on, might be clearer in several places. Near the bottom of page 171, read inverses for inversions. On page 168, (8–5) is the general primitive solution of $a^2+b^2=c^2$ only with a even; and (8–6) is not the general solution in rationals (as stated), since in its derivation it was assumed that $v\neq 0$. Indeed, the solution $a=0$, $b=1=c$ cannot be obtained with r and t rational.

GORDON PALL

A Concise History of Mathematics. Two volumes. (Dover Series in Mathematics and Physics). By D. J. Struik. New York, Dover Publications, 1948. 18+299 pages. $1.50 per volume.

To write on the history of mathematics has been, from old times to our days, a very attractive task. To write a history of mathematics is a tremendous undertaking because it is identical with a presentation of mathematics itself immersed in its human surroundings. As an approximate completeness is here even more unthinkable than in other historic fields, it is laudable to write a concise history as this is presented.

This work has the great advantage of being written by an accomplished mathematician as well as a reliable historian. Of course, the choice of what is described and what is omitted depends on individual taste. In the beginning we find, caused by recent discoveries about pre-Greek mathematics, a relatively detailed report on the beginnings. The report on the nineteenth century, on the other hand, is obviously influenced by F. Klein's inspiring work on the development of mathematics during this period.

The reviewer regrets that the author does not mention the beginning of Axiomatics and does not give the name of the basic work of the "Father of Axiomatics," M. Pasch. This trend in the last quarter of the nineteenth century is important because of its continuation in the present century, particularly in this country.

There are, of course, many such points, more or less important, with which some people might disagree. However, this cannot change the impression that it

is a very valuable book, especially for three reasons. First, the reader will find many interesting details, difficult to find in other, much more voluminous treatises. Second, the author gives ample text references. Also, there are many portraits, drawings and text reproductions. It is only regrettable that, apparently, it was not possible to print the reproductions on a more adequate paper.

M. DEHN

A Philosophy of Mathematics. By L. O. Kattsoff. Ames, Iowa, Iowa State College Press, 1948. 9+266 pages. $5.00.

The aim of this book is to "give the beginning student an introduction to the many problems raised by the queen of the sciences." The author considers that existing treatments of the subject are either too difficult or else not sufficiently comprehensive for this purpose. The book is intended for students whose knowledge of mathematics and philosophy is that of an undergraduate senior. The author is a philosopher, and the work has naturally a primarily philosophical cast.

The book begins with a discussion of the definition of mathematics and its objects. It then considers the notion of natural number, with emphasis on the various ways of introducing it. The next topic is the extensions of the number system to rational, real, and complex numbers. This leads to an elementary discussion of the theory of abstract sets and transfinite numbers. An exposition of the paradoxes then follows. The next four chapters are devoted to the various methods of founding a system of mathematics, viz.: the logistic, formalistic, elementalistic (*i.e.* the theory of Church), and the intuitionistic foundations. Then there is a chapter on the Gödel theorem and its significance; and one on the "signific" standpoint of Mannoury. Finally there are three chapters devoted to an analysis of mathematical symbolism, of the methods of mathematical proof, of various sorts of definitions, of postulational procedures, and of the relations of mathematics to reality.

The treatment is rather sweeping in scope. Among the topics fitted into the above outline are the following: the notion of magnitude according to Bolzano; the propositional algebra and predicate calculus of the *Principia Mathematica*, as well as the definitions relating to cardinal number; Frege's symbolism; the "empirico-postulational" definition of Pasch; an axiom set for the natural numbers due to Neder, but not the original system of Peano; the denumerability of certain aggregates and the diagonal process; the theory of types and the axiom of reducibility; the theories of Chwistek; Heyting's formalization of intuitionistic mathematics, and the main ideas of intuitionism; and a brief treatment of modal logic. Several of these topics are somewhat unusual for a book of this kind. Moreover the book goes into great detail on certain topics, such as the Church theory of metads, which are hardly suited to its purpose, both on the ground that they are highly difficult and technical and because their importance is somewhat controversial. On the other hand there are, as is naturally to be expected, several topics which the reviewer thinks are slighted. Among those not men-

tioned at all, which nevertheless seem of major importance for the purpose of such a book, are the theory of recursive functions, the rule-theoretic methods of Gentzen, the stratification methods of Quine, and the semiotical methods of Carnap.

The reviewer finds the style of the book exceedingly obscure. This obscurity is coupled with carelessness in regard to matters of detail. The book contains a number of sloppy mistakes of which the definition of the sum of two cardinals on page 84 is typical: the author states that if m is the cardinal number of a class M, and n is the cardinal number of N, then $m+n$ is the cardinal number of $M+N$, without the obvious restriction that M and N be mutually exclusive. The reader who wishes to see the extent and gravity of these errors should read the able review by Canon Feys in the *Journal of Symbolic Logic*, vol. 13, pp. 208–212 (1948). Certainly the book is not to be recommended to any mathematician, young or old, who is seeking information on the foundations of mathematics, and is not in a position to have a critical judgment of his own.

H. B. CURRY

Modern Operational Calculus. By N. W. McLachlan. Cambridge, at the University Press; New York, The Macmillan Company, 1948. 12+218 pages, 29 illustrations. $5.00.

This book presents a concise treatment of the Laplace transform method of solution of ordinary and partial differential equations. It is illustrated by typical examples taken from electric circuits, electric transmission lines, heat flow, acoustics, and so on. The book is intended primarily for postgraduate engineers. Laplace transforms are defined in Chapter 1 and special rules for their formation are derived in Chapter 2. Chapter 3 explains the use of Laplace transforms in solving ordinary differential equations, and Chapter 4 deals with partial differential equations. These four chapters account for one-half of the book; the rest is devoted to evaluation of integrals, derivation of Laplace transforms, and appendices containing various definitions and theorems pertaining to infinite series and infinite integrals. At the end of the book there is a large collection of problems and a short table of Laplace transforms.

The author has included a great deal of useful information in this small volume and many engineers will find it a convenient reference book.

It should be mentioned however that the author uses the "p-multiplied transform" defined by

$$\phi(p) = p \int_0^\infty e^{-pt} f(t) dt$$

instead of the more conventional transform

$$F(p) = \int_0^\infty e^{-pt} f(t) dt.$$

While it is easy to change from one type of transform to the other, the difference must constantly be kept in mind when referring to other books and tables of transforms. The author's reasons for the change of definition are two: (1) to make Laplace transforms identical with Heaviside's operational forms, and (2) to obtain the dimensional equivalence of time functions and their Laplace transforms. The price of these two items is the loss of the simple interpretation of the conventional transform as the relative amplitude of a typical frequency component of the spectrum of the corresponding time function.

S. A. SCHELKUNOFF

Cours de Mécanique Rationelle. Third Edition. By Jean Chazy. Paris, Gauthier-Villars, 1947–8. Tome I, *Dynamique du Point Matériel*, 1947. 5+482 pages. 900 francs. Tome II, *Dynamique des Systèmes Matériels*, 1948. 6+511 pages. 1100 francs.

Professor Chazy's *Cours* provides another lucid, and in this case refreshing, introduction to theoretical dynamics. The work now appears in the third edition. The two volumes first appeared in 1933 and were reviewed by Professor W. R. Longley in the *Bulletin*, *39*, 491, 1933 and *41*, 13, 1935, respectively.

The first volume is devoted to particle dynamics and the second to the dynamics of rigid bodies and dynamical systems in general. Together they comprise the course of lectures which the author gave at the Faculté des Sciences de Paris from 1928 to 1941, and, excepting kinematics, represent the program for the *Certificat de Mécanique Rationnelle*. Explicitly recognizing the need for exercises, which seem to be customarily omitted from French treatises on mechanics, Professor Chazy has again inserted the lists of the *Certificat* questions, now, in seventy pages, for the years 1928 to 1946. This collection is appended to the first volume, but the questions themselves in large part call for a knowledge of the material which is reserved to the second volume.

The subject matter of the second volume is largely the traditional one, together with chapters on hydrostatics, hydrodynamics, and the elements of the theory of Newtonian attraction and potential. There is, among other things, an instructive chapter on impulsive motion. Variational principles are neglected, although Hamilton's principle is briefly considered. But for the reader who wishes an account of these matters there is Professor Chazy's own treatment in the first chapter of his admirable *La Théorie de la Relativité et la Mécanique Céleste* (Paris, 1928–30).

Professor Longley (*loc. cit.*) dealt sharply with the second chapter on "the principles of mechanics" in the first volume of the present work. Somewhere one must choose a fundamental reference frame of rest in order to get on with the subject. Professor Chazy places the origin at "the center of gravity of the solar system," the direction of axes being fixed with respect to the "fixed stars." These he calls the "axes of Copernicus," and a coordinate system in uniform rectilinear translation with respect to these defines the "axes of Galileo"; motion with respect to either of these reference frames is called "absolute." An

axiomatic treatment is attempted: (1) "the principle of inertia or the principle of Kepler" ("An isolated particle has uniform rectilinear motion, or zero acceleration"); (2) "the principle or axiom of Galileo or of initial conditions"; (3) "the principle of the equality of action and reaction"; (4) "the principle of the composition of accelerations and of forces" (parallelogram law). The second of these caused Professor Longley undue concern. It was Galileo's discovery that it is acceleration and not velocity (as Aristotle and the Scholastics had taught) which requires a force to initiate and maintain it. In other words the fundamental equations of motion are ordinary differential equations of the *second* order, with compatible boundary conditions. It is this vital fact which Professor Chazy wishes to emphasize, together with the consideration of the existence and uniqueness of the solution of the differential system. In the course of this "rationalizing" the author for some inexplicable reason never chooses to mention the name of Newton, but that is another matter. Shortly the motion of a particle on the earth's surface is obtained relative to the rotating earth, the centrifugal and Coriolis forces being introduced at once.

Many instructors in mechanics will prefer a different approach to the fundamental concepts of mechanics. At least the attention of promising students may be called to this book, and they might advisedly be invited to discuss critically and elaborate on Professor Chazy's stimulating account of the matter, in class or in their mathematics club.

S. G. HACKER

NEW BOOKS RECEIVED

Les Grands Courants de la Pensee Mathematique. Introduction by F. le Lionnais. Cahiers du Sud, 1948. 533 pp. 840 francs.

Calculus. 2nd Edition. By L. M. Kells. New York, Prentice-Hall, 1949. 12+508 pp. $4.00.

A Short Course in Differential Equations. By E. D. Rainville. New York, Macmillan, 1949. 10+210 pp. $3.00.

Analytic Geometry. By J. J. Corliss, I. K. Feinstein and H. S. Levin. New York, Harper, 1949. 14+370 pp. $3.25.

Plane and Spherical Trigonometry With Tables. 3rd Edition. By J. Shibli. Boston, Ginn and Company, 1949, 12+94 pp. $3.00.

College Algebra. 3rd Edition. By J. B. Rosenbach and E. A. Whitman. Boston, Ginn and Company, 1949. 10+523+42 pp. $3.00.

Table of Sines and Cosines to Fifteen Decimal Places at Hundredths of a Degree. (Applied Mathematics Series, no. 5). Prepared by the Computation Laboratory of the National Bureau of Standards. Washington, D. C., U. S. Government Printing Office, 1949. 8+95 pp. 40 cents.

Precis de Mathematiques Economiques et Fiscales. By Henri Eyraud. Lyon, France, Faculté des Sciences, Lyon, 1948. 72 pp.

Modern Algebraische Geometrie. Die Idealtheoretischen Grundlagen. By Wolfgang Fröbner. Springer Verlag, 1949, 12+212 pp., 1949. $5.70.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS, 1948–49

White Mathematics Club, University of Kentucky

The *White Mathematics Club* met monthly during the year except for the months of September and December. Papers presented include:

Use of the analytical triangle in curve tracing, by Virginia Baskett

The use of complex numbers in linear network analysis, by J. C. Flack

Integral domains, by A. E. Foster

Jobs for mathematics majors, by T. K. Dyer.

A birthday party for Dr. H. H. Downing, Head of the Department of Mathematics and Astronomy, was held in November.

A social get-together was held at the end of each regular meeting and refreshments were served. The annual Club Picnic was held in May.

The officers elected were: President, Fay Hays; Vice-President, Elizabeth Napier; Secretary-Treasurer, Eugene Miller; Chairman of Publicity Committee, Franz Ross.

Zeno Club, Alfred University

The following talks were given at the bimonthly meetings of the *Zeno Club* of Alfred University:

The normal curve, by Mr. H. S. Graf

Multiple surfaces, by Dr. E. Rhodes

Kirkman's schoolgirl problem, by Mr. R. Beals

Pick a number, by Prof. V. Nevins

Differential equations of applied mathematics, by Mr. I. Miller

The mil system, by Mr. H. Munson

Interpolation errors, by Dr. E. Rhodes

The Lorentz transformation, by Prof. J. Freund

An introduction to the solution of indeterminate problems, by Prof. R. Polan

Non-euclidean geometry, by Mr. L. Shershoff.

The first annual mathematics contest for freshmen and sophomores, sponsored by the *Zeno Club*, was won by Mr. H. L. Stoll. Mr. H. Munson was chosen as the outstanding mathematics student of the year at Alfred University.

Officers elected for 1949–50 are: President, L. Shershoff; Vice-President, I. Miller; Secretary, J. Freund.

Ricci Mathematics Academy, Boston College

The following talks were presented to the *Ricci Mathematics Academy*, Boston College, during 1948–49:

The theory and use of the slide rule, by Thomas Colbert

The philosophy of mathematics, by Dr. Fakhri Maluf

Mathematics as the language of science, by Rev. J. A. Tobin, S.J.

On the Ricci Mathematical Journal, by Patrick Leonard and John McClay

Curves and determinants, by Dr. F. E. White

Fields open to the graduate in mathematics, by George Donaldson

The lighter side of mathematics, by Rev. J. F. X. Murphy, S.J.

The first Annual Mid-term Social and Dance, sponsored by the Academy, was held at the Hotel Commander in Cambridge.

Officers for the year of 1949–50 were: President, Anthony Minnichelli; Vice-President, John Monahan; Secretary, Anthony Lemos; Treasurer, Edmund Murphy; Moderator, Joseph Krebs.

Mathematics-Physics Club, College of Saint Teresa

In addition to a Christmas party and a Spring picnic, four meetings were held at which the following papers were presented by members of the club:

The atomic age and some of its social and moral implications

Popular astronomy

Time and its measurement

Science and its impact on the literary imagination, by Sister M. Emmanuel, O.S.F.

Other activities included a field trip to the Mayo Research Foundation, the Medical Science Building, and the Mayo Clinic in Rochester, Minnesota.

Officers for the year 1949–50 are: President, Mary Ellen Tighe; Vice-President, Rita Kulas; Secretary, Eleanor Wise; Treasurer, Nancy Tighe.

Mathematics Club, Carleton College

The *Mathematics Club* of Carleton College held monthly meetings which included a picnic, a Christmas party, and a program of mathematical recreations. The movie *A triple integral* was also shown. The talks presented during the year 1948–49 were:

Calculating machines, by Dr. K. May

Graphical representation of complex roots, by Prof. L. Beasley

Mathematical implications of Zeno's paradoxes, by Prof. M. Capek

Wire puzzles, by Prof. C. Hatfield

Review of "Flatland," by E. A. Abbott, by Miss Jeane M. Baldwin.

Officers for the year 1948–49 were: President, Joan Snapper; Vice-President, Kinsey Anderson; Secretary, Yolanda Stork; Faculty Advisor, Prof. Kenneth May.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

RESEARCH FELLOWSHIPS IN PSYCHOMETRICS

The Educational Testing Service is offering for 1950–51 its third series of research fellowships in psychometrics leading to the Ph.D. degree at Princeton University. Open to men who are acceptable to the Graduate School of the University, the two fellowships each carry a stipend of $2,375 a year and are normally renewable.

Fellows will be engaged in part-time research in the general area of psychological measurements at the offices of the Educational Testing Service and will, in addition, carry a normal program of studies in the Graduate School. Competence in mathematics and psychology is a prerequisite for obtaining these fellowships. Information and application blanks may be obtained from: Director of Psychometric Fellowship Program, Educational Testing Service, Box 592, Princeton, N. J.

PERSONAL ITEMS

Cornell University announces: Assistant Professor G. L. Walker of Purdue University has been appointed Visiting Assistant Professor of Mathematics; Dr. Bertram Yood has been promoted to an assistant professorship; Assistant Professor G. B. Robison of Sampson College has been appointed to a teaching fellowship.

Marietta College reports that Mr. R. E. Eberhard has been appointed to an instructorship and that Miss Margaret Bootz has resigned to continue graduate study at Ohio State University.

Oberlin College makes the following announcements: Acting Chairman E. P. Vance has been promoted to the position of Chairman of the Department of Mathematics; Assistant Professor R. W. Wagner has been promoted to an associate professorship; Dr. Bryant Tuckerman of Cornell University has been appointed to an assistant professorship.

At the University of Colorado: Professor A. J. Kempner has retired and has accepted a visiting professorship at Pomona College; Dr. A. B. Farnell of Princeton University has been appointed to an assistant professorship.

Wayne University announces: Assistant Professor Russell Ackoff has been transferred from the Department of Philosophy to the Department of Mathematics; Mr. Samuel Conte has returned from a leave of absence; Mr. Bertram Eisenstadt has been appointed Assistant Professor of Mathematics.

Dr. Milton Abramowitz of the Computation Laboratory of the National Bureau of Standards is now associated with the Numerical Mathematics Service of New York City.

Instructor H. P. Atkins, Jr., of the University of Rochester has been pro-

moted to an assistant professorship.

Assistant Professor A. H. Bailey of Georgia Institute of Technology has been promoted to an associate professorship.

Dr. T. A. Bancroft, formerly director of the Statistical Laboratory of Alabama Polytechnic Institute, has accepted a position at the Statistical Laboratory, Iowa State College.

Lecturer J. D. Bankier of McMaster University has been promoted to an associate professorship.

Dr. W. E. Barnes of Cornell University has been appointed to an assistant professorship at the College of William and Mary.

Mr. D. Y. Barrer of Northwestern University has been promoted to an instructorship.

Assistant Professor I. L. Battin of Drew University has been promoted to an associate professorship.

Mr. A. V. Bauser, formerly engineer, Elliott Company, Ridgeway, Pennsylvania, is now teaching at J. W. Cooper High School, Shenandoah, Pennsylvania.

Associate Professor J. W. Beach of the New Mexico School of Mines has been appointed Assistant Professor of Mathematics at the University of New Mexico.

Mr. R. W. Beals has been appointed to an instructorship at Alfred University.

Assistant Professor S. Louise Beasley of Carleton College has been appointed to an assistant professorship at Lindenwood College.

Instructor R. F. Bell has been promoted to an assistant professorship at Eastern Washington College of Education.

Assistant Professor F. L. Celauro of Newark College of Engineering has been appointed Associate Professor and Head of the Department of Mathematics at Western State College of Colorado.

Dr. Sarvadaman Chowla of the Institute for Advanced Study has been appointed to an acting professorship at the University of Kansas.

Professor J. W. Clawson has been appointed Dean of Ursinus College.

Assistant Professor V. F. Cowling of Lehigh University has been appointed to an associate professorship at the University of Kentucky.

Instructor R. R. Croxton of the University of South Carolina has been promoted to the position of Adjunct Professor of Mathematics.

Associate Professor J. C. Currie of Alabama Polytechnic Institute has accepted an associate professorship at Georgia Institute of Technology.

Mr. J. E. Darraugh of the Consolidated Edison Company, New York City, has been appointed to an instructorship at Case Institute of Technology.

Professor D. C. Dearborn, Catawba College, has been appointed Dean.

Assistant Professor F. G. Dressel of Duke University has been promoted to an associate professorship.

Mr. W. M. Duke, chairman of Research Division, Cornell Aeronautical Laboratory, has been promoted to the position of Assistant Director.

Dean W. H. Durfee has been promoted to the position of Provost of the Colleges of the Seneca.

Reverend L. E. Ernsdorff of Loras College has been promoted from an instructorship to an associate professorship.

Professor F. A. Ficken of the University of Tennessee has been granted a leave of absence for the year 1949–50 and is spending the year at the Institute of Mathematics and Mechanics, New York University.

Dr. C. D. Firestone, Rutgers University, has been promoted to an assistant professorship.

Dr. M. M. Flood of the American Statistical Association has accepted a position as project officer with the Rand Corporation, Santa Monica.

Associate Professor C. B. Gass of Nebraska Wesleyan University is now Professor of Mathematics and Dean of Men.

Mr. B. T. Goldbeck, Jr., graduate student at Texas Christian University, has been appointed to an instructorship at the University of Wyoming.

Assistant Professor Parker Hamilton, formerly of Boston University, has accepted an appointment at Antioch College.

Mr. R. A. Harrison, formerly master at St. Mark's School, Southborough, Massachusetts, has been appointed Chairman of the Department of Mathematics of The Peddie School, Hightstown, New Jersey.

Professor M. A. Hill, Jr., of the University of North Carolina is now Associate Dean of the General College.

Miss Ruth I. Hoffman who has been teaching at North High School, Denver, is now Dean at Byers Jr. High School, Denver.

Mr. W. C. Hoffman, Iowa State College of Agriculture and Mechanic Arts, has accepted a position as assistant social scientist with the Rand Corporation, Santa Monica.

Assistant Professor C. H. Holton has been promoted to an associate professorship at Georgia Institute of Technology.

Associate Professor G. B. Huff of the University of Georgia has been promoted to a professorship.

Assistant Professor M. Gweneth Humphreys of Newcomb Memorial College, Tulane University, has been appointed to an associate professorship at Randolph-Macon Woman's College.

Dr. Solomon Hurwitz, City College of the City of New York, has been promoted to an assistant professorship.

Instructor B. C. Horne, Jr., Virginia Polytechnic Institute, has been promoted to an assistant professorship.

Professor J. B. Jeffries, Agricultural and Technical College of North Carolina, has accepted a position as assistant director with the Midway Technical School, New York City.

Associate Professor R. E. Johnson of Yale University has been appointed to an associate professorship at Smith College.

Mr. Winfield Keck of Lafayette College has been promoted to an assistant professorship.

Instructor W. J. Klimczak of the University of Rochester has been promoted to an assistant professorship.

Mr. J. A. LaRue, formerly teaching fellow at West Virginia University, has been appointed to an instructorship at Morris Harvey College.

Mrs. Sally I. Lipsey, teacher at Taft High School, New York City, has been appointed Lecturer at Hunter College.

Assistant Professor Ella Marth, Harris Teachers College, is now Associate Professor of Mathematics and Dean of Women.

Mr. A. L. Mayerson, Institute of Life Insurance, has accepted a position as actuarial assistant with the National Surety Corporation, New York City.

Associate Professor F. J. Murray of Columbia University has been promoted to a professorship.

Mr. J. D. Newburgh of the Massachusetts Institute of Technology has been appointed a member of the Institute for Advanced Study.

Miss Margaret Owchar, formerly instructor at Rockford College, has been appointed to an instructorship at the University of Minnesota.

Mr. C. L. Perry of the University of Michigan has been appointed to an assistant professorship at the University of Arkansas.

Professor W. G. Pollard has been appointed Executive Director of the Oak Ridge Institute of Nuclear Studies.

Associate Professor D. H. Porter is on leave of absence from Marion College and has a position as teaching fellow at Indiana University.

Dr. R. C. Prim of Princeton University has accepted a position as mathematician with the Bell Telephone Laboratories.

Assistant Professor A. L. Putnam of the University of Chicago has been appointed to a professorship at New York University.

Dr. H. J. Ryser of the Institute for Advanced Study has been appointed to an assistant professorship at Ohio State University.

Dr. W. R. Scott, University of Michigan, has received an appointment as assistant professor at the University of Kansas.

Assistant Professor W. T. Scott, Northwestern University, has been promoted to an associate professorship.

Mr. K. M. Siegel is now Research Associate at the Aeronautical Research Center, University of Michigan, Willow Run Airport, Ypsilanti.

Assistant Professor W. C. Taylor, University of Cincinnati, has accepted a position as mathematician with the Ballistics Laboratory, Aberdeen Proving Ground, Maryland.

Mr. E. F. Trombley, formerly assistant instructor at the University of Chicago, has been appointed to an instructorship at Illinois Institute of Technology.

Professor J. A. Ward, who has been serving as assistant head of the Department of Mathematics of the University of Georgia, has been appointed to a professorship at the University of Kentucky.

Professor K. L. Warren of Millsaps College has been appointed Associate Professor of Physics, Kent State University.

Dr. V. M. Wolontis of Harvard University has been appointed to an assistant professorship at the University of Kansas.

Instructor H. M. Zerbe, Hazleton Center of Pennsylvania State College, has been promoted to an assistant professorship.

Reverend J. E. Case, S.J., director of the Department of Mathematics of St. Louis University, died on August 5, 1949.

Dr. J. K. L. MacDonald of the Naval Ordnance Test Station, China Lake, died February 3, 1949.

Professor A. H. Mowbray of the University of California died on January 7, 1949.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE THIRTY-FIRST SUMMER MEETING OF THE ASSOCIATION

The thirty-first summer meeting of the Mathematical Association of America was held at the University of Colorado, Boulder, Colorado, on Monday and Tuesday, August 29–30, 1949 in conjunction with the summer meetings of the American Mathematical Society, the Institute of Mathematical Statistics, and the Econometric Society. A total of seven hundred and seventy adults were registered, including the following three hundred and thirty-five members of the Association:

V. W. Adkisson, University of Arkansas
O. W. Albert, University of Redlands
E. B. Allen, Rensselaer Polytechnic Institute
R. D. Anderson, University of Pennsylvania
R. V. Anderson, Colorado A & M College
R. V. Andree, University of Oklahoma
K. J. Arnold, University of Wisconsin
W. L. Ayres, Purdue University
Joshua Barlaz, Rutgers University
I. A. Barnett, University of Cincinnati
C. F. Barr, University of Wyoming
D. Y. Barrer, Northwestern University
D. L. Barrick, University of Colorado
H. G. H. Bartram, Cornell University
M. A. Basoco, University of Nebraska
P. T. Bateman, Institute for Advanced Study
J. W. Beach, University of New Mexico
J. E. Bearman, University of Minnesota
E. F. Beckenbach, University of California at Los Angeles
May M. Beenken, Immaculate Heart College
Alice K. Bell, Fresno State College
J. H. Bell, Michigan State College
B. C. Bellamy, Bellamy & Sons
Theodore Bennett, Marietta College
S. F. Bibb, Illinois Institute of Technology
F. C. Biesele, University of Utah
E. E. Blanche, United States Army
H. D. Block, Iowa State College
R. P. Boas, Mathematical Reviews
H. F. Bohnenblust, California Institute of Technology
A. W. Boldyreff, University of New Mexico
T. A. Botts, University of Virginia

J. W. Bradshaw, University of Michigan
H. E. Bray, Rice Institute
J. R. Britton, University of Colorado
J. C. Brixey, University of Oklahoma
B. K. Brown, James Millikin University
M. C. Brown, University of Kentucky
R. C. Buck, Brown University
C. E. Buell, Sandia Laboratory
P. B. Burcham, University of Missouri
Herbert Buseman, University of Southern California
J. H. Bushey, Hunter College
Jewell H. Bushey, Hunter College
C. H. Butler, Western Michigan College
S. S. Cairns, University of Illinois
W. D. Cairns, Oberlin College
J. W. Calkin, Rice Institute
E. A. Cameron, University of North Carolina
C. C. Camp, University of Nebraska
H. H. Campaigne, United States Navy
F. M. Carpenter, Colorado School of Mines
W. B. Carver, Cornell University
L. G. Chelius, Atomic Energy Commission
R. V. Churchill, University of Michigan
A. G. Clark, Colorado A & M College
Helen E. Clarkson, Creighton University
Nathaniel Coburn, University of Michigan
C. J. Coe, University of Michigan
Nancy Cole, Syracuse University
E. P. Coleman, Highland Falls, N. Y.
L. A. Colquitt, Texas Christian University
E. G. H. Comfort, Illinois Institute of Technology
J. A. Cooley, University of Tennessee
N. A. Court, University of Oklahoma
W. R. Cowell, Kansas State College
K. W. Crain, Purdue University
E. L. Crow, U. S. Naval Ordnance Test Station
J. H. Curtiss, National Bureau of Standards
J. A. Daum, Texas A & M College
P. H. Daus, University of California at Los Angeles
M. W. De Jonge, Purdue University
R. F. Deniston, Iowa State College
W. W. Denton, University of Arizona
R. P. Dilworth, California Institute of Technology
Bernard Dimsdale, Aberdeen Proving Ground
Roy Dubisch, Fresno State College
W. L. Duren, Tulane University
J. M. Earl, University of Omaha
E. D. Eaves, University of Tennessee
Paul Eberhart, Washburn University
P. D. Edwards, Ball State Teachers College
W. E. Ekman, University of South Dakota
Terrell Ellis, North Texas State College
Paul Erdös, University of Illinois
G. C. Evans, University of California
H. P. Evans, University of Wisconsin
Howard Eves, Oregon State College
G. M. Ewing, University of Missouri
A. B. Farnell, University of Colorado
William Feller, Cornell University
H. H. Ferns, University of Saskatchewan
F. N. Fisch, Colorado State College of Education
C. H. Fischer, University of Michigan
L. R. Ford, Illinois Institute of Technology
W. C. Foreman, University of Kansas
J. S. Frame, Michigan State College
Thornton C. Fry, Bell Telephone Laboratories
L. E. Fuller, University of Wisconsin
M. G. Gaba, University of Nebraska
H. M. Gehman, University of Buffalo
F. C. Gentry, Arizona State College
R. E. Gilman, Brown University
J. W. Givens, University of Tennessee
A. M. Gleason, Harvard University
Michael Golomb, Purdue University
R. N. Goss, Iowa State College
W. H. Gottschalk, University of Pennsylvania
Cornelius Gouwens, Iowa State College
J. W. Green, University of California at Los Angeles
R. E. Greenwood, University of Texas
Edison Greer, Kansas State College
F. L. Griffin, Reed College
V. G. Grove, Michigan State College
H. T. Guard, Colorado A & M College
D. F. Gunder, Cornell University
W. S. Gustin, Indiana University
Beatrice L. Hagen, Pennsylvania State College
Franklin Haimo, Washington University
P. R. Halmos, University of Chicago
O. H. Hamilton, Oklahoma A & M College
Clara L. Hancock, Virginia Junior College
J. R. Hanna, University of Wichita
W. R. Hanson, City College of San Francisco
Katharine E. Hazard, Rutgers University
E. A. Hazlewood, Texas Technological College
I. L. Hebel, Colorado School of Mines
G. A. Hedlund, Yale University
E. R. Heineman, Texas Technological College
E. D. Hellinger, Illinois Institute of Technology

ANN S. HENRIQUES, University of Utah
GERTRUDE A. HERR, Iowa State College
J. G. HERRIOT, Stanford University
FRITZ HERZOG, Michigan State College
M. R. HESTENES, University of California at Los Angeles
T. H. HILDEBRANDT, University of Michigan
A. G. HILL, North Dakota Agricultural College
J. J. L. HINRICHSEN, Iowa State College
P. G. HOEL, University of California at Los Angeles
D. L. HOLL, Iowa State College
CARL HOLTOM, U.S.A.F. Institute of Technology
LEROY HOLUBAR, University of Colorado
R. E. HORTON, Los Angeles City College
L. AILEEN HOSTINSKY, Temple University
E. MARIE HOVE, Hofstra College
J. M. HOWELL, Los Angeles City College
H. K. HUGHES, Purdue University
P. F. HULTQUIST, University of Colorado
M. GWENETH HUMPHREYS, Randolph-Macon Woman's College
N. C. HUNSAKER, Utah State College
BURROWES HUNT, University of Colorado
C. C. HURD, International Business Machines Corp.
W. R. HUTCHERSON, University of Florida
C. A. HUTCHINSON, University of Colorado
H. H. IRWIN, State College of Washington
T. W. JACKSON, Jamestown College
B. W. JONES, University of Colorado
G. K. KALISCH, University of Minnesota
IRVING KAPLANSKY, University of Chicago
LOIS KARR, Lindenwood College
M. W. KELLER, Purdue University
J. L. KELLEY, University of California
CLARIBEL KENDALL, University of Colorado
P. W. KETCHUM, University of Illinois
D. E. KIBBEY, Syracuse University
E. C. KIEFER, James Millikin University
W. J. KIRKHAM, Oregon State College
J. R. KLINE, University of Pennsylvania
H. L. KRALL, Pennsylvania State College
MAX KRAMER, University of Illinois
G. M. KUZNETS, University of California
O. E. LANCASTER, United States Navy
JOSEPH LANDIN, University of Illinois
R. E. LANGER, University of Wisconsin
H. D. LARSEN, Albion College
C. G. LATIMER, Emory University
D. H. LEAVENS, Colorado Springs, Colo.
W. G. LEAVITT, University of Nebraska
SOLOMON LEFSCHETZ, Princeton University
D. H. LEHMER, University of California
L. C. LEITHOLD, Phoenix College
W. T. LENSER, University of Nebraska
W. J. LEVEQUE, University of Michigan
HARRY LEVY, University of Illinois
A. J. LEWIS, University of Denver
C. F. LEWIS, Kansas State College
F. A. LEWIS, University of Alabama
S. B. LITTAUER, Columbia University
W. S. LOUD, University of Minnesota
R. G. LUBBEN, University of Texas
SAUNDERS MACLANE, University of Chicago
M. L. MADISON, Colorado A & M College
MORRIS MARDEN, University of Wisconsin
ANNA MARM, Bethany College
D. C. B. MARSH, University of Arizona
M. H. MARTIN, University of Maryland
J. R. MAYOR, University of Wisconsin
R. B. MCCLENON, Grinnell College
ELOISE MCCORD, University of Wichita
DOROTHY MCCOY, Wayland College
N. H. MCCOY, Smith College
L. H. MCFARLAN, University of Washington
E. J. MCSHANE, University of Virginia
A. E. MEDER, JR., Rutgers University
A. S. MERRILL, Montana State University
B. C. MEYER, University of Arizona
H. L. MEYER, JR., University of Chicago
R. R. MIDDLEMISS, University of Washington
R. A. MILLER, University of Mississippi
T. W. MOORE, U. S. Naval Academy
F. R. MORRIS, Fresno State College
DOROTHY J. MORROW, Civil Aeronautics Administration
S. B. MYERS, University of Michigan
W. K. NELSON, University of Colorado
GRETA NEUBAUER, University of Wyoming
C. V. NEWSOM, University of the State of New York
IVAN NIVEN, University of Oregon
M. L. NORDEN, Johns Hopkins University
RUTH E. O'DONNELL, Duquesne University
E. G. OLDS, Carnegie Institute of Technology
EMMA J. OLSON, Kent State University
R. L. O'QUINN, Louisiana State University
J. C. OXTOBY, Bryn Mawr College
H. C. PETERSON, University of Denver
R. P. PETERSON, University of California at Los Angeles
T. S. PETERSON, University of Oregon
A. D. PIERSON, JR. College of Kansas City

Margaret M. Pihlblad, University of Kansas
George Piranian, University of Michigan
J. C. Polley, Wabash College
Florence E. Pool, University of Nebraska
G. B. Price, University of Kansas
A. L. Putnam, University of Chicago
E. D. Rainville, University of Michigan
J. F. Randolph, University of Rochester
Ruth B. Rasmusen, Chicago City College
L. T. Ratner, Vanderbilt University
L. L. Rauch, University of Michigan
C. B. Read, University of Wichita
M. O. Reade, University of Michigan
O. H. Rechard, University of Wyoming
O. W. Rechard, Ohio State University
J. K. Reckzeh, New Jersey State Teachers College
Mina S. Rees, United States Navy
P. K. Rees, Louisiana State University
R. F. Reeves, Iowa State College
W. T. Reid, Northwestern University
Irving Reiner, University of Illinois
J. G. Renno, University of Wisconsin
C. N. Reynolds, West Virginia University
H. L. Rice, University of Omaha
F. A. Rickey, Louisiana State University
P. R. Rider, Washington University
L. G. Riggs, Northwestern University
L. A. Ringenberg, Eastern Illinois State College
D. D. Rippe, University of Michigan
E. K. Ritter, University of Michigan
B. D. Roberts, New Mexico Highlands University
Fred Robertson, Iowa State College
Raphael M. Robinson, University of California
Florence V. Rohde, University of Kentucky
Arthur Rosenthal, Purdue University
J. B. Rosser, Institute for Numerical Analysis
M. F. Rosskopf, Syracuse University
R. G. Sanger, Kansas State College
A. C. Schaeffer, Purdue University
Edith R. Schneckenburger, University of Buffalo
K. C. Schraut, University of Dayton
Nathan Schwid, University of Wyoming
E. J. Shapiro, Brooklyn College
H. C. Shaub, Washington & Jefferson College
C. R. Sherer, Texas Christian University
L. W. Sheridan, College of St. Thomas
Sister M. Nicholas Arnoldy, Marymount College
Sister M. Pachomia Larkey, College of St. Teresa
Sister M. Teresine Lewis, Fontbonne College
Sister Rose Margaret Cook, Loretto Heights College
M. F. Smiley, State University of Iowa
A. H. Smith, Purdue University
F. C. Smith, College of St. Thomas
G. W. Smith, University of Kansas
H. L. Smith, Louisiana State University
L. C. Snively, University of Colorado
Elizabeth S. Sokolnikoff, University of Wisconsin
I. S. Sokolnikoff, University of California at Los Angeles
T. H. Southard, Wayne University
F. W. Sparks, Texas Technological College
E. J. Specht, Emmanuel Missionary College
Vivian E. Spencer, U. S. Bureau of Census
C. E. Springer, University of Oklahoma
K. H. Stahl, University of Colorado
L. W. Stark, Atlantic Christian College
B. M. Stewart, Michigan State College
Ruth W. Stokes, Syracuse University
E. B. Stouffer, University of Kansas
C. J. Stowell, McKendree College
A. G. Swanson, Gustavus Adolphus College
Otto Szasz, University of Cincinnati
A. Helen Tappan, Western College
A. H. Taub, Institute for Advanced Study
V. B. Temple, Louisiana College
H. P. Thielman, Iowa State College
J. M. Thomas, Duke University
F. B. Thompson, University of California
R. M. Thrall, University of Michigan
Leonard Tornheim, University of Michigan
E. P. Tovani, University of Colorado
A. W. Tucker, Princeton University
J. W. Tukey, Princeton University
J. L. Ullman, University of Michigan
Gilbert Ulmer, University of Kansas
R. S. Underwood, Texas Technological College
W. R. Utz, University of Missouri
H. S. Vandiver, University of Texas
V. J. Varineau, University of Wyoming
R. W. Veatch, University of Tulsa
J. F. Wagner, University of Colorado
Earl Walden, New Mexico College of A & M A
G. L. Walker, Cornell University
R. J. Walker, Cornell University
J. L. Walsh, Harvard University

Lillie C. Walters, University of Colorado
K. W. Wegner, Carleton College
A. L. Whiteman, University of Southern California
L. R. Wilcox, Illinois Institute of Technology
F. B. Wiley, Denison University
S. S. Wilks, Princeton University
R. L. Wilson, University of Tennessee
G. M. Wing, Cornell University
R. M. Winger, University of Washington
Clement Winston, Department of Commerce
H. E. Wolfe, Indiana University
C. R. Wylie, Jr., University of Utah
P. M. Young, Kansas State College
J. H. Zant, Oklahoma A & M College

The Association held it sessions on Monday afternoon and Tuesday morning in Room 201W, Arts Building, with President R. E. Langer presiding. The Tuesday morning session was held jointly with the Institute of Mathematical Statistics and the Econometric Society. The Program Committee for the meeting consisted of S. S. Cairns, chairman, Wladimir Seidel, and C. R. Wylie, Jr.

FIRST SESSION OF THE ASSOCIATION

"What Has Happened to Algebraic Geometry?", by Professor R. J. Walker, Cornell University.

"The Problems Section of the Monthly," by Professor Howard Eves, Oregon State College.

"The Monte Carlo Method," by Dr. S. M. Ulam, Los Alamos Scientific Laboratory.

SECOND SESSION OF THE ASSOCIATION

Symposium: "Mathematical Training for Social Scientists," with Professor Jacob Marschak, University of Chicago as Chairman, and Professors R. L. Anderson, North Carolina State College, T. W. Anderson, Columbia University, G. C. Evans, University of California, F. L. Griffin, Reed College, Harold Gulliksen, Educational Testing Service, Harold Hotelling, University of North Carolina, William Jaffe, Northwestern University, and G. M. Kuznets, University of California, as participants.

At the conclusion of the symposium, it was voted that the officers of the Mathematical Association of America, the Institute of Mathematical Statistics, and the Econometric Society be requested to appoint a joint committee, in cooperation with other interested societies, to study the need for better mathematical training of social scientists and the ways and means of improving their mathematical preparation. It was suggested that this committee report at the next joint meeting of the three organizations.

MEETING OF THE BOARD OF GOVERNORS

The Board met on Monday evening in the Southwest Recreation Room of the Third Dormitory. Twenty-seven members of the Board were present.

Progress reports were received from several committees. No action was taken on the matter of proposed translations of publications of the Association into foreign languages, but the concensus was that everything possible should be

done to encourage the development of mathematics in foreign countries, especially in South America.

The Board voted to authorize the printing of two new Carus Monographs. Number 9 is entitled: "The Theory of Algebraic Numbers" by Harry Pollard and number 10 is: "The Arithmetic Theory of Quadratic Forms" by B. W. Jones. The Executive and Finance Committees were empowered to act on the method of distribution of Monographs 9 and 10.

MEETING OF SECTION SECRETARIES

A meeting of secretaries of the Sections of the Association was held on Tuesday evening in the Southwest Lounge. Twenty-two of the twenty-five Sections were represented.

The general topic of the meeting was the proposed handbook of instructions for Section Secretaries. Many helpful suggestions were made concerning the conduct of section meetings. Representatives of several sections reported on special projects being carried out by their sections.

MEETINGS OF OTHER ORGANIZATIONS

The sessions of the American Mathematical Society began on Tuesday afternoon and continued through Friday afternoon. Professor G. A. Hedlund of Yale University delivered the Colloquium lectures on "Topological Dynamics." Professor B. Jessen of the University of Copenhagen spoke on "Some recent investigations in almost periodic functions" and Professor F. B. Jones of the University of Texas spoke on "Aposyndetic and non-aposyndetic continua."

The Institute of Mathematical Statistics held its sessions from Monday afternoon through Thursday afternoon.

Sessions of the Econometric Society began on Monday morning and continued through Friday morning.

From Monday through Thursday, the Ninth Summer Meeting of the National Council of Teachers of Mathematics was held at the University of Denver in Denver, Colorado. Many members of the Council and of the Association took advantage of this opportunity to attend the summer meetings of both organizations.

ARRANGEMENTS, ENTERTAINMENT AND RECREATION

The New Dormitory Group of the University of Colorado was available to all attending the meeting and to their families. Meals were served in the dining halls of the Third Dormitory.

On Tuesday afternoon a tea for the visiting mathematicians and their families was given at the President's house by the ladies of the mathematics departments.

On Wednesday afternoon an excursion was held along the Trail Ridge Road in Rocky Mountain National Park to the Museum and Store at Fall River Pass. Supper was eaten at Glacier Basin Camp Ground.

On Thursday evening a steak fry was held on Flagstaff Mountain. Afterwards the group gathered in the amphitheatre on the top of the mountain overlooking the University and the city of Boulder. Professor C. A. Hutchinson acted as master of ceremonies. Vice-President W. F. Dyde greeted the guests on behalf of the University of Colorado. Dr. C. V. Newsom responded on behalf of the four organizations. A telegram from Professor A. J. Kempner was read expressing his regrets at being unable to be present at the meeting and the group voted to send Professor Kempner a telegram of greetings and best wishes. Mr. A. B. Patterson, Episcopal student chaplain at the University of Colorado, led the singing and sang several solos.

At the conclusion of Dr. Newsom's talk, a motion was enthusiastically adopted expressing appreciation to the University of Colorado for the hospitality it had extended to the visitors, and thanking all those who had given so generously of their time in making the meetings such a noteworthy success. Special mention was made of the members of the departments of Mathematics and of Applied Mathematics, and of the Committee on Arrangements, and finally of C. A. (Hutch) Hutchinson, himself, "who had done everything from endorsing checks to ringing the dinner bell."

H. M. GEHMAN, *Secretary-Treasurer*

THE MARCH MEETING OF THE PACIFIC NORTHWEST SECTION

The third annual meeting of the Pacific Northwest Section of the Mathematical Association of America was held at Oregon State College, Corvallis, on Friday and Saturday, March 25–26, 1949.

Eighty-five persons attended, including the following fifty-seven members of the Association: Jessie V. Allhands, Tyler Allhands, H. A. Antosiewicz, R. A. Beaumont, R. F. Bell, S. E. Boselly, L. G. Butler, W. B. Caton, Richard Cebula, Paul Civin, C. L. Clark, C. M. Cramlet, J. H. Curtiss, Douglas Derry, W. W. Dolan, J. L. Ericksen, H. W. Eves, K. S. Ghent, E. G. Goman, F. L. Griffin, S. G. Hacker, V. E. Hoggatt, Jr., H. H. Irwin, S. A. Jennings, J. M. Kingston, W. J. Kirkham, Celia E. Klotz, M. S. Knebelman, J. C. R. Li, J. J, Livers, A. T. Lonseth, R. E. Lowney, C. F. Luther, L. H. McFarlan, W. E. Milne, A. F. Moursund, B. N. Moyls, D. C. Murdoch, Ivan Niven, Andrewa R. Noble, T. G. Ostrom, T. S. Peterson, A. R. Poole, H. F. Price, R. A. Rosenbaum, R. B. Saunders, W. G. Scobert, W. L. Shepherd, W. H. Simons, M. C. Stippes, W. M. Stone, D. B. Tillotson, J. R. Vatnsdal, G. A. Williams, L. B. Williams, R. M. Winger, F. E. Wood.

At the business meeting of the Section the following officers were elected for the year 1949–50: Chairman, R. M. Winger, University of Washington; Vice-Chairman, A. F. Moursund, University of Oregon; Secretary-Treasurer, S. G. Hacker, State College of Washington. The Section voted to accept the invitation of the University of Washington to hold the next annual meeting in Seattle in the spring of 1950.

The Friday afternoon session was opened by Professor W. E. Milne, Chair-

man of the Section, and Professor Moursund was invited by him to preside. By invitation the two papers by Professors Eves and Rosenbaum were assigned twenty minutes and each was followed by an interesting discussion. Subsequently, on motion of Professor Eves, a representative committee from this Section, with Professor Rosenbaum as chairman, was appointed to extend and implement the suggestions made in Professor Rosenbaum's paper; these suggestions are outlined below in the abstract of his paper. Professor C. M. Cramlet, University of Washington, delivered the annual invited hour address, the title being *On Algebraic and Differential Invariant Theory.*

Following the business meeting on Friday night Professor H. H. Irwin, State College of Washington, gave a report of the findings of the Committee on Improvements in Qualifications of Mathematics Teachers in Secondary Education in the Pacific Northwest. This committee was appointed last year by this Section, and it was asked to continue to serve for the year 1949–50. Professor Irwin reported on the results of three surveys conducted by his committee: (1) the requirements of Pacific Northwest institutions for a bachelor's degree in mathematics or for a degree in education with a major or minor in mathematics; (2) undergraduate mathematics courses considered by college instructors and by secondary school teachers to be helpful to the latter in their subsequent teaching; (3) the amount of college training in mathematics required by each of the forty-eight states in order for a secondary school teacher to qualify to teach mathematics in its accredited secondary schools.

On Saturday morning the newly elected Chairman, Professor Winger, invited Professors Griffin and Knebelman to preside. Dr. J. H. Curtiss of the National Applied Mathematics Laboratories, National Bureau of Standards, read an invited paper on *The Mathematical Programs of the National Bureau of Standards*. Dr. Curtiss spoke regarding the background, administration, and certain objectives of the Laboratories.

The first eight of the following papers, including Professor Cramlet's address, were presented at the Friday afternoon session. The other thirteen papers, were read on Saturday morning.

1. *A sum transformation*, by Professor W. M. Stone, Oregon State College.

A sum transformation, defined by

$$LF(kh) = h\sum_{n=0}^{\infty} F(nh)(1 + sh)^{-n-1}$$

where k is a positive integer and h and s are arbitrary complex parameters, reduces to the classical Laplace transformation when $|h| \to 0$, $kh \to t$. Various properties of the sum transformation were given, together with applications to the solution of difference equations, particularly those which arise by replacing differential operators by difference operators.

2. *Weighted trigonometric interpolation*, by Professor Paul Civin, University of Oregon.

For a continuous function $f(x)$, a comparison was made between the convergence behavior of the set of trigonometric polynomials of order n which take the values $f(x_n^i)$ at $x_n^i = 2\pi i/(2n+1)$, and

the set which takes values

$$\alpha(n)[f(x_n^{i-1}) + f(x_n^{i+1})] + [1 - 2\alpha(n)]f(x_n^i)$$

at x_n^i.

3. *A parametric treatment of polar tangent curves*, by Professor R. M. Winger, University of Washington.

These curves have been studied by Stratton, utilizing mainly polar coordinates, in this MONTHLY, vol. 43 (1936), p. 398. Professor Winger derived a compact parametric representation, permitting the use of the technique for dealing with rational curves, and he obtained and employed to advantage the collineation groups which leave the curves invariant.

4. *The rank of a curve of real order 3 in 3-space*, by Professor Douglas Derry, University of British Columbia.

Denoting by C_3 a closed curve in real projective 3-space which is cut by a plane in at most three real points, for each point of which a tangent and osculating plane are defined, the author proved the rank of C_3 to be 4. An extension of the proof to a C_4 was indicated.

5. *An experiment in intellectual stimulation of gifted high school students*, by Professor L. B. Williams, Reed College.

During the past year members of the staff of the Science Division of Reed College have given monthly lectures on scientific subjects, followed by discussions, to a group of about sixty Portland high school students selected on the basis of scientific aptitude and interest. Professor Williams reported on the way in which students were selected and the lecture subjects, together with a partial evaluation of the response and the results.

6. *Invited Annual Address: On algebraic and differential invariant theory*, by Professor C. M. Cramlet, University of Washington.

Professor Cramlet traced the development of the theory of algebraic invariants from the early formal theory to the theory of group representations. It was explained that the latter provides an abstract basis on which the foundations of tensor algebra are securely built, algebraic invariant theory becoming an aspect of tensor algebra, and differential invariant theory a natural extension from the theory of algebraic forms to the theory of differential forms.

7. *Some unsolved "elementary" problems submitted to the Monthly*, by Professor H. W. Eves, Oregon State College.

The Elementary Problem Department of this MONTHLY made its first appearance in the October 1932 issue, and has since published over 850 proposals. It is interesting that among the first 800 of these problems there are eight for which no solutions have been received, thirteen for which only partial solutions have been received, and nine which contain unanswered related questions raised either by a solver or by the editor. The problem numbers of the proposals in the respective groups are: (1) E 9, E 28, E 518, E 534, E 570, E 585, E 604, E 774; (2) E 185, E 208, E 246, E 379, E 410, E 442, E 496, E 590, E 708, E 735, E 764, E 765, E 777; (3) E 400, E 401, E 476, E 644, E 696, E 724, E 747, E 750, E 791. Professor Eves' paper was devoted to a discussion of selected problems from this list.

8. *Discussion of a problem book of a particular type*, by Professor R. A. Rosenbaum, Reed College.

Professor Rosenbaum suggested that this Section might possibly be interested in the compilation of a small book of serious problems of definite mathematical content which could prove a

challenge and a stimulating experience to undergraduates of ability in the course of their study of mathematics. The speaker gave several interesting illustrations of the types of problems which he has in mind.

9. *Note on formal integration*, by Professor F. L. Griffin, Reed College.

Professor Griffin called attention to systematic elementary procedures for integrating certain expressions of the types considered by H. F. MacNeish in this MONTHLY, vol. 56 (1949), p. 25.

10. *On the integral solutions of the Diophantine equations* $x^2-y^2=at^2$, $y^2-z^2=at^2$, by Mr. Joseph Irwin, Portland, Oregon, introduced by Professor Griffin, read by title.

11. *On the number of primitive* λ*-roots mod* m, by Professor D. C. Murdoch, University of British Columbia.

Professor Murdoch stated that in an unpublished paper H. Griffin and B. Sussman have solved the problem of finding the number of primitive λ-roots modulo a composite integer m, and he gave the following alternative formula obtained by group theoretic methods. If $m=p_0^{e_0}p_1^{e_1}\cdots p_r^{e_r}$, where $p_0=2$ and $p_1, \cdots, p_r$ are distinct odd primes, the number of primitive λ roots mod m is

$$\phi[\phi(m)]\cdot\prod_{i=0}^{r}\left[(1-q_i^{-\nu_i})/(1-q_i^{-1})\right]$$

where the product extends over all prime divisors q_i of $\phi(m)$, $q_0=2$, and the numbers ν_i are defined as follows: (a) if $e_0\leqq 2$ and $q_i^{\beta_i}$ is the highest power of q_i which divides any of the numbers $\phi(p_0^{e_0})$, $\phi(p_1^{e_1}), \cdots, \phi(p_r^{e_r})$, then ν_i is the number of these numbers which are divisible by $q_i^{\beta_i}$; (b) if $e_0>2$ and $q_i^{\beta_i}$ is the highest power of q_i which divides any one of the numbers 2, $\frac{1}{2}\phi(p_0^{e_0})$, $\phi(p_1^{e_1}), \cdots,$ $\phi(p_r^{e_r})$, then ν_i is the number of these numbers which are divisible by q_i.

12. *Hilbert's geometric postulates as theorems*, by Miss Patricia M. Cowan, Reed College, introduced by Professor Griffin.

Miss Cowan, a student at Reed College, discussed the possible deduction of Hilbert's postulates for geometry from the Pieri postulates. Mimeographed copies of the Pieri definitions and postulates were distributed in order to facilitate the discussion.

13. *Numerical treatment of the Laplacian operator*, by Professor W. E. Milne, Oregon State College and The Institute for Numerical Analysis.

Given a plane harmonic function $U(x, y)$ and its values U_i at the nodes of a square mesh, formulas were obtained, with bounds for the error, for the approximate representation of the first and second order partial derivatives of $U(x, y)$, for interpolation on curvilinear boundaries, and for the Laplacian operator. These formulas were expressed in terms of linear combinations of the U_i at a suitable pattern of nodal points.

14. *Some elementary problems in probability*, by Professor Ivan Niven, University of Oregon.

Several examples were given to show that books on college algebra could use problems on probability of the continuous type as well as the usual problems of the discrete type. For example, it might be asked, what is the probability that the distance between two points taken at random on the boundary of a unit square be greater than some constant which could either be specified numerically or left as a parameter.

15. *The mathematical programs of the National Bureau of Standards*, by Dr. J. H. Curtiss, National Applied Mathematics Laboratories, National Bureau of Standards.

16. *Rigidity of Lennes polyhedra*, by Professor H. W. Eves, Oregon State College.

Cauchy has shown that every simple closed convex polyhedral surface is rigid. To date it is unknown whether there exist simple closed non-convex polyhedral surfaces which are non-rigid. A Lennes polyhedron is a simple closed polyhedral surface with triangular faces and such that the line segment joining any two vertices not on a common edge lies either wholly or partly outside the volume bounded by the surface. Such polyhedra are known to be non-convex. The author stated that from paper models of Lennes polyhedra one might be led to conjecture that such polyhedral surfaces are non-rigid, for they seem to possess a fair range of mobility. However it was the purpose of this paper to show that this apparent motion is accompanied by small distortions of the faces and that the polyhedra are really rigid.

17. *A recursion relation involving exponentials*, by Mr. Daniel Drew, Reed College, introduced by Professor Rosenbaum.

Mr. Drew, a student at Reed College, obtained a general solution of lim u_n, $u_n = a^{u_{n-1}}$, $u_0 = a_0$, and applied the result to the problem of lim v_n, $v_n = a_n^{\,n-1}$, $v_0 = a_0$.

18. *An application of Laplace integrals to Hankel functions*, by Professor W. B. Caton, State College of Washington.

Using the principle of rotation of the path of integration of a Laplace integral, together with Abelian theorems for these integrals, the author obtained representations and formulas of the Hankel functions valid in appropriate half-planes. In particular, the expressions for $H_\alpha^{(1)}(ze^{2\pi i})$ and $H_\alpha^{(2)}(ze^{2\pi i})$ involving Laplace integrals were exhibited.

19. *Sample size for Wilks' tolerance limits*, by Professors Z. W. Birnbaum, and H. S. Zuckerman, University of Washington, read by title.

20. *Some consequences of a well-known theorem on conics*, by Professor R. A. Rosenbaum, Reed College, and Mr. Joseph Rosenbaum, The Milford School.

This paper, which was concerned with a generalization of the Desargues involution theorem and a porism related to Steiner's theorem on chains of tangent circles, was read by title at the authors' generous suggestion in view of the limited time left available.

21. *The Canadian congress on applied mathematics, Vancouver, 1949*, by Professor S. A. Jennings, University of British Columbia.

Professor Jennings gave a brief account of the extensive scientific and social arrangements which have been made for the Congress, and extended a cordial invitation to the members of this section to attend.

S. G. HACKER, *Secretary*

THE APRIL MEETING OF THE SOUTHWESTERN SECTION

The ninth annual meeting of the Southwestern Section of the Mathematical Association of America was held at New Mexico College of A. & M. A., State College, New Mexico, April 29 and 30, 1949. Professor Earl Walden, chairman of the Section, presided.

The attendance was forty-one including the following twenty-three members of the Association: J. W. Beach, A. W. Boldyreff, C. R. Buell, J. H. Butchart, Louise H. Chin, W. W. Denton, Ruth A. Fish, F. C. Gentry, R. F. Graesser, Rose A. Grundman, W. P. Heinzman, R. C. Hildner, P. F. Hultquist, H. D. Huskey, Brother Cyprian Luke, D. C. B. Marsh, E. J. Purcell, B. D. Roberts, H. P. Rogers, Earl Walden, R. L. Westhafer, Charles Wexler, and D. L. Webb.

The first forenoon was devoted to a trip to the White Sands Rocket Proving Grounds, where exhibits, lectures, and motion pictures instructed the group. A banquet at the Student Union Building was followed by an address, *Automatical Digital Computing*, by Dr. H. D. Huskey, Chief of the Machine Development Unit, Institute of Numerical Analysis, University of California at Los Angeles. This lecture was a joint presentation of the Association and the Physical Science Laboratory of the host college, Dr. George Gardiner, director.

At the business session the following officers were elected: Chairman, E. J. Purcell, University of Arizona; Vice-Chairman, A. W. Boldyreff, University of New Mexico; Visiting Lecturer, J. H. Butchart, Arizona State College. It was voted by the Section, upon the request of the delegation from the Texas School of Mines, El Paso, Texas, to ask the Board of Governors to change the territorial limits of the Southwestern Section to include El Paso, Texas. It was voted to invite the members at Lubbock, Texas to join in a similar request.

The program consisted of the following papers:

1. *On the definition of an analytical function of a complex variable*, by Professor A. W. Boldyreff, University of New Mexico.

2. *Systems of lines and planes on the quadric hypersurfaces of four and five dimensions*, by Ruth A. Fish, University of Arizona.

This was an expository treatment of systems of flat space on quadric primals in projective spaces of four and five dimensions.

3. *Postulates of projective algebras*, by Louise H. Chin, University of Arizona.

4. *Some characteristics of the altitude quadric*, by Professor J. H. Butchart, Arizona State College.

The speaker discussed some properties of the hyperboloid of one sheet of which the altitudes of a tetrahedron belong to one regulus. In particular, the Monge point is the center of the altitude quadric, and the plane section normal to any ruling is an equilateral hyperbola.

5. *A Cremona involution in n-dimensional space*, by Professor E. J. Purcell, University of Arizona.

6. *Grading and the fluctuations of sampling*, by Professor R. F. Graesser, University of Arizona.

A population of students with given percentages of A, B, C, D, and E students is assumed. From this population classes are drawn as random samples. The fluctuations in their grade distribution was exhibited.

7. *An application of matrix methods to a recent postulate system for m-valued functional calculi*, by Professor D. L. Webb, University of Arizona.

Professor Webb discussed an application of row matrices to the postulate system of J. B. Rosser and A. R. Turquette (*Journal of Symbolic Logic*, vol. 13, pp. 177–192). He compared the properties of the operators of the 2-valued and the m-valued cases.

8. *Further analysis of Fermat's congruence for composite moduli*, by Professor A. W. Boldyreff, University of New Mexico.

A method of finding all values of x satisfying the congruence $x^{n-1} \equiv 1 \pmod{n}$, when n is a product of two odd primes was discussed.

9. *Interpretation of singular solutions of ordinary differential equations of the first order by geometric means*, by Professor R. L. Westhafer, New Mexico College of A. & M. A.

Using the definition of singular and regular line element of the differential equation $F(x, y, p) = 0$ as given by E. Kamke, in which the locus in 3-space of $F(x, y, z) = 0$ is considered, the types of loci which give rise to singular line elements are seen to be isolated points and curves and their limit points, three dimensional regions of points, intersections of surfaces, and boundary points of the projections of the locus on the xy-plane.

10. *On certain analogies between measure and category*, by Manfred Fliess, New Mexico College of A. & M. A., introduced by the Secretary.

11. *On the three dimensional distribution of a bomb*, by M. S. Hendrickson, University of New Mexico, read by R. C. Hildner.

12. *A problem in maximum range for rockets*, by Keith Guard, New Mexico College of A. & M. A., introduced by the Secretary.

13. *Results of placement tests for sectioning college algebra*, by H. P. Rogers, University of New Mexico.

14. *On the high school training in mathematics of our freshmen*, by Professor Earl Walden, New Mexico College of A. & M. A.

B. D. ROBERTS, *Secretary*

THE APRIL MEETING OF THE METROPOLITAN NEW YORK SECTION

The eighth annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at Brooklyn College, Brooklyn, New York, on Saturday, April 9, 1949. Professor T. F. Cope, Collegiate Vice-Chairman of the Section presided at the morning session, and Professor R. A. Johnson, Chairman of the Section, presided at the afternoon session.

One hundred and twenty-five persons attended the sessions, including the following seventy-three members of the Association: Brother Bernard Alfred, R. G. Archibald, H. C. Ayres, Frances E. Baker, Samuel Borofsky, C. B. Boyer, A. D. Bradley, Benjamin Braverman, Paul Brock, A. B. Brown, Jewell Hughes Bushey, Hobart Bushey, Margaret C. Byrne, John Clark, T. F. Cope, J. E. Darraugh, J. G. Deutsch, I. A. Dodes, J. N. Eastham, J. E. Eaton, Samuel Ei-

lenberg, Carolyn Eisele, J. M. Feld, Edward Fleisher, R. M. Foster, Marion C. Gray, Harriet Griffin, George Grossman, G. C. Helme, T. R. Humphreys, Solomon Hurwitz, L. C. Hutchinson, R. A. Johnson, Aida Kalish, L. S. Kennison, H. S. Kieval, M. S. Klamkin, Edna Kramer-Lassar, A. W. Landers, J. A. Larrivee, Nathan Lazar, A. A. LePori, M. E. Levenson, Emanuel Levine, May H. Maria, F. H. Miller, L. T. Moore, A. J. Mortola, D. S. Nathan, C. V. Newsom, M. A. Nordgaard, P. B. Norman, Walter Prenowitz, James Quinn, R. M. Reed, Moses Richardson, G. J. Ross, S. G. Roth, C. T. Salkind, Arthur Schack, Harry Schor, Aaron Shapiro, Edward Shapiro, James Singer, F. E. Smith, E. R. Stabler, Mildred M. Sullivan, Nelly Ullman, Israel Wallach, Alan Wayne, J. M. Wolfe, Margaret Y. Woodbridge, H. J. Zimmerberg.

The officers elected at the business meeting were: Chairman, B. P. Gill, The City College of the College of the City of New York; Collegiate Vice-Chairman, L. F. Ollman, Hofstra College; High School Vice-Chairman, Alan Wayne, Brooklyn High School of Automotive Trades; Secretary, James Singer, Brooklyn College; Treasurer, Aaron Shapiro, Midwood High School. The ninth annual meeting will be held in the Spring of 1950.

The following papers were presented:

1. *Address of welcome*, by Dr. W. R. Gaede, Dean of Faculty, Brooklyn College.

2. *Generalizations of the law of cosines*, by Professor L. W. Cohen, Queens College (introduced by Professor T. F. Cope).

3. *Interference patterns in the teaching of mathematics*, by Dr. Nathan Lazar, Bureau of Reference, Research and Statistics, Board of Education, New York City.

Experienced teachers of mathematics will not let a homework assignment consist of one type of exercise only, but will choose examples of each of several types. Nevertheless it is a common practice when introducing a topic to present only one aspect and to give almost exclusive drill on examples illustrating that new topic before any other topic is presented, no matter how closely related these topics may be.

This writer claims that the repetition of even one exercise without any significant variation in its pattern tends to encourage the student to perform mathematical operations without insight and understanding, and to make adventitious and unjustifiable inductions from the "model example" worked out. Further repetition of the same pattern will encourage the student to believe in the correctness of his procedure, and make it harder for him to adjust himself to a new type of the same pattern where his ad hoc hypothesis will not work.

It is therefore recommended that the successive presentation of examples illustrating the same mathematical concept be so varied as to prevent the formation of undesirable associations. This approach may not yield the immediate feeling of success that the traditional one gives. It may even require at first more time and more careful preparation than one is accustomed to. This effort will, however, be justified by the end result—a real understanding of the nature of mathematical operations, of the purposes underlying them, and of the mathematical laws governing them.

4. *The indebtedness of Greek to Babylonian exact sciences*, by Professor O. Neugebauer, Brown University (introduced by Professor T. F. Cope).

In 1928 J. K. Fotheringham published in the *Monthly Notices of the Royal Astronomical Society*

an article on *The Indebtedness of Greek to Chaldean Astronomy.* In discussing this subject twenty years later, we can add Babylonian mathematics to the comparison. The vast increase of material has also considerably contributed to the complication of the problem and to the realization that there are huge gaps in our records. Greek and demotic papyri show that Babylonian arithmetical methods were used simultaneously with the geometrical astronomical models of Hipparchus and Ptolemy. Babylonian algebra developed methods which are paralleled in Greek "geometrical algebra." Babylonian number theory shows a development previously assumed to be "Pythagorean." The "Pythagorean" theorem was used a thousand years before Pythagoras. Yet it is very difficult to indicate the process by which these discoveries were transmitted to the Greeks. Theoretical astronomy still remains the only field where we are able to see some points of direct contact between Greek and Babylonian science.

5. *Some educational trends in New York State and their significance to the mathematicians*, by Dr. C. V. Newsom, Assistant Commissioner for Higher Education, University of State of New York.

This paper discussed some of the trends in education on all levels in New York State. Particular attention was given to the new program for the training of elementary and secondary teachers, the development of the new syllabus in mathematics for the secondary level, and the expanded program anticipated by the State in the field of higher education. The report contained many personal observations as the result of the author's actual visitation of institutions.

JAMES SINGER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-third Annual Meeting, New York City, December 30, 1949.

International Congress of Mathematicians, Cambridge, Massachusetts, August 30–September 6, 1950.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN

ILLINOIS, Southern Illinois University, Carbondale, May 12–13, 1950.

INDIANA, Wabash College, Crawfordsville, April 29, 1950.

IOWA, State University of Iowa, Iowa City, April 21–22, 1950.

KANSAS, Spring, 1950.

KENTUCKY, University of Kentucky, Lexington, April 29, 1950.

LOUISIANA-MISSISSIPPI, Centenary College, Shreveport, Louisiana, Spring, 1950.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Fall, 1949.

METROPOLITAN NEW YORK, Spring, 1950.

MICHIGAN, March, 1950.

MINNESOTA, Macalaster College, St. Paul, May 6, 1950.

MISSOURI, Spring, 1950.

NEBRASKA, Nebraska Wesleyan University, Lincoln, May 6, 1950.

NORTHERN CALIFORNIA, Berkeley, January 28, 1950.

OHIO, Denison University, Granville, April 22, 1950.

OKLAHOMA

PACIFIC NORTHWEST, University of Washington, Seattle, June, 1950.

PHILADELPHIA, Haverford College, November 26, 1949.

ROCKY MOUNTAIN, University of Denver, April, 1950.

SOUTHEASTERN, University of Florida, Gainesville, March, 1950.

SOUTHERN CALIFORNIA, Immaculate Heart College, Hollywood, March 11, 1950.

SOUTHWESTERN, Spring, 1950.

TEXAS, Abilene, Spring, 1950.

UPPER NEW YORK STATE, Syracuse University, Spring, 1950.

WISCONSIN, Marquette University, Milwaukee, May, 1950.

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By L. J. GREEN, Case Institute of Technology

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MATHEMATICAL NOTES

Edited by E. F. BECKENBACH, University of California, Los Angeles, and Institute for Numerical Analysis of the National Bureau of Standards

CLASSROOM NOTES

Edited by C. B. ALLENDOERFER, Haverford College

PROBLEMS AND SOLUTIONS

Edited by HOWARD EVES, Oregon State College, and E. P. STARKE, Rutgers University

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Numbers in blackface type refer to problems, those in lightface to pages.

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Edited by L. F. OLLMANN, Hofstra College

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Edited by EDITH R. SCHNECKENBURGER, University of Buffalo

GENERAL INFORMATION

PERSONAL INFORMATION

The following persons presented papers at meetings of the Association and its Sections.

Personal Mention. This section contains the names of officers of the Association and its Sections, persons mentioned in the Department of News and Notices, and those conducting the business of the Association. The list does not include names of new members or of attendants at meetings.

NECROLOGY

REPORTS AND ANNOUNCEMENTS OF THE ASSOCIATION AND ITS SECTIONS

MEETINGS AND ANNOUNCEMENTS OF THE ASSOCIATION

MEETINGS OF ITS SECTIONS

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The AMERICAN MATHEMATICAL MONTHLY

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NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the Monthly should be sent to the Secretary-Treasurer, H. M. Gehman, University of Buffalo, Buffalo 14, N. Y. Change of address must reach the Secretary-Treasurer about six weeks before the change can become effective.

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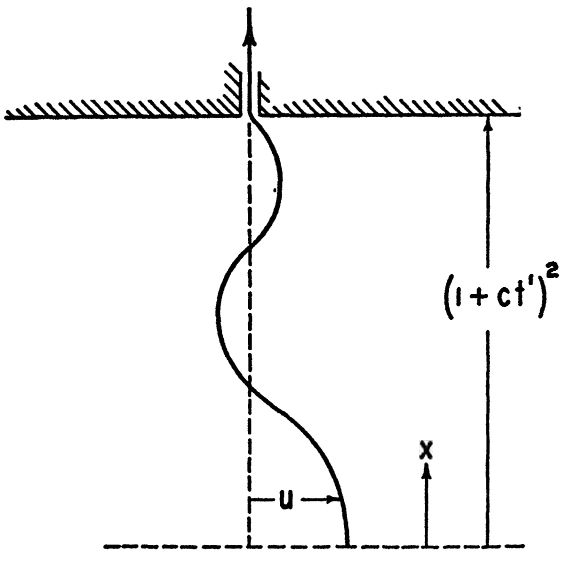
THE SPAGHETTI PROBLEM

G. F. CARRIER, Brown University

1. Introduction. There are two problems concerned with the lateral vibrations of strings which should be of considerable popular and academic interest. They are: (1) the problem of describing the motion of a cord of finite length as it is accelerated vertically through an orifice (this is related in an obvious way to the title), and (2) the guitar string oscillation problem wherein one slides a rigid support along an oscillating elastic string. Experimentally, the results to be expected are well-known. In the first case the amplitude of the oscillation usually increases sufficiently so that the cord frequently "slaps" the plate containing the orifice. In the second, the intensity of the generated acoustic waves seems not to increase but the pitch certainly rises.

Here, we shall discuss the eigenvalue problems associated with these phenomena and shall attempt to predict the proper behaviors. In order that the problem be tractable and the methods elementary, we shall, of course, use a linearized (small oscillation) theory.



2. The Spaghetti Problem. Let us consider a cord of density ρ and cross sectional area A, which is being drawn vertically upward through an orifice with constant acceleration. In fact, let the motion of the cord be such that the distance of the lower end from the orifice is $(1+ct')^2$, so that $2c^2$ is the acceleration of the string and t' is the time. If we now fix the origin of the distance coordinate x (note the figure) at the lower end of the string, its effective unit weight in the accelerating system is $\rho(g+2c^2)=w$, where g is the acceleration of gravity.

The well-known equation governing the small dynamic lateral displacement u of a hanging inextensible cord is

$$(wxu_x)_x = \rho u_{t't'}. \tag{1}$$

If we define $t = t'\sqrt{g+2c^2}$, then (1) becomes

$$(xu_x)_x = u_{tt}. \tag{2}$$

The boundary conditions of our problem require that $u(0, t)$ be finite and* that $u([1+bt]^2, t) = 0$. It is desirable evidently to use a variable substitution [say $\xi(x, t)$, $\eta(x, t)$] such that the boundary conditions take the nicer form $v(0, \eta)$ finite, $v(a, \eta) = 0$, where $v(\xi, \eta) = u(x, t)$. One such substitution which proves convenient is

$$\xi = 2b^2x/(1 + bt)^2, \qquad \eta = \ln (1 + bt).$$

In fact, if this substitution is made and if we look for a solution of (2) in the form $u = v = f(\xi)\, e^{i\beta\eta}$, then substitution into (2) leads to the following ordinary differential equation for $f(\xi)$:

$$- \xi(1 - \xi)f_{\xi\xi} + \left[\left(\frac{3}{2} - i\beta\right)\xi - 1\right]f_\xi - \frac{\beta^2 - i\beta}{4}f = 0. \tag{3}$$

This is the conventional form of the hypergeometric equation, and thus we could write the solutions as hypergeometric functions and anticipate that for certain eigenvalues β_ν (probably complex) the boundary conditions given above would be satisfied. However, in order to get a better insight into the nature of the eigenvalues and eigenfunctions, it is worthwhile to put Equation (3) into a canonical form. To accomplish this, we write,

$$f(\xi) = \xi^{-1/2}(1 - \xi)^{-1/4}e^{i\beta/2 \ln x (1-\xi)}\cdot g(\xi) \tag{4}$$

and when this is placed in (3), the equation for g becomes

$$g'' + \left(\frac{\lambda}{\xi(1 - \xi)^2} + \frac{1}{4\xi^2} + \frac{1}{16\xi(1 - \xi)}\right)g = 0 \tag{5}$$

where $\lambda = (3 - 4i\beta + 4\beta^2)/16$. The boundary conditions are $g(0) = g(2b^2) = 0$.

This system is of the Sturm-Liouville type and its eigenvalues λ_n are real. Furthermore the λ_n tend to ∞ as $n \to \infty$. However,

$$\beta_n = \frac{i}{2} + \sqrt{4\lambda_n - 1}, \tag{6}$$

and thus for all except possibly a finite set of values, each of the β_n has a positive imaginary part. Now if b is negative (the spaghetti is then being drawn in) and $\operatorname{Im} \beta_n > 0$, then $\operatorname{Re} (i\beta\eta)$ is a positive increasing function of t, and, in fact, the amplitude of the motion must go exponentially to infinity as the undrawn por-

* We define b so that $ct' = bt$.

tion of the cord decreases in length to zero. Actually, when the amplitude becomes very large, the linear equation on which the theory is based becomes invalid but the growth of amplitude is nevertheless predicted for the earlier stages of the motion. From the practical point of view, one should note that the air and any other fluids present will produce a damping action which may, for sufficiently small b, actually counteract the growth predicted in the undamped problem. However, for large b this is not to be anticipated.

3. The Guitar String Problem. If we now turn to the second of our problems, we must consider an elastic string whose deflection is governed by the differential equation

$$u_{xx} = u_{tt}, \tag{7}$$

if we choose such a unit system that the propagation velocity is unity. If one end of the string is fixed and taken as the origin, and if a support which allows no lateral motion of that point on the string with which it is in contact is kept at $x = 1 + bt$, then the boundary conditions of the problem are

$$u(0, t) = u(1 + bt, t) = 0. \tag{8}$$

Rather than state explicit initial conditions, however, we shall again merely look for eigenfunctions and any initial conditions can then be realized by choosing an appropriate linear combination of the eigenfunction.

In this problem, it is convenient to choose

$$\xi = \frac{bx}{1 + bt}, \qquad \eta = \ln (1 + bt). \tag{9}$$

Defining $u(x, t) \equiv v(\xi, \eta) = \alpha(\xi)\, e^{i\beta\eta}$ we obtain in place of Equation (7), the equation

$$(1 - \xi^2)\alpha_{\xi\xi} - 2\xi(1 - i\beta)\alpha_\xi + i\beta(1 - i\beta)\alpha = 0. \tag{10}$$

This equation could also be put in hypergeometric form, but again it is convenient to write

$$\alpha = (1 - \xi^2)^{-1/2} e^{i(\beta/2)\ln(1-\xi^2)} h(\xi) \tag{11}$$

and then h must satisfy

$$h'' = \frac{1 + \beta^2}{(1 - \xi^2)^2}\, h = 0 \tag{12}$$

where

$$h(0) = h(b) = 0.$$

Using the arguments of the foregoing section, we see that almost all the β_n are real and that the amplitude of oscillations is this time not an increasing function of the time.

There is an interesting observation which can be made concerning the result of these problems. Solutions of the systems given by Equations (5) and (12) which are differentiable in the interval $0<\xi<2b^2$ and $0<\xi<b$ exist only where $2b^2$ and b, respectively, are less than unity. That this should be true can be observed by referring to the original problems. These problems are associated with certain hyperbolic differential equations. The solutions of such equations are completely determined inside a certain domain of influence* when only the initial conditions and the boundary conditions at $x=0$ are stated. Thus if the curve $x_0=(1+bt)^2$ [in the first problem] or $x_0=1+bt$ [in the second] should lie within this domain then, in general, no continuous solution can exist which obeys the given conditions on $x=x_0(t)$. Physically this states that the support moves along the string at a speed greater than the wave propagation speed associated with the phenomenon under investigation. Thus the lack of existence of solutions outside a certain range of b is consistent with the physical facts.

A RECURRING THEOREM ON DETERMINANTS

OLGA TAUSSKY, National Bureau of Standards

1. Introduction. This note concerns a theorem (Theorem I) on determinants [**0–21, 25–27**] of which proofs are being published again and again; on the other hand, the theorem is not as well known as it deserves to be. The theorem has arisen in many varied connections as is indicated by the titles of the papers quoted. Although it can be proved in a very simple manner, some of the proofs that have been given are very complicated. The theorem deals with determinants of matrices with a "dominant" main diagonal. Such matrices are particularly useful.

In what follows the theorem and several generalizations are discussed. A rather important application to estimating characteristic roots of general matrices with complex elements is mentioned. By applying these estimates to the matrices with a "dominant" main diagonal more general results are obtained.

2. Complex matrices. It will be convenient to denote by A_i the sum of the moduli of the non-diagonal terms of the ith row of a matrix $\mathbf{A}=(a_{ij})$.

THEOREM I. *If (a_{ik}) is an $n\times n$ matrix with complex elements such that*

$$|a_{ii}| > A_i, \qquad i = 1, \cdots, n \tag{1}$$

then $|a_{ik}|\neq 0$.

* In the x, t, plane.

Proof. Assume that $|a_{ik}|=0$. The system of equations

$$
\begin{aligned}
&a_{11}x_1 + \cdots + a_{1n}x_n = 0\\
&\quad\vdots\\
&a_{n1}x_1 + \cdots + a_{nn}x_n = 0
\end{aligned}
\tag{2}
$$

then has a non-trivial solution $x_1, \cdots, x_n$. Let r be one of the indices for which $|x_i|$, $(i=1, \cdots, n)$, is maximum. Consider the rth equation in System (2). It implies

$$
|a_{rr}||x_r| \leqq \sum_{k=1,k\neq r}^{n} |a_{rk}||x_k| \leqq |A_r| x_r|
$$

which is in contradiction with (1). Hence $|a_{ik}|\neq 0$.

THEOREM II. *Let (a_{ik}) be an $n\times n$ matrix with complex elements such that*

$$
|a_{ii}| \geqq A_i, \qquad i = 1, \cdots, n \tag{3}
$$

with equality in at most $n-1$ cases. Assume further that the matrix cannot be transformed to a matrix of the form

$$
\begin{pmatrix} P & U \\ 0 & Q \end{pmatrix}
$$

by the same permutation of the rows and columns, where P and Q are square matrices and 0 consists of zeros. It follows that the determinant $|a_{ik}|\neq 0$.

Proof. The proof is similar to that for Theorem I. Assume, for example, that the first relation in (3) is not an equality. From this it follows that $|x_r|>|x_k|$ for at least one value of k. Hence the rth equation of (2) is in contradiction with (3), provided not all $a_{ri}=0$ for which $|x_r|>|x_i|$. If this, however, is the case, then the rth row contains $n-s$ zeros where s is the number of suffixes j for which $|x_j|=|x_r|$. All the s corresponding rows contain $n-s$ zeros in the same places. It follows that the matrix is of the form which was excluded.

3. Real matrices. If all the relations in (3) become equalities the theorem ceases to hold, as is shown by any real matrix (a_{ik}) with $a_{ik}\leqq 0$ for $i\neq k$ and $\sum_{k=1}^{n} a_{ik}=0$, $i=1, \cdots, n$.

In this connection the following result was established: **[6, 10, 14]**

THEOREM III. *Let (a_{ik}) be a real $n\times n$ matrix such that $a_{ii}\geqq 0$ and $a_{ik}\leqq 0$ for $i\neq k$. Assume in addition that*

$$
a_{ii} \geqq A_i, \qquad i = 1, \cdots, n \tag{4}
$$

and that the matrix is not of the type excluded in Theorem II. The determinant then vanishes if and only if $\sum_{k=1}^{n} a_{ik}=0$, $i=1, \cdots, n$.

Proof. The determinant obviously vanishes if $\sum_{k=1}^{n} a_{ik}=0$. Assume, conversely, that the determinant vanishes. Consider the System (2). From the arguments used for the proof of Theorem I and II it follows that $|x_1|=|x_2| = \cdots = |x_n|$. Any equation of the system (2) then implies $\sum_{k=1}^{n} a_{ik}=0$.

THEOREM IV. *If* (a_{ik}) *is a real* $n \times n$ *matrix such that*

$$a_{ii} > A_i, \qquad i = 1, \cdots, n \tag{5}$$

then $|a_{ik}|>0$.

Proof. This theorem can be proved by induction [12]. A different proof is obtained by using the fact that Theorem IV is obviously true if $a_{ik}=0$ for $i \neq k$. Using this and Theorem I a proof can be obtained by continuity arguments.

4. Generalizations.

THEOREM V. *If* (a_{ik}) *is an* $n \times n$ *matrix such that*

$$|a_{ii}|\,|a_{kk}| > A_i A_k, \qquad i, k = 1, \cdots, n;\ i \neq k, \tag{6}$$

then $|a_{ik}| \neq 0$. [14, 23]

Proof. Note that the relations (6) imply $|a_{ii}|>A_i$ for all i but one. If these inequalities are satisfied for all i, the relations (1) hold. In this case the theorem is known. Assume, for example, that

$$|a_{11}| < A_1; \quad |a_{ii}| > A_i, \qquad i = 2, \cdots, n;$$
$$|a_{ii}|\,|a_{kk}| > A_i A_k, \quad i, k = 1, \cdots, n;\ i \neq k.$$

Without loss of generality it may be assumed that $a_{11}=1$ so that the above relations can be replaced by

$$1 < A_1; \quad |a_{ii}| > A_i, \qquad i = 2, \cdots, n;$$
$$|a_{ii}| > A_1 A_i, \qquad i = 2, \cdots, n;$$
$$|a_{ii}|\,|a_{kk}| > A_i A_k, \quad i, k = 2, \cdots, n;\ i \neq k.$$

Multiply the first column of the matrix (a_{ik}) by A_1. It will be sufficient to prove that this new matrix is non-singular. Denote its elements by a'_{ik} the numbers corresponding to A_i by A'_i. The following inequalities hold:

$$|a'_{11}| = A'_1 = A_1; \qquad |a'_{ii}| > A'_i, \qquad i = 2, \cdots, n.$$

Since the matrix (a'_{ik}) satisfies (3), it follows that $|a'_{ik}| \neq 0$. Hence also $|a_{ik}| \neq 0$.

THEOREM VI. *Theorems I and II are best possible insofar as the inequalities involved cannot be replaced by weaker ones.*

Proof. Suppose that

$$|a_{11}| + \epsilon > A_1; \qquad |a_{ii}| > A_i, \qquad i = 2, \cdots, n,$$

where $\epsilon > 0$, but is arbitrarily small. The result follows because the matrix

$$\begin{pmatrix} \epsilon/2 & \epsilon \\ 1/2 & 1 \end{pmatrix}$$

for which these relations are satisfied is clearly singular.

5. Application. If Theorem II is applied to the characteristic determinant of any $n \times n$ matrix (a_{ik}) with complex coefficients it follows that the characteristic roots must lie inside the circles with centres a_{ii} and radii A_i [**9, 10, 14, 18, 22, 24, 25, 28–30**]. A boundary point can only be a characteristic root if it is also on the boundary of the $n-1$ other circles.

Similarly, the application of Theorem VI shows that the roots lie inside or on the boundary of a set of $n(n-1)/2$ Cassini ovals.

Now apply the circles in particular to a real matrix (a_{ik}) of the type considered in Theorems III and IV. These circles may pass through the origin, but otherwise lie entirely to the right of the imaginary axis. This gives

THEOREM VII. *All the non-zero characteristic roots of matrices with real elements which satisfy* (4) *or* (5) *have positive real parts.*

If none of the non-diagonal elements is positive it has been shown that the root with the smallest real part is real [**10**].

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MATHEMATICAL NOTES

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AN INTEGRAL TEST FOR CONVERGENCE

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In a recent note [3] M. Ward called attention to, and gave a simple proof of, G. H. Hardy's formulation of the integral test for the convergence of series. At about the same time, M. F. Egan [1] gave a more general form of the theorem. Here we give a still shorter proof of the theorem in Egan's form.

THEOREM. *Let $f(x)$ be a complex-valued function of bounded variation on the whole interval $0 \leqq x < \infty$. Then the series $\sum_{n=0}^{\infty} f(n)$ and the integral $\int_0^{\infty} f(x)dx$ converge or diverge together.*

We use the notation $\int_a^b |df(x)|$ for the total variation of $f(x)$ from a to b. By hypothesis $\int_0^x |df(t)|$ is bounded; since it does not decrease, it approaches a finite limit; consequently $\int_x^y |df(t)| \to 0$ as $x, y \to \infty$ and so $|f(x)-f(y)| \leqq \int_x^y |df(t)| \to 0$ as $x, y \to \infty$. Hence $\lim_{x\to\infty} f(x)$ exists; if it is not zero, both the series and the integral diverge. We assume from now on that $f(x) \to 0$.

We have for $0 < c \leqq 1$ and each nonnegative integer n,

$$(1) \qquad \left| \int_n^{n+c} f(x)dx - cf(n) \right| \leqq \int_n^{n+c} |f(x) - f(n)| \, dx \leqq c \int_n^{n+c} |df(x)|.$$

From (1) it follows that, since $f(n) \to 0$, $\lim_{x\to\infty} \int_0^x f(t)dt$ exists if and only if $\lim_{n\to\infty} \int_0^n f(t)dt$ exists. Next, taking $c=1$ in (1), and adding the inequalities corresponding to $n=p, p+1, \cdots, q-1$, we obtain

$$(2) \qquad \left| \int_p^q f(x)dx - \sum_{n=p}^{q-1} f(n) \right| \leqq \int_p^q |df(x)|,$$

so that as $p, q \to \infty$ the left side tends to 0. The theorem follows.

We remark that, when the function $f(x)$ is continuous at $x=0, 1, 2, \cdots$, the inequality (2) can be obtained from the formula

$$\int_p^q f(x)dx - \sum_{n=p}^{q-1} f(n) = \int_p^q (x - [x])df(x),$$

which is a variant of the Euler [2] (or Euler-Maclaurin) summation formula, the last integral being a Riemann-Stieltjes integral. Under this more restrictive hypothesis we have

$$\int_0^\infty f(x)dx - \sum_{n=0}^\infty f(n) = \int_0^\infty (x - [x])df(x)$$

and

$$(3) \qquad \left| \int_0^\infty f(x)dx - \sum_{n=0}^\infty f(n) \right| \leqq \int_0^\infty | df(x) |$$

whenever the series and integral are convergent. The inequality (3) follows from (2) even when $f(x)$ is discontinuous for integer values of x; and (3) is a "best" inequality in the sense that equality in (3) can hold when the right member is positive and finite.

Egan's theorem includes the more familiar case of a nonincreasing $f(x)$, while Ward's theorem does not: Ward assumes that $\int_0^\infty |f'(x)| dx$ converges and that $f'(x)$ is integrable to $f(x)$, which need not be the case for a monotonic $f(x)$.

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SUMMABLE SERIES AND INTEGRALS

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In our reading we frequently encounter the idea of summability of series and integrals. We understand that there are some non-convergent series and integrals to which some sort of "sum" can be attached. But are we able to give, offhand, a few simple examples of such series and integrals? The present note is to supply one method of construction of these series and integrals.

The literature dealing with operations on series is very extensive; we cite but a few references at the end of the paper. There are now many methods of finding the sum of a series, most of which grew out of the method described in this paper.

Most discussions of the subject begin with the series

$$(1) \qquad 1 - 1 + 1 - 1 + \cdots,$$

with partial sums

$$(2) \qquad S_0 = 1, S_1 = 0, S_2 = 1, S_3 = 0, \cdots.$$

The series (1) is non-convergent, but the sequence of arithmetic means,

$$S_0, \frac{S_0 + S_1}{2}, \frac{S_0 + S_1 + S_2}{3}, \cdots,$$

does converge to the "sum" 1/2. We say that the series is summable $(H, 1)$, the Hölder method, to the value 1/2.

Let us write*

$$(3)\qquad \begin{cases} \overset{m+1}{\sigma_n} = \dfrac{\overset{m}{\sigma_0} + \overset{m}{\sigma_1} + \cdots + \overset{m}{\sigma_n}}{n+1}, & m = 0, 1, 2, \cdots; n = 0, 1, 2, \cdots; \\ \overset{0}{\sigma_n} = S_n, & n = 0, 1, 2, \cdots. \end{cases}$$

If σ_n^k converges to σ as $n \to \infty$ we say that the series is summable (H, k) to σ. Suppose we desire an example of a series summable $(H, 2)$ to the value 1/2, but not summable $(H, 1)$. We can let

$$\overset{1}{\sigma_0} = 1,\ \overset{1}{\sigma_1} = 0,\ \overset{1}{\sigma_2} = 1,\ \overset{1}{\sigma_3} = 0, \cdots,$$

which leads to

$$S_0 = 1,\ S_1 = -1,\ S_2 = 3,\ S_3 = -3,\ S_4 = 5, \cdots,$$

which in turn gives the series

$$(4)\qquad 1 - 2 + 4 - 6 + 8 - 10 + \cdots.$$

This series is, then, $(H, 2)$ summable to 1/2, but is not $(H, 1)$ summable.

In general, we can construct a series which is (H, k) summable to 1/2, but is not $(H, k-1)$ summable, by letting the $(k-1)$st means take on successively the values in the sequence

$$1, 0, 1, 0, 1, \cdots.$$

The cases $k=3$ and 4 are easily worked out to be

$$(5)\qquad 1 - 4 + 14 - 32 + 58 - 92 + 134 - 184 + \cdots,$$

$$(6)\qquad 1 - 8 + 46 - 156 + 386 - 784 + 1398 - 2276 + \cdots,$$

respectively.

We note that the series (1), (4), (5), and (6) can be expressed by substituting the value $x=1$ in the series expansion for

$$\frac{1}{1+x},\quad \frac{1+x^2}{(1+x)^2},\quad \frac{1-x+5x^2-x^3}{(1+x)^3},\quad \text{and}\quad \frac{1-4x+20x^2-16x^3+7x^4}{(1+x)^4},$$

respectively. This immediately suggests the following theorem:**

---

* It is traditional to describe this as Hölder's method, although there is some difference of opinion on this matter. It is held that the method properly should be credited to Frobenius (see Crelle's Journal. Bd. 89, 1880, pp. 262–264).

** This theorem was originally a "conjecture." A referee furnished a proof which, in slightly modified form, appears here.

THEOREM. *Let $f(x)$ be a polynomial in x such that $f(-1)\neq 0$. Then the series expansion for*

$$f(x)/(1+x)^k$$

with x put equal to 1, *is (H, k) summable to $f(1)/2^k$, but is not $(H, k-1)$ summable.*

Proof: We shall replace H in the statement by C (Cesàro). This is equivalent by Schnee's theorem (see [4], p. 481).

Consider

$$F(x) = f(x) - \frac{f(1)}{2^k}(1+x)^k.$$

This is a polynomial in x. Moreover, $F(1)=0$. Hence,

$$F(x) = (1-x)P(x),$$

where $P(x)$ is a polynomial in x. Therefore, we can write

$$\sum_{n=0}^{\infty} u_n x^n = \frac{f(x)}{(1+x)^k} = \frac{f(1)}{2^k} + \frac{(1-x)P(x)}{(1+x)^k},$$

or

$$(1-x)^{-(k+1)}\sum_{n=0}^{\infty} u_n x^n = \frac{f(1)}{2^k}(1-x)^{-(k+1)} + P(x)(1-x^2)^{-k}.$$

As a result, $\sum_{u=0}^{\infty} u_n$ is (C, k) summable to $f(1)/2^k$ (see [2], p. 314). It remains to show that $\sum u_n$ is not $(C, k-1)$ summable.

Let us write

$$\sum u_n x^n = a_0(1+x)^{-k} + a_1(1+x)^{-k+1} + a_2(1+x)^{-k+2} + \cdots, \tag{7}$$

where $a_0=f(-1)\neq 0$. There will be a finite number of the a_i. In place of (7) we can write

$$\begin{aligned}\sum u_n x^n = {} & a_0 + a_1 + a_2 + \cdots \\ & - \left[a_0\binom{k}{1} + a_1\binom{k-1}{1} + a_2\binom{k-2}{1} + \cdots\right]x \\ & + \left[a_0\binom{k+1}{2} + a_1\binom{k}{2} + a_2\binom{k-1}{2} + \cdots\right]x^2 \qquad (8) \\ & - \left[a_0\binom{k+2}{3} + a_1\binom{k+1}{3} + a_2\binom{k}{3} + \cdots\right]x^3 + \cdots.\end{aligned}$$

The series $\sum u_n$ then is the series (8) with x set equal to 1.

Suppose the series $\sum u_n$ is $(C, k-1)$ summable. Then (by [4], p. 484) we

must have, for $n\to\infty$,

$$u_n = o(n^{k-1}). \tag{9}$$

But,

$$u_n \sim a_0 n^{k-1}/(k-1)! + a_1 n^{k-2}/(k-2)! + \cdots .$$

Hence

$$\frac{u_n}{n^{k-1}} \to \frac{a_0}{(k-1)!} \neq 0,$$

and the fact that (9) does not hold completes the proof of the theorem.

In the case of integrals, the Hölder analogues are the successive integrals

$$H_0(x) = \int_0^x f(t)dt.$$

$$H_1(x) = \frac{1}{x}\int_0^x H_0(t)dt,$$

$$\cdots\cdots\cdots\cdots\cdots,$$

$$H_{k+1}(x) = \frac{1}{x}\int_0^x H_k(t)dt.$$

If $H_n(x)\to H$ as $x\to\infty$, then the integral $\int_0^\infty f(t)dt$ is said to be summable (H, n) to H.

We can build up examples as indicated for series. The following are some results of such procedure:

$$\int_0^\infty \sin 2t dt,$$

$$\int_0^\infty (2\sin 2t + 2t\cos 2t)dt,$$

$$\int_0^\infty (4\sin 2t + 10t\cos 2t - 4t^2\sin 2t)dt.$$

These integrals are each summable to the value 1/2, by the respective methods $(H, 1)$, $(H, 2)$, and $(H, 3)$, and are not summable by any lower Hölder method.

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CLASSROOM NOTES

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LIMITS BY "CONSECUTIVE RATIONALS"

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Most mathematicians have seen simple demonstrations of the monotonicity of $(1+1/n)^n$ or $(1+1/n)^{n+1}$, where n increases through integral values. Even with the existence of these limits established, it usually remains to discuss $(1+h)^{1/h}$ with h rational, or real. The following direct demonstration of the existence of lim $(1+h)^{1/h}$ when h tends to zero through positive rational values, may be of interest. It is desired to prove that if $0<r<s$, r and s being rational, then

$$(1 + r)^{1/r} > (1 + s)^{1/s}. \tag{1}$$

Any two rational numbers r and s may be taken over a common denominator, as p/q and $(p+h)/q$, and by considering consecutive pairs $(p+i)/q$ $(i=0, 1, \cdots, h)$, we see that (1) will follow from

$$(1 + p/q)^{q/p} > (1 + (p + 1)/q)^{q/(p+1)}, \tag{2}$$

where p and q are positive integers. Now (2) is equivalent to

$$(1 + p/q)^{p+1} > (1 + (p + 1)/q)^{p}. \tag{3}$$

We expand each member by the binomial theorem. The first two terms coincide, and each remaining term on the left exceeds the corresponding term on the right, if

$$\binom{p+1}{r+1}\frac{p^{r+1}}{q^{r+1}} > \binom{p}{r+1}\frac{(p+1)^{r+1}}{q^{r+1}} \qquad (r = 1, \cdots, p - 1),$$

hence if $p^{r+1}>(p-r)(p+1)^r$. The last inequality is true since the powers of p (below p^r) in the expansion of the right member have negative coefficients.

To complete the proof of the existence of $e=\lim (1+h)^{1/h}$ when h tends to zero through positive rationals, it is still necessary to prove that $(1+h)^{1/h}$ is bounded. This now follows from the boundedness of $(1+1/n)^n$ with n a positive integer, which can be demonstrated as usual by expanding this by the binomial theorem. For k negative, one may use

$$\frac{(1 - k)^{-1/k}}{(1 + k)^{1/k}} = (\phi(k))^{\psi(k)}, \text{ where } \phi(k) = \left(1 + \frac{k^2}{1 - k^2}\right)^{(1-k^2)/k^2} \to e,$$

$$\text{and } \psi(k) = k/(1 - k^2) \to 0.$$

This method, which we may call the *method of consecutive rationals* can also be used to demonstrate directly the existence of

$$g(a) = \lim_{h\to 0} (a^h - 1)/h,$$

(whence the derivative of a^x is $a^x g(a)$). We will suppose here that $a>1$, and that h tends to zero through positive rationals. Since $(a^h-1)/h$ is bounded below (by 0), it suffices to prove that $(a^h-1)/h$ is monotone decreasing as $h\to 0+$, and hence by our method, it suffices to prove that

$$\frac{a^{(p+1)/q} - 1}{(p+1)/q} > \frac{a^{p/q} - 1}{p/q},$$

where p and q are positive integers. Put $b = a^{1/q}$. Our inequality reduces to

$$p(b^{p+1} - 1) > (p+1)(b^p - 1),$$

hence to $pb^p(b-1) > b^p - 1$, or to

$$pb^p > b^{p-1} + b^{p-2} + \cdots + 1,$$

which is evident since $b>1$.

A NOTE ON DIFFERENTIAL EQUATIONS

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The following very elementary methods may be used to motivate or replace the "D Operator" methods for solving linear differential equations with constant coefficients, and the substitution $y = e^{mx}$ often used in their solution. We illustrate by solving

$$y'' + ay' + by = f(x), \tag{1}$$

where primes denote x derivatives. (The methods apply also to higher order equations). If r, s are the roots, distinct or not, of the auxiliary equation $m^2 + am + b = 0$, then we can write

$$y'' + ay' + by = y'' - (r+s)y' + rsy = y'' - ry' - s(y' - ry),$$

which shows that the substitution $u = y' - ry$ will reduce the order of equation (1), giving in fact the first order equation

$$u' - su = f(x), \tag{2}$$

whose solution is

$$u = e^{sx}\int e^{-sx} f(x)dx + Ce^{sx} = y' - ry. \tag{3}$$

Solving the second equation in (3) gives

$$y = C_1e^{rx} + Ce^{rx}\int e^{-rx}e^{sx}dx + e^{rx}\int e^{-rx}e^{sx}\int e^{-sx}f(x)dx^2.$$

The middle term, for example, simplifies in two ways, according as $r \neq s$ or $r = s$, so that the case of equal roots is handled quite naturally.

The solution of (3) given above corresponds to the "operator iteration" method. The so-called "partial fraction solution" is obtained still more easily. By symmetry we rewrite the second equation in (3), interchanging r and s, and subtract the resulting equation from (3), thus solving algebraically.

WITH, OR WITHOUT, MOTIVATION?*

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The following lines present the same proof twice, first briefly without motivation, then broadly with motivation. I think that the comparison of these two presentations may clarify a few not quite trivial points of class-room technique.

1. Deus ex machina. A mathematical lecture should be, first of all, correct and unambiguous. Still, we know from painful experience that a perfectly unambiguous and correct exposition can be far from satisfactory and may appear uninspiring, tiresome or disappointing, even if the subject-matter presented is interesting in itself. The most conspicuous blemish of an otherwise acceptable presentation is the "deus ex machina." Before further comments, I wish to give a concrete example.†

2. Example. I wish to present the proof of the following elementary, but not too elementary, theorem: *If the terms of the sequence* $a_1, a_2, a_3, \cdots$ *are nonnegative real numbers, not all equal to* 0, *then*

$$\sum_1^\infty (a_1a_2a_3 \cdots a_n)^{1/n} < e\sum_1^\infty a_n.$$

Proof. Define the numbers $c_1, c_2, c_3, \cdots$ by

$$c_1c_2c_3 \cdots c_n = (n+1)^n$$

for $n = 1, 2, 3, \cdots$. We use this definition, then the inequality between the arithmetic and the geometric means, and finally the fact that the sequence defining e, the general term of which is $[(k+1)/k]^k$, is increasing. We obtain

$$\sum_1^\infty (a_1a_2 \cdots a_n)^{1/n} = \sum_1^\infty \frac{(a_1c_1a_2c_2 \cdots a_nc_n)^{1/n}}{n+1}$$

* Presented at the meeting of the Northern California Section of the Mathematical Association of America, San Francisco, January 29, 1949.

† I may be excused if I choose an example from my own work. See G. Pólya, Proof of an inequality, Proceedings of the London Mathematical Society (2) v. 24, 1925, p. LVII. The theorem proved is due to T. Carleman.

$$\leqq \sum_{1}^{\infty} \frac{a_1c_1 + a_2c_2 + \cdots + a_nc_n}{n(n+1)}$$

$$= \sum_{k=1}^{\infty} a_kc_k \sum_{n \geqq k} \frac{1}{n(n+1)}$$

$$(1) \qquad = \sum_{k=1}^{\infty} a_kc_k \sum_{n=k}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \sum_{k=1}^{\infty} a_k \frac{(k+1)^k}{k^{k-1}} \frac{1}{k}$$

$$< e \sum_{k=1}^{\infty} a_k.$$

3. Motivation. The crucial point of the proof is the definition of the sequence $c_1, c_2, c_3, \cdots$. This point appears right at the beginning without any preparation, as a typical "deus ex machina." What is the objection to it?

"It appears as a rabbit pulled out of a hat."

"It pops up from nowhere. It looks so arbitrary. It has no visible motive or purpose."

"I hate to walk in the dark. I hate to take a step, when I cannot see any reason why it should bring me nearer to the goal."

"Perhaps the author knows the purpose of this step, but I do not and, therefore, I cannot follow him with confidence."

"Look here, I am not here just to admire you. I wish to learn how to do problems by myself. Yet I cannot see how it was humanly possible to hit upon your . . . definition. So what can I learn here? How could I find such a . . . definition by myself?"

"This step is not trivial. It seems crucial. If I could see that it has some chances of success, or see some plausible provisional justification for it, then I could also imagine how it was invented and, at any rate, I could follow the subsequent reasoning with more confidence and more understanding."

The first answers are not very explicit, the later ones are better, and the last is the best. It reveals that an intelligent reader or listener desires two things:

First, to see that the present step of the argument is correct.

Second, to see that the present step is appropriate.

A step of a mathematical argument is appropriate, if it is essentially connected with the purpose, if it brings us nearer to the goal. It is not enough, however, that a step *is* appropriate: it should *appear so* to the reader. If the step is simple, just a trivial, routine step, the reader can easily imagine how it could be connected with the aim of the argument. If the order of presentation is very carefully planned, the context may suggest the connection of the step with the aim. If, however, the step is visibly important, but its connection with the aim is not visible at all, it appears as a "deus ex machina" and the intelligent reader or listener is understandably disappointed.

In our example, the definition of c_n appears as a "deus ex machina." Yet this step is certainly appropriate. In fact, the argument based on this definition proves the proposed theorem, and proves it rather quickly and clearly. The trouble is that the step in question, although vindicated in the end, does not appear as justified from the start.

Yet how could the author justify it from the start? The complete justification takes some time; it is supplied by the following proof. What is needed is, not a complete, but an *incomplete justification*, a *plausible provisional ground*, just a hint that the step has some chances of success, in short, some heuristic *motivation*.

In many similar cases, the motivation can be given in a few words, but this is not always so. In some cases a plausible story of the discovery supplies an attractive motivation. Such stories are much more suitable for oral presentation than for print, but just for once I take the liberty of printing such a story, even if it is not quite short. It is almost unnecessary to remind the reader that the best stories are not true; they contain, however, some elements of truth.

4. Another presentation of the example. The theorem proved in section 2 is surprising in itself. We should be less surprised, if we would know, how it was discovered. We are led to it naturally in trying to prove the following: *If the series with positive terms*

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is convergent, the series

$$a_1 + (a_1a_2)^{1/2} + (a_1a_2a_3)^{1/3} + \cdots + (a_1a_2a_3 \cdots a_n)^{1/n} + \cdots$$

is also convergent. I shall try to emphasize some motives which may help us to find the proof.

A suitable known theorem. It is natural to begin with the usual questions.‡

What is the hypothesis? We assume that the series $\sum a_n$ converges—that its partial sums remain bounded—that

$$a_1 + a_2 + \cdots + a_n \text{ not large.}$$

What is the conclusion? We wish to prove that the series $\sum(a_1, a_2 \cdots a_n)^{1/n}$ converges—that

$$(a_1a_2 \cdots a_n)^{1/n} \text{ small.}$$

Do you know a theorem that could be useful? What we need is some relation between the sum of n positive quantities and their geometric mean. *Have you seen something of this kind before?* If you ever have heard of the inequality be-

‡ About the rôle of such questions see the author's booklet, How to Solve It, Princeton, 5th enlarged printing, 1948.

tween the arithmetic and the geometric means, it has a good chance to occur to you at this juncture:

$$(A)\qquad (a_1 a_2 \cdots a_n)^{1/n} \leqq \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

This inequality shows that $(a_1, a_2 \cdots a_n)^{1/n}$ is small when $a_1+a_2+\cdots+a_n$ is not large. It has so many contacts with our problem that we can hardly resist the temptation of applying it:

$$(2)\qquad \begin{aligned} \sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} &\leqq \sum_{n=1}^{\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} \\ &= \sum_{k=1}^{\infty} a_k \sum_{n=k}^{\infty} \frac{1}{n} \end{aligned}$$

—complete failure! The series $\sum 1/n$ is divergent, the last line of (2) is meaningless.

Learning from failure. It is difficult to admit that our plan was wrong. We would like to believe that at least some part of it was right. The useful questions are: *What was wrong with our plan? Which part of it could we save?*

The series $a_1+a_2+\cdots+a_n+\cdots$ converges. Therefore, a_n is small when n is large. Yet the two sides of the inequality (A) are different when $a_1, a_2, \cdots a_n$ are not all equal, and they may be very different when $a_1, a_2, \cdots, a_n$ are very unequal. In our case, a_1 is much larger than a_n, and so there may be a considerable gap between the two sides of (A). This is probably the reason that our application of (A) turned out to be insufficient.

Modifying the approach. The mistake was to apply the inequality (A) to the quantities

$$a_1, a_2, a_3, \cdots, a_n$$

which are too unequal. Why not apply it to some related quantities which have more chance to be equal? We could try

$$1a_1, 2a_2, 3a_3, \cdots, na_n.$$

This may be the idea! We may introduce such increasing compensating factors as $1, 2, 3, \cdots n$. We should, however, not commit ourselves more than necessary, we should reserve ourselves some freedom of action. We should consider perhaps, more generally, the quantities

$$1^\lambda a_1, 2^\lambda a_2, 3^\lambda a_3, \cdots, n^\lambda a_n.$$

We could leave λ *indeterminate* for the moment, and choose the most advantageous value later. This plan has so many good features that it seems ripe for action:

$$
\begin{aligned}
\sum_{1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} &= \sum_{1}^{\infty} \frac{(a_1 1^{\lambda} \cdot a_2 2^{\lambda} \cdots a_n n^{\lambda})^{1/n}}{(1 \cdot 2 \cdots n)^{\lambda/n}} \\
&\leqq \sum_{n=1}^{\infty} \frac{a_1 1^{\lambda} + a_2 2^{\lambda} + \cdots + a_n n^{\lambda}}{n(n!)^{\lambda/n}} \qquad (3) \\
&= \sum_{k=1}^{\infty} a_k k^{\lambda} \sum_{n=k}^{\infty} \frac{1}{n(n!)^{\lambda/n}} \,.
\end{aligned}
$$

We run into difficulties. We cannot evaluate the last sum. Even if we recall various relevant tricks, we are still obliged to work with "crude equations" (notation $\approx$, instead of $=$):

$$
\begin{aligned}
(n!)^{1/n} &\approx ne^{-1}, \\
\sum_{n=k}^{\infty} \frac{1}{n(n!)^{\lambda/n}} &\approx e^{\lambda} \sum_{n=k}^{\infty} n^{-1-\lambda} \\
&\approx e^{\lambda} \int_{k}^{\infty} x^{-1-\lambda} dx \\
&= e^{\lambda} \lambda^{-1} k^{-\lambda}.
\end{aligned}
$$

Introducing this into the last line of (3) we come very close to proving

$$
\sum_{1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} \leqq C \sum_{1}^{\infty} a_k \qquad (3')
$$

where C is some constant, perhaps $e^{\lambda}\lambda^{-1}$. Such an inequality would, of course, prove the theorem in view.

Looking back at the foregoing reasoning we are led to repeat the question: "Which value of λ is most advantageous?" Probably the λ that makes $e^{\lambda}\lambda^{-1}$ a minimum. We can find this value by differential calculus:

$$
\lambda = 1.
$$

This suggests strongly that the most obvious choice is the most advantageous: the compensating factor multiplying a_n should be $n^1 = n$, or some quantity not very different from n when n is large. This may lead to the simple value $C = e$ in (3′).

More flexibility. We left λ indeterminate in our foregoing reasoning (3). This gave our plan a certain *flexibility:* the value of λ remained at our disposal. Why not give our plan still more flexibility? We could leave the compensating factor that multiplies a_n quite indeterminate; we call it c_n, and we will dispose of its value later, when we shall see more clearly what we need. We embark upon this further modification of our original approach:

$$
\begin{aligned}
\sum_{1}^{\infty}(a_1a_2\cdots a_n)^{1/n} &= \sum_{n=1}^{\infty}\frac{(a_1c_1\cdot a_2c_2\cdots a_nc_n)^{1/n}}{(c_1c_2\cdots c_n)^{1/n}}\\
&\leqq \sum_{n=1}^{\infty}\frac{a_1c_1+a_2c_2+\cdots+a_nc_n}{n(c_1c_2\cdots c_n)^{1/n}} \qquad (4)\\
&= \sum_{k=1}^{\infty}a_kc_k\sum_{n=k}^{\infty}\frac{1}{n(c_1c_2\cdots c_n)^{1/n}}.
\end{aligned}
$$

How should we choose c_n? This is the crucial question and we can no longer postpone the answer.

First, we see easily that a factor of proportionality must remain arbitrary. In fact, the sequence $cc_1, cc_2, \cdots, cc_n, \cdots$ leads to the same consequences as $c_1, c_2, \cdots, c_n, \cdots$.

Second, our foregoing work suggests that both c_n and $(c_1, c_2 \cdots c_n)^{1/n}$ should be asymptotically proportional to n:

$$c_n \sim Kn, \qquad (c_1c_2\cdots c_n)^{1/n} \sim e^{-1}Kn = K'n.$$

Third, it is most desirable that we should be able to effect the summation

$$\sum_{n=k}^{n=\infty}\frac{1}{n(c_1c_2\cdots c_n)^{1/n}}.$$

At this point, we need whatever previous knowledge we have about simple series. If we are familiar with the series

$$\sum\frac{1}{n(n+1)} = \sum\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

it has a good chance to occur to us at this juncture. This series has the property that its sum has a simple expression not only from $n=1$ to $n=\infty$, but also from $n=k$ to $n=\infty$—a great advantage! This series suggests the choice

$$(c_1c_2\cdots c_n)^{1/n} = n+1.$$

Now, visibly $n+1\sim n$ for large n—a good sign! What about c_n itself? As

$$c_1c_2\cdots c_{n-1}c_n = (n+1)^n, \qquad c_1c_2\cdots c_{n-1} = n^{n-1},$$

$$c_n = \frac{(n+1)^n}{n^{n-1}} = \left(1+\frac{1}{n}\right)^n n \sim en;$$

the asymptotic proportionality with n is a good sign. And the number e arises—a very good sign!

We choose this c_n and, after this choice, we take up again the derivation (1)—with more confidence than before.

Now, we may understand how it was humanly possible to discover that definition of c_n which appeared in section 2 as a "deus ex machina." The derivation (1) became also more understandable. It appears now as the last, and the only successful, attempt in a chain of consecutive trials, (2), (3), (4) and (1). And the origin of the theorem itself is elucidated. We see now how it was possible to discover the rôle of the number e which appeared so surprising at the outset.§

5. Demonstrative conclusions and heuristic motives. The two presentations, in section 2 and in section 4, are very different. The most obvious difference is that one is short and the other long. The most essential difference is that one gives proofs and the other plausibilities. One is designed to check the *demonstrative conclusions* justifying the successive steps. The other is arranged to give some insight into the *heuristic motives* of certain steps. The demonstrative presentation follows the accepted manner, usual since Euclid; the heuristic presentation is extremely unusual in print. Yet an ambitious teacher can use both manners of exposition. In fact, he should teach his students two things:

First, to distinguish a valid demonstration from an invalid attempt, a proof from a guess.

Second, to distinguish a more reasonable guess from a less reasonable guess.

The first point is generally recognized and I need not stress it. The second point is, in my opinion, even more important, but much more subtle. If my long presentation can serve this subtle second aim to some little degree, its length is amply justified.

Of course, various transitions or compromises are possible between the two manners of presentation.** An alert teacher should be able to find out how much stress on motivation suits his audience, how much suits himself personally, and how much time he has for motivation.

I cannot omit a final remark on logic. Some authors distinguish two branches of logic, deductive logic and inductive logic. Yet these two branches differ widely. Deductive logic is a firmly established branch of science, and became in its latest development, as symbolic logic, practically a branch of mathematics. Inductive logic is an interesting subject of philosophical discussion, but can scarcely be regarded as an established science. Deductive logic is concerned with the validity of proofs. Inductive logic which I would prefer to call *heuristic logic*, in order to emphasize its wider scope, is concerned with plausible inference only. That deductive logic is closely connected with mathematics, is widely recognized; some modern authors think, that its proper object is the analysis of the deductive structure of mathematical theories. Now I come to my point: I

§ Of course, many different heuristic motives could have guided us to the same solution. Especially, we could have raised the question: "Is $C=e$ the best (smallest) value of C for which the inequality (3′) holds?" This question (to which the answer is affirmative) could have suggested further grounds for the choice of c_n. See l.c. †).

** For a presentation intermediate between section 2 and section 4 see G. H. Hardy, J. E. Littlewood and G. Pólya, Inequalities, pp. 249–250.

think that also heuristic logic is closely connected with mathematics, but not with mathematical theories and their deductive structure, rather with mathematical problems and the invention of their solution. In fact, I think that heuristic logic could make serious progress in studying such plausible motives of the solution as were emphasized in the long presentation of our example.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 891. *Proposed by Richard and Josephine Andree, University of Oklahoma*

Given that $E^2=H$ and that each symbol represents a unique digit, find, and prove unique, the numerical solution of the following addition problem.

$$\begin{array}{rrrrrr} & & & D & \& & J \\ & A & N & D & R & E & E \\ & & & S & E & N & D \\ \hline & & & C & H & E & E & R \end{array}$$

E 892. *Proposed by J. P. Ballantine and G. E. Ulrich, University of Washington*

Let T be a given triangle, U the triangle whose vertices are the centroids of equilateral triangles described externally on the sides of T, and V the triangle whose vertices are the centroids of equilateral triangles described internally on the sides of T. Show that area $T=$area $U-$area V.

E 893. *Proposed by N. J. Fine, University of Pennsylvania*

Find

$$\lim_{x\to 1} \sum_{n=0}^{\infty} (-1)^n \tan^{-1} x^{2n+1}.$$

E 894. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let perpendiculars through vertex A of tetrahedron $ABCD$ to the faces ABC, ACD, ADB cut the circumsphere of $ABCD$ in B', C', D' respectively. Show that the volumes of the polyhedra $A\text{-}B'C'D'\text{-}B$, $A\text{-}B'C'D'\text{-}C$, $A\text{-}B'C'D'\text{-}D$ are equal to that of $ABCD$.

E 895. *Proposed by L. Fejes Tóth, Budapest, Hungary*

Let the *incircle* of a convex polygon be defined as the largest circle whose interior lies in the interior of the polygon. Show that the sum of the squares of the edges of a convex polyhedron is at least twelve times the square of the diameter of the least incircle of the faces.

SOLUTIONS

Evaluation of a Determinant

E 857 [1949, 179]. *Proposed by M. S. Knebelman, Washington State College*

Evaluate the s by s determinant whose element in the $(i+1)$st row and $(j+1)$st column is $d^{m+j}x^{n+i}/dx^{m+j}$.

Solution by S. T. Parker, Kansas State College. We suppose that $n \geqq m$, since $n<m$ makes the value of the determinant zero. Let $\Delta(s)$ be the value of the s by s determinant and let D represent d/dx.

It is seen that a factor $(n+i)!/(n+i-m)!$ can be removed from the $(i+1)$st row. Let

$$K = \prod_{i=0}^{s-1} (n+i)!/(n+i-m)!.$$

Then

$$\Delta(s) = K \begin{vmatrix} x^{n-m} & Dx^{n-m} & D^2x^{n-m} & \cdots \\ x^{n-m+1} & Dx^{n-m+1} & D^2x^{n-m+1} & \cdots \\ x^{n-m+2} & Dx^{n-m+2} & D^2x^{n-m+2} & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ x^{n-m+s-1} & Dx^{n-m+s-1} & D^2x^{n-m+s-1} & \cdots \end{vmatrix} = K\delta(s), \quad \text{say.}$$

In $\delta(s)$, multiply the $(s-1)$st row by $-x$ and add it to the sth row, multiply the $(s-2)$nd row by $-x$ and add it to the $(s-1)$st row, *etc.* The result is

$$\delta(s) = \begin{vmatrix} x^{n-m} & Dx^{n-m} & D^2x^{n-m} & D^3x^{n-m} & \cdots \\ 0 & x^{n-m} & 2Dx^{n-m} & 3D^2x^{n-m} & \cdots \\ 0 & x^{n-m+1} & 2Dx^{n-m+1} & 3D^2x^{n-m+1} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ 0 & x^{n-m+s-2} & 2Dx^{n-m+s-2} & 3D^2x^{n-m+s-2} & \cdots \end{vmatrix} = x^{n-m}(s-1)!\delta(s-1).$$

We see that $\delta(1) = x^{n-m}$ and hence $\delta(s) = x^{s(n-m)} \prod_{i=0}^{s-1} i!$. Therefore

$$\Delta(s) = x^{s(n-m)} \prod_{i=0}^{s-1} (n+i)!i!/(n-m+i)!.$$

Also solved by J. L. Ericksen, N. D. Lane, Roger Lessard, Norman Miller, Azriel Rosenfeld, and the proposer.

For an application of the special case $m=0$ see the proposer's article *The Wronskian for linear differential equations*, this MONTHLY, [1949, 252–254].

An Interesting Even Function

E 858 [1949, 179]. *Proposed by Henry Scheffé, University of California at Los Angeles*

Give an example of an even function with continuous derivatives up to order n, which has neither a maximum nor a minimum at $x=0$.

Solution by J. C. C. McKinsey, Santa Monica, California. Take

$$f(x) = x^{2n}\cos 1/x, \qquad x \neq 0,$$
$$f(0) = 0.$$

The continuity of the derivatives follows as in the solution of E 770 [1948, 97]. The function, however, has no extreme at the origin since it assumes both positive and negative values in any neighborhood of the origin.

Also solved by J. L. Ericksen, Vern Hoggatt, M. S. Klamkin, Norman Miller, Leo Moser, C. S. Ogilvy, Mary Payne, C. F. Pinzka, and the proposer.

Several solvers pointed out that $f(x)=\text{constant}$ satisfies the requirements. Others used $f(x)=x^{2n+1}\sin 1/x$, $x\neq 0$; $f(0)=0$.

Ratio of Inradii of Hexahedron and Octahedron

E 859 [1949, 180]. *Proposed by C. W. Trigg, Los Angeles City College*

If the faces of a hexahedron are equilateral triangles congruent to the faces of a regular octahedron, then the radii of the inscribed spheres are in the ratio 2:3.

Solution by William Douglas, Courtenay, British Columbia. The hexahedron is composed of two equi-edge triangular pyramids (regular tetrahedra), base to base, and the octahedron is composed of two equi-edge square pyramids, base to base. The incenters of the two solids are located at the centroids of the bases of the pyramids. If we let the edges of the two solids be of unit length, then, by simple use of the Pythagorean theorem, we find that the inradii of the solids are $\sqrt{6}/9$ and $1/\sqrt{6}$, which are in the ratio 2:3.

Also solved by W. E. Buker, Ragnar Dybvik, N. D. Lane, Leo Moser, Azriel Rosenfeld, C. M. Sandwick, and the proposer.

For the same edge, it is well known that the equi-edge tirangular pyramid has half the volume of the equi-edge square one, whence $V_h/V_o=1/2$. Also, the surface of the octahedron is 8/6 that of the hexahedron, or $S_o/S_h=4/3$. Since $V_h=(r_hS_h)/3$, $V_o=(r_oS_o)/3$, it follows that

$$r_h/r_o = (V_h/V_o)(S_o/S_h) = 2/3.$$

That the equi-edge triangular pyramid has half the volume of the equi-edge square one is interestingly verified by familiar results on the piling of shot. A

triangular pyramid of n courses contains $t_n = n(n+1)(n+2)/6$ shot, and a square pyramid of n courses contains $s_n = n(n+1)(2n+1)/6$ shot. Now $\lim_{n\to\infty} t_n/s_n = 1/2$.

A Volume Dissection

E 860 [1949, 180]. *Proposed by Leo Moser, University of Manitoba*

Show that if all the faces of a polyhedron have central symmetry then it can be dissected by a finite number of plane cuts and the pieces fitted together to form a solid cube.

Solution by the Proposer. First dissect the faces into parallelograms. Then, from the vertices to one 'side' of a fixed zone, drop lines into the polyhedron, equal and parallel to the edges of the fixed zone. (See solution to problem 4176 [1947, 169].) These determine a layer of parallelepipeds which when removed leave a polyhedron all of whose faces are parallelograms but whose number is smaller than before. Proceeding in this way we can dissect the given polyhedron into a number of parallelepipeds. Each parallelepiped we can then dissect into a rectangular parallelepiped by the following method. From one of the vertices drop a perpendicular plane to one of the opposite faces. There will always be at least one such perpendicular plane cutting through the parallelepiped. Fit the two pieces together to form a parallelepiped with at least one right angle. Proceeding in this way we can make all the angles right angles.

Now suppose the volume of the whole polyhedron is 1, and the dimensions of one of the rectangular parallelepipeds are a, b, c. Making cuts perpendicular to the plane of the rectangle $a\times b$, dissect this rectangle into a rectangle $1\times(ab)$. (See, *e.g.*, Kraitchik, Mathematical Recreations, Ch. VIII.) Then by cuts perpendicular to the plane of the rectangle $(ab)\times c$ dissect this into a rectangle $1\times(abc)$. Thus we obtain a rectangular parallelepiped $1\times 1\times(abc)$. If we do this to all the parallelepipeds we can clearly put all the rectangular paralelepipeds together to form a cube.

Editorial Note. It is a remarkable and well known fact that any two polygons having equal areas are such that one of them can be dissected into a finite number of polygonal pieces which can be rearrangėd to form the second polygon. On the other hand it is not always possible to dissect a given polyhedral solid into a finite number of polyehdral pieces which can be fitted together to form a second given polyhedral solid of the same volume. This interesting fact is known as Dehn's theorem and was first established by M. Dehn in *Nachr. der k. Gesellschaft der Wissenach. zu Göttingen*, 1900. Also see Dehn's article "Über den Rauminhalt," *Math. Ann.*, 1902, p. 465. A proof of this theorem may be found in H. G. Forder, The Foundations of Euclidean Geometry, pp. 288–290.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4370. *Proposed by H. F. Sandham, Trinity College, Ireland*

A', B', C' are points on the opposite sides of a triangle ABC. The circles through $B'C'A$, $C'A'B$, $A'B'C$ intersect in M, the Miguel point, whose isogonal conjugate is M'. Prove that M, M' are corresponding points under the direct circular transformation set up by A, A'; B, B'; C, C'.

4371. *Proposed by Albert Wilansky, Lehigh University*

Define $\theta(a, h)$ as the largest number θ satisfying

$$\text{(i)} \qquad 0 < \theta < 1$$

$$\text{(ii)} \qquad f(a + h) = f(a) + hf'(a + \theta h)$$

where $f(x) = x^2 \sin (1/x)$ for $x \neq 0$, $f(0) = 0$.

Now set $\lambda(h) = [h \cdot \theta(0, h)]^{-1}$. Prove that as h tends to zero, $\lambda(h)$ tends to infinity in a step function manner; specifically, given $\epsilon > 0$, there is a number $H(\epsilon)$ such that for every h with $|h| < H$ there is an integer $n(h)$ such that

$$|\lambda(h) - (n + \tfrac{1}{2})\pi| < \epsilon.$$

4372. *Proposed by Ky Fan, University of Notre Dame*

For what real values of x does the sequence

$$f_n(x) = \sin 7^n \pi x$$

converge, and what is the limit?

4373. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In every system of numeration having a base B which is the product of distinct prime factors, each to the first power, and which is not divisible by a given odd prime number m nor by any prime number of the form $2km + 1$, the mth powers of integers which terminate in the digits 0 to $B - 1$ all have distinct units digits.

4374. *Proposed by Paul Erdös, Syracuse University*

Let $d_n > 0$, $D_n = \sum_{k=1}^{n} d_k$, $D_n \to \infty$. The well known theorem of Abel-Dini

states that $\sum_{n=1}^{\infty} d_n/D_n = \infty$. Now let $f(n) > 0$ be any increasing function which satisfies $\sum_{n=1}^{\infty} d_n/f(n) = \infty$, then

$$\sum_{n=1}^{\infty} \frac{d_n}{\max\{D_n, f(n)\}} = \infty.$$

SOLUTIONS

Volume of Polar Tetrahedron a Minimum

4248 [1947, 419], corrected. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Having given a tetrahedron $ABCD$, place a sphere (S) of given radius in such a manner that the volume of the polar tetrahedron of $ABCD$ with respect to (S) will be a relative minimum.

*Solution by Robert Bouvaist, Vincelles, Saône-et-Loire, France.** We shall establish first the following proposition: Given a tetrahedron $ABCD$ of volume V and a central quadric of center O and of semi-axes a, b, c, the volume V' of the tetrahedron $A'B'C'D'$ obtained from $ABCD$ by polar reciprocation with respect to the quadric is given by

$$V' = \frac{-V^3}{V_1V_2V_3V_4}\left(\frac{abc}{6}\right)^2,$$

where V_1, V_2, V_3, V_4 are the volumes of the tetrahedrons $OBCD$, $OCDA$, $ODAB$, $OABC$. Let the equation of the quadric be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

and the coördinates of A, B, C, D be x_i, y_i, z_i $(i=1, 2, 3, 4)$. Then

$$V = \frac{1}{6}\left| x_1y_2z_31 \right|.$$

The equation of $B'C'D'$ is:

$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} + \frac{z_1z}{c^2} - 1 = 0.$$

Since the volume of tetrahedron $A'B'C'D'$, for which the faces have the equations

$$u_ix + v_iy + w_iz + s_i = 0, \qquad (i = 1, 2, 3, 4),$$

is

$$V' = \frac{1}{6} \frac{\left| u_1v_2w_3s_4 \right|^3}{\left| u_2v_3w_4 \right| \cdot \left| u_3v_4w_1 \right| \cdot \left| u_4v_1w_2 \right| \cdot \left| u_1v_2w_3 \right|},$$

* Translated by W. E. Byrne, Virginia Military Institute.

it follows that

$$V' = \frac{-V^3}{V_1V_2V_3V_4}\left(\frac{abc}{6}\right)^2$$

since $V_1 = -|x_2y_3z_4|$, etc.

If we designate by A, B, C, D the areas of the faces BCD, CDA, DAB, ABC and by x, y, z, t the absolute normal tetrahedral coordinates of O with respect to $ABCD$, we have

$$V' = -\frac{81V^3}{AxByCzDt}\left(\frac{abc}{6}\right)^2.$$

But we have $Ax+By+Cz+Dt=3V$. The maximum of $AxByCzDt$ occurs for $Ax=By=Cz=Dt$, so that the point O should coincide with the centroid of $ABCD$.

Genetic Algebra

4277 [1948, 34]. *Proposed by C. D. Olds, San Jose State College, California*

In a non-associative algebra, it is necessary to distinguish the possible interpretations of x^n. Thus, for example, in a non-commutative non-associative algebra x^3 can mean $x\cdot x^2$ or $x^2\cdot x$. In a general non-commutative non-associative algebra the number of interpretations of x^n is $2(2n-3)!/n!(n-2)!$ Is there a formula for the number of interpretations of x^n in a general commutative non-associative algebra?

Discussion by H. W. Becker, Santa Monica, California. The expression

$$N_{n+1} = (2n)!/n!(n+1)! = \binom{2n}{n}/(n+1) \tag{1}$$

has a wide variety of distinct combinatorial and graphical interpretations, of at least five different types. That the types are distinct is made evident upon attempting to enumerate the symmetrically different sequences or graphs of each type, for these in general yield unequal numbers.

Wedderburn [1] demonstrated the well known

$$N_1 = N_2 = 1, \qquad N_n = \sum_{m=1}^{n} N_{n-m}\cdot N_m, \tag{2}$$

and showed that, upon omitting all terms of (2) having like form, we have

$$N_1' = N_3' = 1, \qquad N_{2n+1}' = \sum_{m=1}^{n} N_{2n+1-m}'\cdot N_m', \tag{3}$$

$$N_2' = 1, \qquad N_{2n}' = N_n'(N_n'+1)/2 + \sum_{m=1}^{n-1} N_{2n-m}'\cdot N_m'. \tag{4}$$

Here N_n' and N_n are the numbers of commutative and non-commutative, non-associative products of n factors.

Etherington [2] independently found generating functions for N_n and N_n' and gave an explanation as to why they yield a simple formula for the first but not the second. His further work [3, 4] founded a tradition in *genetic algebra* whose development at the hands of R. D. Schafer [5, 6, 7] reached a wide audience.

In lieu of an exact formula for N_n, we seek an approximation. By Stirling's approximation to $n!$, (1) becomes

$$N_{n+1} \approx 4^n/(n+1)\sqrt{\pi n}, \tag{5}$$

whence N_{n+1}/N_n approaches 4 with increasing n. If we calculate N_n and N_n' for values of n through $n=25$, and utilize an idea due to Motzkin [8], the following results are suggested empirically:

$$N_{n+1}/N'_{n+1} = R_n, \qquad R_n/R_{n-1} = r_n, \qquad r_n \to r = 1.612. \tag{6}$$

Thus

$$N'_{n+1}/N'_n = k_n, \qquad k_n \to k = 4/r = 2.48. \tag{7}$$

If the number h_n is defined so that the following equation is true, it appears as though there exists a number h, the limit of h_n, in terms of which the desired approximation of N'_{n+1} may be expressed. We have then $h_n \approx 0.812$ and approximately

$$N'_{n+1} = h_n k^n/(n+1)\sqrt{n} \approx N_{n+1}/(0.7)r^n. \tag{8}$$

This formula is suggested for what it may be worth, based as it is on inspection of a few values of n, and lacking any proof of the existence of the limits r, k, or h. The detailed figures are shown in the table below. Note, however, that (5) and (8) have the same form, $U_n \approx ab^n/n^{3/2}$, as the approximations for the number of n-branch series-parallel passive circuits [9], and of n-branch root-trees [10].

Other interpretations of the function (1) should be of interest. N_n is the number of planar rhyme schemes [11], such that there are no crossovers in the Puttenham diagram [12]. N_{n+1} is the number of ballot sequences in a two party election, such that the non-loser gets $n-1$ votes and is never behind his opponent, who may get anywhere from 0 to $n-1$ votes, Lucas [13], p. 164, *le scrutin du ballotage.* (See also p. 14, *marches du pion du jeu de dames*; p. 86, *les deux files de soldats*; and p. 87, *déplacements de la tour sur l'échiquier triangulaire.*)

N_{n+1} is the number of ways of decomposing an $(n+2)$-gon into triangles by $n-1$ non-intersecting diagonals, Lucas [13], pp. 90–96, 489. N_{n+1} is also the number of ways of joining $2n$ points around a circle by n non-intersecting chords, the c_n of Motzkin [8], which linearize to the configurations superieures of Touchard [14].

This problem 4277 may also be regarded as a sequel to problem 3954 of O. Ore [1941, 564], whose solutions provide a helpful background.

n	N_n	N'_n	r_n	k_n	h_n
1	1	1	1	1	.806
2	1	1	2	1	.69
3	2	1	1.25	2	.909
4	5	2	1.867	1.5	.791
5	14	3	1.5	2	.856
6	42	6	1.714	1.8333	.809
7	132	11	1.558	2.0909	.842
8	429	23	1.663	2	.83
9	1430	46	1.595	2.1304	.824
10	4862	98	1.635	2.1122	.814
11	16796	207	1.607	2.1787	.8175
12	58786	451	1.624	2.1796	.8119
13	2 08012	983	1.612	2.2167	.8139
14	7 42900	2179	1.617	2.2258	.811
15	26 74440	4850	1.612	2.2485	.8118
16	96 94845	10905	1.615	2.2587	.8108
17	353 57670	24631	1.613	2.274	.8109
18	1296 44790	56011	1.613	2.2837	.8105
19	4776 38700	1 27912	1.612	2.2949	.8108
20	17672 63190	2 93547	1.612	2.3034	.8108
21	65641 20420	6 76157	1.612	2.3121	.811
22	2 44662 67020	16 63372	1.6121	2.3194	.8112
23	9 14825 63640	36 26149	1.6118	2.3265	.8115
24	34 30596 13650	84 36379	1.6118	2.3328	.8119
25	128 99041 47324	196 80277	1.6117	2.3387	.8121

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RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

Applied Mathematics for Engineers and Scientists. By S. A. Schelkunoff. New York, D. Van Nostrand Company, 1948. 11+472 pages. $6.50.

Most of the first sixteen chapters of this book deal with topics which might find a place in a course in advanced calculus; the next five chapters deal with special transcendental functions; the last chapter, entitled "Formulation of equations," gives some examples of types of equations arising in various physical problems. The author states that his purpose is to bring "Higher Mathematics for Scientists and Engineers" up to date. This is certainly a laudable purpose, and several of the chapters admirably fulfill it. For example, the chapters on ordinary differential equations emphasize linear equations and approximate methods for nonlinear equations, while the special methods for handling equations which occur only in textbooks are dismissed with the contempt which they deserve. There are good chapters on special conformal mappings and on the Laplace transform. The chapters on special functions form a convenient short collection of formulas for the gamma function, for exponential and sine and cosine integrals and for Bessel and Legendre functions. Other topics treated include complex numbers, integration (real and complex, with applications), power and Fourier series, vector analysis, numerical solution of equations (only the simplest methods are considered), and partial differential equations, with particular attention to wave equations and their applications in electrical problems.

If used as a textbook, this book would seem to require supplementing with more numerous exercises, especially in the later chapters. A student could learn much useful mathematical technique from it. However, there are so many misleading mathematical statements that it should hardly be used except under competent guidance. For example, although the author refrains from saying that every function, or even every function encountered in practice, can be represented by a Fourier series, he says (incorrectly) that every piecewise continuous function can be so represented; it would have been no harder, and even more suitable for practical purposes, to say (correctly) that every function made up of a finite number of monotonic pieces can be represented by a Fourier series. The author claims that the sum of a conditionally convergent series is changed by *any* rearrangement of the terms, and that a power series converges uniformly in the interior of its circle of convergence. He seems to consider it satisfactory to obtain the derivatives of $\exp(-1/x^2)$ at $x=0$ as the limits of the derivatives for $x\neq 0$; this process would give an incorrect result if applied to

$x^2 \sin(1/x)$. He claims that "those functions which are given by the same set of arithmetic operations on each value of the variable z *as a whole* are clearly monogenic functions; and it can be shown that all monogenic functions can be represented in this way." The definition of improper integral as given in the book would include Cauchy principal values, which the author does not seem to want. The ambiguity of the notation e^z for the exponential function (so that a^z is multiple-valued but e^z is not) is not recognized. The author misses the chance to clarify the confusing (though standard) notation "δ-function." To call it a function instead of admitting it as a convenient notational device unnecessarily complicates the notion of function; the author actually introduces (in another connection) the Stieltjes integrals which he could have used to make the situation clear. Many of the author's inaccuracies are mere disregard of mathematical refinements, and unlikely ever to cause difficulty to engineers and scientists. Some, however, such as those just mentioned, might well lead to mistakes. It seems to the reviewer that students are entitled to be given accurate statements, or at the very least warned that exceptions may occur to the inaccurate statements which are presented for convenience.

There is one terminological innovation which might confuse anyone who looks at any other mathematical text: the author calls the absolute value of a complex number its *amplitude*. While this term may seem desirable for applications, it is unfortunate to use it when the same word is so often used to denote the angle of the complex number (for which this book uses the satisfactory term *phase*).

R. P. Boas, Jr.

CLUBS AND ALLIED ACTIVITIES

Edited by L. F. Ollmann, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS, 1948–49

Mathematics Club, Iowa State College

Topics of discussion at the meetings of the Iowa State *Mathematics Club* for the year 1948–49 were:

The mysteries of addition-subtraction logarithms, by J. E. K. Smith
Logarithms to the base twelve, by Gerald Helmstadter
Scientific satire in Gulliver's Travels, by Norris Chapman
Geometry of paper folding, by George Fox
Sundial, by Dale Hiedeman

Mathematics and imagination, by John Weisen

Mapping, by Larew Collister

Land location, by Charles Fritz

Application of mathematics in actuarial science, by Prof. Floyd Harper of Drake University

Mathematics of eclipses, by James McDonald

Student life in India, by Dr. B. R. Seth, Visiting Professor at Iowa State College from Hindu College at New Delhi, India.

Officers for 1948–49: President, Mary Ann Paulu; Program Committee, Jane Lloyd, Dolores A. Larson, R. J. Schrimper and G. K. Johnson; Faculty Advisers, Fred Robertson and Ruth L. Royer.

Kappa Mu Epsilon, Central Michigan College

The program for the academic year, 1948–49, of *Michigan Beta* Chapter *of Kappa Mu Epsilon* was as follows:

The Russian peasant system of multiplication, by Glenn Clark

The game of nim, by Mary Wright

Topics from Mathematics and The Imagination, by Guy Coykendall

Calculations of pi, by Don Chinnery

Certain postulates in mathematics, by William Kumbier

Use of mathematics in determining the frequency and location of the proposed local radio station, by Walter Maxwell, Station Manager and Chief Engineer

Report of the national meeting of Kappa Mu Epsilon in Topeka, Kansas, by Dr. C. Richtmeyer, National Historian, and the student representatives, Mary Janet Booth and Edward Czarnecki.

How to Add Fractions and *Percentages*, two movies of interest to prospective teachers, were shown.

Officers elected for the year 1949 were as follows: President, Raymond Heminger; Vice-President, Margaret King; Secretary, Margaret Geukes; Treasurer, Edward Czarnecki; Corresponding Secretary and Advisor, Mr. Dana R. Sudborough.

Graduate Mathematics Club, Indiana University

During the third year of the *Graduate Mathematics Club* at Indiana University, a series of lectures was given by members of the department. The topics were:

The problem of Plateau (with soap film demonstrations), by Dr. D. Gilbarg and Dr. G. W. Whaples

Anecdotes on the development of non-euclidean geometry, by Dr. V. Hlavaty

The three-body problem, by Dr. E. Hopf

The theory of games, by Dr. J. W. T. Youngs.

A picnic was held in a nearby state park in May.

The Executive Committee for the year was: David Van Tijn, Joseph Sullivan, and Charles Tyler.

Pi Mu Epsilon, New York University

Lectures held during the past year by the *New York Delta* Chapter of *Pi Mu Epsilon* at Washington Square College of New York University were:

A geometrical approach to self-excited oscillations, by Prof. K. O. Friedrichs

New electronic devices, by William Sollfrey

Mathematical concepts and reality, by Prof. Hofstadter

A geometrical approach to ballistics problems, by Samuel Karp

Casualty actuary: salesman of probabilities, by Mr. T. O. Carlson, Actuary of the National Bureau of Casualty Underwriters and Members of the Casualty Actuarial Society

Harmony of the world, by Prof. Morris Kline.

An annual banquet and an initiation of 28 new members were part of the activities.

The following officers were elected for 1949–50: Director, Paul Grey; Vice-Director, Solomon Ciolkowski; Secretary-Treasurer, Arlene Layton, Faculty Advisor and Permanent Secretary, John Schoonmaker.

Mathematics Club, University of Kansas

The following papers were presented by members of the *Mathematics Club* of the University of Kansas:

Diaphantine analysis of $x^4-y^4=z^4$, by Arthur Kruse

Relative motion, by Jack Hollingsworth

Predicting elections, by Leslie Pihlblad

Unsolved problems in mathematics, by Richard Harrington

The Euler ϕ-function—Number theory, by Sidney Lida

Planetary motion, by George Cole

Some identities for binomial coefficients, by Dr. R. Schatten

A discussion of the new requirements for mathematics majors, by Dr. G. W. Smith, Dr. G. B. Price, and Dean Gilbert Ulmer.

Extraction of number roots, by Bert Parsons

Principles of nomographs, by Dale Rummer

Visual aids in mathematics, by William Rinner

The abacus, by Hsei Chang.

All papers were presented by undergraduates except those by Dr. Schatten, Dr. Smith, Dale Rummer, and Hsei Chang.

Special events during the year included a social meeting and the annual picnic. Refreshments were served after each meeting.

Prizes were given for the best undergraduate talks of the year. Sidney Lida was awarded *Men of Mathematics* by E. T. Bell, for first prize; and Arthur Kruse received as second prize, *Analytical Functions of a Complex Variable*, by D. R. Curtiss.

Officers elected to serve for 1949–50 are: President, Ralph Simmons; Vice-President, Claire Grothusen; Secretary-Treasurer, Zelina Higgenbottom.

Kappa Mu Epsilon, Bowling Green State University

The program of the *Ohio Alpha* Chapter of *Kappa Mu Epsilon* included the following:

Mathematics in construction engineering, by Mr. H. Janzer of a local engineering firm

The paradoxes of infinity, by Dr. F. C. Ogg

The historical development of mathematics, by Prof. Gryting

Some application of group theory to molecular structure, by Prof. W. E. Singer

Some practical aspects of mathematics, by Prof. W. F. Cornell.

Initiations were held in January and in May.

Officers for 1949–50 are: President, Harry Ling; Vice-President, Gerald Carrier; Secretary, Ilona Pohlod; Corresponding Secretary, W. R. Cornell; Faculty Advisor, Dr. F. C. Ogg.

Pi Mu Epsilon, Duke University

The program of the *North Carolina Alpha* chapter of *Pi Mu Epsilon* for the school year 1948–49 was:

Mathematical analysis of mechanisms used in automatic machinery, by Mr. John May of Wright's Automatic Machinery Company. This talk was followed by a tour of W. W. Rankin's Mathematics Laboratory.

Maxima and minima, by Dr. W. M. Whyburn, Head of the Mathematics Department of the University of North Carolina.

There were 57 new members initiated at the above two meetings.

The officers elected for 1949–50 are: President, John Putnam; Vice-President, Edwin Webb; Secretary, Janet Henchie; Treasurer, Charles Gassett; Membership Chairman, Pat Collins.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

TENTH CHRISTMAS MEETING OF THE N.C.T.M.

The Wichita Mathematics Association will be hosts to the National Council of Teachers of Mathematics for the Tenth Christmas Meeting to be held in Wichita, Kansas, December 28–30, 1949. Sectional meetings will be held for teachers in elementary schools, junior high schools, secondary schools, and colleges. There will be discussion groups on topics related to problems representing

various phases of mathematics teaching, film forums on recent films and filmstrips, and an exhibit of mathematical models, instructional aids and instruments. In addition, plans are being made to hold a tour of leading industries in Wichita.

PRELIMINARY ACTUARIAL EXAMINATIONS

The Society of Actuaries has announced the winners of the prize awards offered to the nine undergraduates ranking highest on the score of Part 2 of the 1949 Preliminary Actuarial Examinations. The first prize of $200 was awarded to J. W. Moran, Yale University. Additional prizes of $100 were awarded to each of the following: Ariel Zemach, Harvard University; J. P. Mayberry and A. D. Murch of the University of Toronto; J. D. Lordan, Massachusetts Institute of Technology; W. A. White, Dartmouth College; T. P. Farmer, Jr., State University of Iowa; D. L. Haakenstad and W. V. Hauke of the University of Michigan.

The Society of Actuaries has authorized a similar set of nine prize awards for the 1950 Examinations on Part 2, which is a general mathematics examination covering algebra, trigonometry, coordinate geometry, differential and integral calculus. Part 1 of the Preliminary Actuarial Examinations is a language aptitude examination. Part 3 is a mathematics examination covering finite differences, probability and statistics.

The 1950 Preliminary Actuarial Examinations will be administered by the Educational Testing Service at centers throughout the United States and Canada on May 19, 1950. The closing date for applications is March 15, 1950. Detailed information concerning the Examinations can be obtained from the Society of Actuaries, 208 South LaSalle Street, Chicago 4, Illinois.

THE TENTH ANNUAL PUTNAM COMPETITION

The tenth annual William Lowell Putnam Mathematical Competition will be held on Saturday, March 25, 1950. This competition, made possible by the trustees of the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, is open to undergraduate students in universities and colleges of the United States and Canada who have not received a degree. The examination will consist of two parts of three hours each. The questions will be taken from the fields of calculus (elementary and advanced) with applications to geometry and mechanics not involving techniques beyond the usual applications, higher algebra (determinants and theory of equations), elementary differential equations, and geometry (advanced plane and solid analytic geometry). Any college or university wishing to enter a team or individual contestants may secure an application blank from Professor L. E. Bush, 112 Albertus Magnus Hall, College of St. Thomas, St. Paul 1, Minnesota, by a postcard request. All applications must be filed with the Director not later than March 1, 1950. If three candidates are presented from a college or university, they are to constitute a team; if more than three are presented from

any one college or university, the team of three must be named on the application. Fewer than three from one college or university may compete as individuals.

The examination may be given at any place where a team, or at least three candidates, can be assembled. Exceptions to this rule may be made by the Director in cases where it would entail unusual inconvenience to a contestant. Sealed copies of the examinations will be sent to the supervisor of the examination in time for the examination day and are not to be opened before the hour set.

The prizes to be awarded to the departments of mathematics of the institutions with the winning teams are $400, $300, $200, and $100, in the order of their rank. In addition, there will be prizes of $40, $30, $20 and $10 awarded to the members of these teams according to the rank of the team; a prize of $50 to each of the five highest contestants and a prize of $20 to each of the succeeding five highest contestants. Each of the winners will receive a suitable medal. Honorable mention will be given to several teams next in order after the four winning teams and to several individuals next in order after the ten individual winners. For further encouragement of the Competition, there will be awarded at Harvard University (or at Radcliffe College in the case of a woman) an annual $1500 William Lowell Putnam Prize Scholarship to one of the first five contestants, this to be available either immediately or on the completion of the student's undergraduate work.

Reports on the nine previous competitions and examination questions will be found in this MONTHLY for May, 1938, 1939, 1940, 1941, 1942, October, 1946. August–September, 1947, December, 1948, and August–September, 1949.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE APRIL MEETING OF THE ROCKY MOUNTAIN SECTION

The thirty-second annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at the Colorado School of Mines, Golden, Colorado, April 22 and 23, 1949. There were three sessions with Professor I. L. Hebel presiding at each.

Among the ninety-eight persons who registered were the following forty-nine members of the Association: R. V. Anderson, W. G. Brady, Leonard Bristow, G. L. Burton, F. M. Carpenter, Nancy V. Cheney, A. G. Clark, G. S. Cook, G. A. Culpepper, David DeVol, Mary C. Doremus, W. R. Eikelberger, O. J. Falkenstern, F. N. Fisch, Katherine C. Garland, R. H. Glass, H. T. Guard,

Leota C. Hayward, I. L. Hebel, H. K. Hilton, Ruth I. Hoffman, LeRoy Holubar, R. J. Howerton, Burrowes Hunt, J. A. Hurry, C. A. Hutchinson, B. W. Jones, A. J. Kempner, Claribel Kendall, A. J. Lewis, M. L. Madison, W. K. Nelson, Greta Neubauer, K. L. Noble, O. H. Rechard, A. W. Recht, L. W. Rutland, Nathan Schwid, S. R. Smith, W. N. Smith, L. C. Snively, K. H. Stahl, J. M. Staley, J. F. Stockman, E. P. Tovani, V. J. Varineau, W. W. Varner, J. F. Wagner, Lillie C. Walters.

At the business meeting, the following officers were elected for the coming year: Chairman, Professor A. J. Lewis, University of Denver; Vice-Chairman, Professor A. E. Mallory, Colorado State College of Education; Secretary-Treasurer, Professor J. R. Britton, University of Colorado.

The program of papers presented was:

1. *The Lill circle*, by Professor (Emeritus) W. J. Hazard, University of Colorado, introduced by A. J. Kempner.

The Lill circle for finding the roots of $ax^2+bx+c=0$, mentioned by Maurice d'Ocagne in his "Calcul Simplifie et Nomographie" as a special case of a general graphic approximation to the roots of the nth degree equation, is here shown to be the XY-plane section of the paraboloid of which the XZ section is the usual parabola plotted from $y=f(x)=x^2+bx/a+c/a$. Both the circle and the parabola show the roots of the equation as the points where the X axis pierces the surface of the paraboloid.

2. *A four-number game*, by Mr. Burrowes Hunt, University of Colorado.

In *Scripta Mathematica* (March 1948) Benedict Freedman showed that if from an ordered set of four positive integers one forms the set of the absolute values of their differences taken in cyclic order, and repeats this process, one arrives at the set 0, 0, 0, 0, in a finite number of steps. Mr. Hunt considered the same game for sets of four positive real numbers. The result is that, in general, any set of real numbers leads to the set 0, 0, 0, 0, but there is a double infinity of exceptional sets all of whose differenced sets consists of positive numbers. If x is the positive root of either of the equations $x^3=1\pm(x^2+x)$, then the set 0, 1, $1+x$, $1+x+x^2$ is exceptional, as is any set derived from it by replacing each element a by $ka+m$.

3. *Is mathematics practical?* by Mr. R. J. Howerton, Regis College.

The author discussed the question of "pseudo-practicality" of problems in textbooks, and the growing tendency for mathematics teachers to be on the defensive with respect to their subject. The problem of teaching the "why" of mathematics rather than the "how" from elementary classes through college work was discussed. Placing mathematics on a plane with history, psychology, literature, and other cultural subjects was proposed. The author maintained that, for the average liberal arts college student, mathematics was highly impractical, and that the only true justification for the subject was on a cultural basis.

4. *The construction of a statistical quality control chart and some interpretations to be made from it*, by Professor J. F. Wagner, University of Colorado.

The bases of a statistical quality control chart are these: (1) Variation in the manufacture or measure of a product quality is to be expected; (2) The data themselves determine the expected spread of the data; (3) The control, or action, limits are then set at such values as to minimize, economically, the waste of looking for trouble when there is none and not looking for trouble when there is some; (4) We shall use the so-called 3σ limits which include all but about one out of every 400 variates in the distribution, unless there is a significant departure from normalcy; (5) The

occurrence of a point outside the control limits shall then be the signal to look for an assignable cause of variation beyond the natural variability of the process.

With the aid of a table of data and a control chart grid which was distributed to each member of the audience, an actual example of this technique taken from industry was discussed. The points lying outside of the control limits were identified with their assignable cause. The meaning of a run or loss of random scattering of the points on a control chart was presented.

5. *A construction for a monoidal quartic*, by Mr. W. G. Brady, University of Wyoming.

In this paper a 4:1 correspondence between the points of two conics is shown to lead to a monoidal quartic, and configurations leading to various types of triple points are discussed.

6. *The central limit concept in an elementary course in statistics*, by Professor H. T. Guard, Colorado A. and M. College.

The author discussed some of the pedagogical problems encountered in the teaching of elementary statistics to students having little mathematical background. Sampling experiments for the verification of the central limit theorem were discussed.

7. *Delta-V, a conical shell*, by Professor F. M. Carpenter, Colorado School of Mines.

For certain types of volumes of revolution the use of a cylindrical element leads to the correct numerical result because of compensating errors. Often an exercise can be analyzed and solved correctly in cartesian coordinates by using a conical shell for the element of volume.

8. *Degenerate conics*, by Professor A. J. Lewis, University of Denver.

The author shows some of the elementary methods of determining when the general equation of the second degree in two variables will give a degenerate conic.

9. *Linkages in relation to certain aspects of college geometry*, by Professor M. L. Madison, Colorado A. and M. College.

The author gave a brief historical sketch of linkages from the time James Watt patented his "parallel motion" in 1874. The use of linkwork models, a number of which were exhibited, as aids in teaching college geometry, analytic geometry, and plane geometry was discussed.

10. *Materials for teaching mathematical meanings in the elementary school*, by Professor Lucy L. Rosenquist, Colorado State College of Education, introduced by A. E. Mallory.

The mathematical meanings that need to be taught in the elementary school are the various relationships between "groups." These groups are the chance groupings met in everyday experience, and the standard groupings of our number system. The processes of addition, subtraction, multiplication, and division are methods of changing chance groupings into standard groupings. Children learn to handle groups with progressively more mature methods as their understanding of groups and group relationships develops. Concrete materials which aid in developing this understanding should have the following characteristics: (1) Compact contours; (2) Patterned arrangement, or capability of being easily arranged in patterns; (3) Freedom from elements that embed the number ideas. These materials are not to be used as demonstration materials by the teacher. Pupils should have opportunities to manipulate materials in discovering solutions to their problems, and in recognizing constant relationships between groups. The explanation of these individual discoveries to the class affords opportunity for clarification of the ideas, and stimulates insight into the meaning of the number system and the computational processes.

11. *Looking backward and forward,* by Professor A. J. Kempner, University of Colorado.

After the program of papers, a joint meeting was held with the Mathematics Section, Eastern Division, Colorado Education Association. There were raised problems relating to the reorganization of mathematics training in the schools of Colorado. Later, a panel, consisting of representatives from elementary, secondary, and college levels attempted to give solutions to these problems.

W. K. NELSON, *Acting Secretary*

THE APRIL MEETING OF THE KANSAS SECTION

The thirty-fourth annual meeting of the Kansas Section of the Mathematical Association of America was held at Kansas State College in Manhattan, on Saturday, April 2, 1949. Sessions were held in the morning and afternoon. Professor R. G. Sanger presided at these sessions. The morning session was a joint meeting with the Kansas Association of Teachers of Mathematics.

The attendance was one hundred fifty-five including the following forty-two members of the Association: Sister M. Nicholas Arnoldy, R. W. Babcock, Wealthy Babcock, Florence L. Black, Frances N. Breneman, Virginia L. Chatelian, W. R. Cowell, L. E. Curfman, Lucy I. Dougherty, Paul Eberhart, Walter Fleming, Albert Furman, W. H. Garrett, Laura Z. Greene, Edison Greer, J. R. Hanna, K. C. Hsu, Emma Hyde, W. C. Janes, L. E. Laird, C. F. Lewis, Anna Marm, Margaret E. Martinson, Thirza A. Mossman, E. P. Northrop, S. T. Parker, P. S. Pretz, G. B. Price, O. M. Rasmussen, C. B. Read, C. A. Reagan, L. M. Reagan, R. G. Sanger, G. W. Smith, R. G. Smith, W. T. Stratton, C. B. Tucker, Gilbert Ulmer, E. B. Wedel, A. E. White, Ferna E. Wrestler, P. M. Young.

At the business meeting the following officers were elected for next year: Chairman, R. G. Smith, Kansas State Teachers College; Vice-Chairman, L. M. Reagan, University of Wichita; Secretary-Treasurer, Anna Marm, Bethany College.

The following papers were presented:

1. *The role of mathematics in general education,* by Professor E. P. Northrop, College of the University of Chicago.

The speaker deplored the fact that teachers of mathematics, in attempting to formulate courses within a program of general education, are in the habit of thinking in terms of subject-matter (content) alone, or at most of deriving ends (aims, objectives) from subject-matter. He pointed to general lack of agreement among teachers concerning what subject-matter is most appropriate, and expressed skepticism of the meaningfulness of courses constructed from the standpoint of subject-matter. He proposed an approach to the problem through initial consideration of ends—of the kinds of abilities and understanding the student ought to acquire from a general education. He listed those ends which he regarded as important and which he felt could best be achieved through the study of mathematics, and pointed out that at least some of them could be served equally well by alternative choices of subject-matter. Briefly put, the speaker argued for an ends-to-means rather than a means-to-ends (or means alone) approach to the formulation of mathematics courses

for general education. He confessed, however, to a belief that ends, subject-matter, and method of presentation must ultimately be considered together.

2. *Some famous problems of modern mathematics*, by Professor G. B. Price, University of Kansas.

This paper contains an account of Waring's problem, the four color problem, the Jordan curve theorem, and the problem of Plateau. It is illustrated with various models and demonstrations, including paper and rubber models of surfaces, a map on a torus requiring seven colors for its coloring, and soap film models for the problem of Plateau. The four problems are used to emphasize the great progress that has taken place in mathematics in recent times, to illustrate the nature and source of problems in mathematics, and to point out the difference between a proof in mathematics and a proof in physics.

3. *A problem in thermodynamics*, by Boris Leaf, Kansas State College, introduced by the Secretary.

A mathematical formalism extending the treatment of J. Willard Gibbs on equilibrium of heterogeneous systems to partial chemical systems is described, a partial system being the substance of a particular chemical species in an infinitesimal region of matter, considered as a continuum. The results of the treatment indicate that each chemical species at a given density and temperature should have the same thermodynamic properties independent of the presence of other species.

4. *Notes on plane cubic curves*, by Laura Z. Greene, Washburn Municipal University.

The paper was a discussion of the condition necessary to determine a third degree curve, and of the double points that may arise.

5. *Evaluation of a complex improper integral*, by Walter Fleming, Fort Hays Kansas State College.

The present paper deals with the evaluation of the integral

$$H(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sec \sqrt{-iv}\, e^{iuv} dv. \tag{1}$$

This integral arises in the evaluation of double Wiener integrals. It is found that (1) can be expressed in terms of θ-functions (Whittaker and Watson, *Modern Analysis*, p. 464) as follows:

$$\begin{aligned} H(u) &= \frac{\pi^{3/2}}{\sqrt{2}} \theta_1'(0, e^{-\pi^2 u}) = \frac{1}{\sqrt{2}\, u^{3/2}} \theta_1'(0, e^{-1/u}), && u > 0 \\ H(u) &= 0, && u < 0 \end{aligned} \tag{2}$$

where

$$\theta_1'(z, q) = \frac{\partial}{\partial z} \theta(z, q).$$

Finally it is established that for small values of R

$$\frac{1}{\sqrt{2\pi}} \int_0^{R^2} H(u) du \sim \frac{4R}{\sqrt{\pi}} e^{-1/4R^2} \{1 - 2R^2 + \cdots \}, \tag{3}$$

and that for large values of R

$$1 - \frac{1}{\sqrt{2\pi}} \int_0^{R^2} H(u) du \sim \frac{4}{\pi} e^{-\pi^2 R^2/4}. \tag{4}$$

The left member of (3) gives the measure of the set of points (x, y) in double Wiener space such that

$$\int_0^1 ([x(t)]^2 + [y(t)]^2)dt < R^2.$$

6. *Requirements for a Bachelors Degree with a major in mathematics*, by Sister M. Nicholas Arnoldy, Marymount College.

A comparative study of the requirements of the twenty-one Kansas colleges granting Bachelor of Arts and Bachelor of Science degrees with a major in mathematics was made. The report stated the total number of semester hours of mathematics, as well as the semester hours above the calculus, required for mathematics majors. The semester hours of science that a person majoring in mathematics is expected to earn was also given. The Kansas colleges are quite uniform in the requirements in mathematics but differ in the science requirements.

7. *Proof by contradiction*, by S. T. Parker, Kansas State College.

This paper dealt with the classic "reductio ad absurdum" method of proof. Several references were made to *A Mathematician's Apology* by G. H. Hardy, with emphasis on the desire for simplicity, power, and generality. Illustrations included the proof that there are infinitely many primes, and that $\sqrt{2}$ is irrational. It was also shown that certain recurrence formulas give all the solutions of the Pellian equation $x^2 = 3y^2 + 1$.

ANNA MARM, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-third Annual Meeting, New York City, December 30, 1949.

International Congress of Mathematicians, Cambridge, Massachusetts, August 30–September 6, 1950.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN

ILLINOIS, Southern Illinois University, Carbondale, May 12–13, 1950.

INDIANA, Wabash College, Crawfordsville, April 29, 1950.

IOWA, State University of Iowa, Iowa City, April 21–22, 1950.

KANSAS, Spring, 1950.

KENTUCKY, University of Kentucky, Lexington, April 29, 1950.

LOUISIANA-MISSISSIPPI, Centenary College, Shreveport, Louisiana, Spring, 1950.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Fall, 1949.

METROPOLITAN NEW YORK, Spring, 1950.

MICHIGAN, March, 1950.

MINNESOTA, Macalester College, St. Paul, May 6, 1950.

MISSOURI, Spring, 1950.

NEBRASKA, Nebraska Wesleyan University, Lincoln, May 6, 1950.

NORTHERN CALIFORNIA, Berkeley, January 28, 1950.

OHIO, Denison University, Granville, April 22, 1950.

OKLAHOMA

PACIFIC NORTHWEST, University of Washington, Seattle, June, 1950.

PHILADELPHIA

ROCKY MOUNTAIN, University of Denver, April, 1950.

SOUTHEASTERN, University of Florida, Gainesville, April 7–8, 1950.

SOUTHERN CALIFORNIA, Immaculate Heart College, Hollywood, March 11, 1950.

SOUTHWESTERN, Spring, 1950.

TEXAS, Abilene, Spring, 1950.

UPPER NEW YORK STATE, Syracuse University, Spring, 1950.

WISCONSIN, Marquette University, Milwaukee, May, 1950.

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DECEMBER 1949

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H. S. M. COXETER	1945–1947	F. W. OWENS	1941–1943
D. R. CURTISS	1934, 1937–1942	W. V. PARKER	1943–1945
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H. J. ETTLINGER	1941–1943	W. R. RANSOM	1944–1946
G. C. EVANS	1936–1941	O. H. RECHARD	1942–1944
PHILIP FRANKLIN	1940–1942	A. W. RECHT	1946–1948
CORNELIUS GOUWENS	1941–1943	H. A. ROBINSON	1942–1944
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E. S. HAMMOND	1946–1949	G. W. SMITH	1944–1946
R. C. HUFFER	1945–1947	H. L. SMITH	1945–1947
RALPH HULL	1946–1948	J. M. THOMAS	1937–1941
C. G. JAEGER	1943–1945	MORGAN WARD	1944–1946
W. C. KRATHWOHL	1941–1943	C. W. WATKEYS	1946–1948
GILLIE A. LAREW	1945–1947	K. W. WEGNER	1943–1945
J. W. LASLEY, JR.	1944–1946	G. T. WHYBURN	1940–1942
C. G. LATIMER	1942–1944	R. L. WILDER	1942–1944
MAYME I. LOGSDON	1940–1942	F. B. WILEY	1940–1941
SAUNDERS MACLANE	1943–1945	K. P. WILLIAMS	1945–1947
		W. L. WILLIAMS	1946–1949

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This list gives the following information, in so far as it is available, for each person who is a member of the Association as of October 15, 1949: (1) Surname with initials, or the given name when there is but one, except that a given name of every woman member is given in full. In case of a married woman, the husband's initials are given. (2) Highest academic degree received, with name of institution conferring the degree. (3) Official title or position. (4) Name and address of institution or organization. (5) Address to be used for mail if different from (4). The name of city and postal zone number are not repeated if they are the same as in (4).

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Wayne Univ. Allen, Baldwin, Borgman, Clatworthy, Epstein, Folley, Harrison, Loweke, Mandelbaum, Minas, Morrow, Nelson, Patterson, Pixley, Scibiorski, Southard, Vuylsteke.
EAST LANSING.
Michigan State Coll. Barbour, Baten, Bell, Carr, Coy, Frame, Grindall, Grove, Hill, Kelly, Lapidus, Musselman, Nordhaus, Payne, Plant, Powell, Sander, Stewart, Sweetland, Wells.
FENTON. Tryon.

FLINT.
General Motors Inst. DeMoss, Grotts, Raker, Schaefer, Straw.
GRAND RAPIDS. Bellardo.
HART. Burdick.
HOLLAND. Lampen.
HOUGHTON.
Michigan Coll. of Mining & Tech. Boggs, Park, Stipe.
IRONWOOD. Field.
KALAMAZOO. Walton.
Western Michigan Coll. Bartoo, Beeler, Blair, Butler, Cain, Everett, Ford, Hannon.
MARQUETTE. Spooner.
MILFORD. McNeal.
MT. PLEASANT.
Central Michigan Coll. Bye, Foust, Pratt, Richtmeyer, Sudborough.
PLYMOUTH. Frankel.
SAULT STE. MARIE. Otis.
YPSILANTI.
Michigan State Normal Coll. Erikson, Goings, Lindquist, Pate.

MINNESOTA

BEMIDJI. Colson, Milkovitch.
COLERAINE. Kearney.
COLLEGEVILLE. Danzl, Kalinowski.
DULUTH. Cothran, McEwen, Mercedes, Morin.
MINNEAPOLIS. Flanary.
Univ. of Minnesota. Amundson, Barnett, Bearman, Brink, Brooke, Bussey, Cameron, Carlson, Doeringsfeld, Eggers, Fischer, Gibbens, Hall, Hart, Hartig, Hartman, Hatfield, Ito, D. A. Johnson, G. P. Johnson, Johnston, Kalisch, Kirchner, Kirmser, Koehler, Laws, Loud, McCutcheon, McHugh, McIntosh, Munro, Ohnsorg, Olmsted, Owchar, Priester, Quaid, Shumway, Smith, Stoner, Thornton, Thorp, Turrittin, Wang, Williams, Zeitlin.
MOORHEAD. Mundhjeld.
NORTHFIELD. Carlson.
Carleton Coll. Crum, Francis, Gingrich, May, Wegner.
ROCHESTER. Dubbert.
ST. JOSEPH. Muggli.
ST. PAUL. Bracewell, Brown, Hill, Morgan.
Macalester Coll. Blakely, Camp, Steinberg.
Coll. of St. Catherine. Berger, Boehm, Gibbons.
Coll. of St. Thomas. Bush, Godderz, Montgomery, Norris, Reuber, Sheridan, Smith, Speltz, Taylor, Terami.
ST. PETER.
Gustavus Adolphus Coll. Anderson, Kaufmanis, Swanson.
SLEEPY EYE. Kloyda.
VIRGINIA. Henning.
WINONA. De LaSalle, Lokensgard, Schrader, Schulte.

MISSISSIPPI

BLUE MOUNTAIN. Gillis.
CLEVELAND. Walters.
COLUMBUS. Erickson.
HATTIESBURG.
Mississippi Southern Coll. Davis, Felder, Foote, Jones, Phillips.
JACKSON, Babbitt, Mitchell.
MERIDIAN. Cleveland.
STARKVILLE. Temple.
STATE COLLEGE.
Mississippi State Coll. Goen, Grimes, Hopkins, Murray, Ollivier, Pettis.
UNIVERSITY.
Univ. of Mississippi. Bickerstaff, Hume, Miller, Samuels, Trott.
WINONA. Witty.

MISSOURI

BOONVILLE. Ford.
CAPE GIRARDEAU. Michel.
COLUMBIA. Cosby, Moore.
Univ. of Missouri. Blumenthal, Burcham, Cummings, Ewing, Gaddum, Haynes, Hogan, Kelley, Sawyer, Snider, Stamey, Utz.
FAYETTE.
Central Coll. Barrow, Denny, Helton.
FLAT RIVER. Galloway.
FULTON. Ekstrom, Lacy.
HANNIBAL. Williams.
JEFFERSON CITY. Jason, Muse.
KANSAS CITY. Baxter, Cutting, Doyle, Lackay, Pierson, Rosen.
KIRKSVILLE. Jamison.
LIBERTY. Jones.
MARYVILLE. Lafferty.
ROLLA.
Missouri School of Mines. Erkiletian, Evans, Goodhue, Lee, Pagano, Rankin.
ST. CHARLES. Beasley, Karr.
ST. LOUIS. Gove, Lewis, Marth, Proctor, Willerding, Van Schaack.
St. Louis Univ. Andrews, Bold, Collins, Felling, Regan.
Washington Univ. Bridger, Dunkel, Haimo, Leighton, Mathews, Middlemiss, Rider, Roever, Schriro, Stephens, Thron.
SPRINGFIELD. Fronabarger, Graves, H'Doubler.
WARRENSBURG. Brown.

MONTANA

BOZEMAN.
Montana State Coll. Antosiewicz, Glauz, Hurst, Livers, Lowney.
BUTTE. Smith.
GARRISON. Canning.
MISSOULA. Merrill, Ostrom.

NEBRASKA

CHADRON. Berry.
CRETE. Johnson.
HASTINGS. Lowry, McDill.
KEARNEY. Larsen.

LINCOLN. Blank, Gass, Ogden, Perisho.
Univ. of Nebraska. Basoco, Brenke, Camp, Clarke, Cox, Gaba, Heath, Leavitt, Lenser, Pool, Runge.
OMAHA. Becker, Earl, Rice.
Creighton Univ. Bettinger, Clarkson, Dansky, Tegels.
PERU. Huck.
SCOTTSBLUFF. Wood.
WAYNE. Boyce.
YORK. Feemster.

NEVADA

RENO. Beesley, Wood.

NEW HAMPSHIRE

DURHAM. Slobin.
EXETER. Pennell.
Phillips Exeter Acad. Adkins, Funkhouser, Lynch.
HANOVER. Morgan, Nordstrom.
Dartmouth Coll. Brown, Doyle, Durfee, Forsyth, Fraser, Mathewson, Perkins, Robinson, Silverman, Wilder.
KEENE. Goodrich, Peters.
MANCHESTER. O'Leary.
PLYMOUTH. Smith.

NEW JERSEY

BAYONNE. Koren, Quinn.
BELMAR. Borsuk.
BLOOMFIELD. Oergel.
CALDWELL. Anita.
CLIFTON. Struyk.
CONVENT STATION. Kenna.
EAST ORANGE. LePori, Nordgaard.
FORT MONMOUTH. Norwood.
HIGHLAND PARK. Hamilton.
HIGHTSTOWN. Harrison, Litterick.
HOBOKEN. Murray, Reeks, Rose.
JERSEY CITY. Ayres, Kruse, Reckzeh.
LAKEWOOD. Wallick.
LAWRENCEVILLE. Kiernan, Kimball.
MADISON. Battin.
MAPLEWOOD. Hazeltine.
MONTCLAIR. Kays.
MURRAY HILL.
Bell Telephone Labs. Gray, Hamming, Raisbeck, Shewhart, Schelkunoff.
NEWARK. Mosesson, Strock.
Newark Coll., Rutgers Univ. Ammerman, Davids, Hellman, Henry, McCarthy, Sherak.
Newark Coll. of Engg., Jaffe, Mainardi, Molina, Vedova, Wasson.
NEW BRUNSWICK.
Rutgers Univ. Barlaz, Biser, Brown, Bunyan, Cherlin, Cohn, Firestone, Galbraith, Grant, Hazard, Klein, LeLeiko, Makarov, Meder, Morris, Nelson, Ott, Phelps, Starke, Walter, Zimmerberg.
ORANGE. Chacalos.
PATERSON. Daugherty.
PRINCETON. Houghton, Meier.
Inst. for Advanced Study. Alexander, Bateman, Chowla, Goldstine, Montgomery, Morse, Pettis, Veblen, von Neumann, Wong.
Princeton Univ. Artin, Blackett, Gilbert, Lefschetz, Michael, Snapper, Tucker, Tukey, Wilks.
RAMSEY. Stuckey.
SOUTH ORANGE. Davis, Glusman, Stanwick.
SUMMIT. MacNeille.
TEANECK. Rayher.
TRENTON. Levine, Shuster.
UPPER MONTCLAIR. Campbell.
State Teachers Coll. Clifford, Davis, Humphreys, Mallory, Sensale.
WEST ORANGE. Edison.

NEW MEXICO

ALBUQUERQUE. Bauer, Buell, Carey, Mathany, Rogers.
Univ. of New Mexico. Beach, Boldyreff, Hendrickson, Hildner, Lane, LaPaz.
LAS VEGAS.
New Mexico Highlands Univ. Roberts, Rodgers, Slechticky.
LOS ALAMOS. Benson, Blum, Hammer, Stark.
ROSWELL. Harp.
SANTA FE. Luke.
SOCORRO. Glass, Reece.
STATE COLLEGE.
New Mexico Coll. of A. & M.A. Branson, Heinzman, Kramer, Walden, Westhafer.

NEW YORK

ALBANY. Dumont, Newsom, Noel Marie.
New York State Coll. for Teachers. Beaver, Birchenough, Butler, Lester, Stokes, Turner.
ALFRED.
Alfred Univ. Beals, Freund, Nevins, Polan, Rhodes, Seidlin, Whitford.
AURIESVILLE. Lewis.
AURORA. Hollcroft, Rusk.
BALDWIN. Bowden, Grove, Mitchell.
BAYSIDE. Cohen.
BINGHAMTON. Greene.
BRONX. Gordon, Lipsey, Orshansky, Tucker.
BROOKLYN, Appuhn, Byrne, Epsell, Finkel, First, Francis Xavier, Gerdes, Gerst, Honig, Karnow, Kramer-Lassar, Lavoie, Lazar, Leeds, Lieber, Lonner, McKenna, Miller, Odin, Raines, Reimer, Ross, Rush, Salkind, Sandler, Sarno, Shapiro, Tolle, Waite, Wallach.
Brooklyn Coll. Borofsky, Boyer, Fleisher, Forman, Griffin, Johnson, Karlin, Kennison, Kieval, Landers, Levenson, Maria, Moore, Prenowitz, Richardson, Shapiro, Singer, Smith, Wolfe, Woodbridge.
Poly. Inst. of Brooklyn. Forray, Foster, Hutchinson, Kalish, Klamkin, Lowe, Terzuoli, Ullman, Whitford.
Pratt Inst. Beckman, Cowles, Helme, Kohlmeyer, Levitt, Moore, Norman, Thompson.

BUFFALO. Baez, Benson, Browne, Buchman, Duke, Hand, Hickman, Maloney, Newell, Podmele, Potts, Rapp, Rowley, Saltarelli, Scholl, Sharpe, Tuthill, Walker, Wilkins.
Univ. of Buffalo. Baeumler, Behrns, Farber, Feidner, Fountain, Gehman, Gordon, Gough, Haslam, Hill, Hoffman, Kurland, McArtney, Meyer, Montague, Montgomery, Noller, Pound, Schillo, Schneckenburger, Smurthwaite, Strebe, Warner, Welmers.
CANISTEO. Longley.
CANTON. Boak.
St. Lawrence Univ. Bates, Limpert, Peters, Smith.
CAZENOVIA. Howe.
CLEVELAND. Morenus.
CLINTON.
Hamilton Coll. Gere, Patterson, van Alstyne.
ELMIRA. Suffa.
ENDICOTT. Kent, Wright.
FAR ROCKAWAY. Moore.
FLORAL PARK. Clark.
FLUSHING.
Queens Coll. Archibald, Brown, Cope, Dean, Eaton, Feld, Raudenbush, Sard, Sullivan.
FOREST HILLS. Frank, Hertzig.
FORT SCHUYLER. Kinney.
GENEVA.
Hobart & Wm. Smith Colls. Beinert, Durfee, Hubbs, Milliman, Mosey.
HAMILTON.
Colgate Univ. Aude, Downie, Munshower, Wardwell.
HAYT CORNERS. Ford.
HEMPSTEAD.
Hofstra Coll. Charlesworth, Draudt, Hawthorne, Hove, Jordan, Juelich, Ollmann, Stabler.
HIGHLAND FALLS. Coleman.
HOUGHTON. Luckey.
ITHACA.
Cornell Univ. Agnew, Bartram, Carver, Feller, Gunder, Hurwitz, Kac, Pollard, Robison, Snyder, G. L. Walker, R. J. Walker, Yood.
KINGS POINT. Keyes, Nickl.
LONG ISLAND CITY. Williams.
LOUDONVILLE.
Siena Coll. Garrett, Hanhauser, Kuhn.
MEXICO. Reddick.
NEWBURG. Weinert.
NEW LEBANON. Pflaum.
NEW ROCHELLE. Kiely.
NEW YORK. Alfieri, Berger, Bernard Alfred, Boehm, Braverman, Burgess, Coleman, Conlan, Crane, Croci, Darraugh, D'Atri, Deutsch, Dodes, Gray, Grossman, Harris, Heath, Hlavaty, Hobbs, Jablonower, Jeffries, Joffe, Jordan, Katz, Keeler, Kruskal, Levine, Maltenfort, Matteson, Mayerson, McGrath, McMahon, Mirick, Nehrbas, Nooger, Peiser, Quilty, Roll, Ruderman, Salerno, Sasuly, Schor, Schwartz, Silversten, Skelding, Steinhaus, Tom, Vitale, Wayne, Weaver, Young.
Bell Telephone Labs. Clos, Fry, Jones, MacColl, Mead, Riordan.
City Coll. of the City of New York. Chernofsky, Fagerstrom, Gill, Hibbard, Hubert, Hurwitz, Linehan, MacEwen, Mortola, Nathan, Post, Robinson, Schach, Schwartz, Singer, Weingarten, Wirth, Wright.
Columbia Univ. Aurora, Bolton, Eilenberg, Fehr, Fite, Gentzler, Goldman, Kasner, Kaufman, Lewis, Littauer, Lorch, Mullins, Murray, Ogilvy, Plethides, Reeve, Ritt, Scheffe, Siceloff, Upton, Zilber.
Cooper Union. Anderson, Eastham, Lehman, Miller, Tanzola.
Fordham Univ. Kirby, Kubis, Oehler.
Hunter Coll. Anderson, Aroian, Bradley, Brock, J. H. Bushey, Jewell H. Bushey, Cooper, Darkow, Eisele, Hill, Kutman, Landers, Tuller, Weisner, White.
International Business Machines. Herrick, Hurd, Seeber.
New York Univ. Adler, Bakst, Bernardi, Carl, Cooley, Courant, Edison, Ficken, Graham, Hirsch, Isaacson, F. W. John, Fritz John, Kline, McConnell, Palladino, Payne, Peters, Putnam, Rehberg, Roth, Rubin, Schlauch, Shephard, Stone, Tilley, Wahlert, Wolinsky, Yanosik.
Yeshiva Univ. Block, Frank, Ginsburg, Rosenfeld.
NIAGARA FALLS. Abbey, Welmers.
NIAGARA UNIVERSITY.
Niagara Univ. Banks, Egan, Newell.
NORTH CHILI. Smith.
ONEONTA. Callahan, Sanford.
ORANGEBURG. Beberman.
OVID. Casey.
PAUL SMITHS. Buxton.
PLATTSBURGH. Kotler, Skinner.
POTSDAM. Waltz.
POUGHKEEPSIE. Phillips.
Vassar Coll. Asprey, Baker, Dolciani, McDonald, Newton, Wells.
QUEENS. Gould.
RANDOLPH. Horak.
ROCHESTER. Chesna, Dorothea, Foard, Harding, Merrill.
Univ. of Rochester. Atkins, Barton, Bernstein, Betz, Danese, Gale, Gunderson, Klimczak, Marchand, Randolph, Seidel, Small, Watkeys.
ST. ALBANS. Deutsch.
ST. BONAVENTURE. Scheier.
SAMPSON. Caulum, Lane.
SARATOGA SPRINGS. Williams.
SCHENECTADY. Poritsky, Warr.
Union Coll. Bates, Burkett, Fox, Holt, Maddaus, Male, Morse, Snyder.
SPRINGVILLE. Harrington.
STATEN ISLAND. Stinetorf.
STONY BROOK. Berry.

SYRACUSE. Betz.
Syracuse Univ. Bruns, Carroll, Cole, Decker, Erdos, Gelbart, Goheen, Harwood, Hetzelt, Kibbey, Loewner, Protter, Rosenbloom, Rosskopf, Stokes, Taylor
TROY.
Rensselaer Poly Inst. Allen, Biggerstaff, Burger, Campbell, Guilford, Hendler, Huston, Jones, Maly, Nash, Nickol, Tear, Vrooman, Warnock.
UTICA. Greeley, Jenkins, Tiller.
WEST POINT.
U. S. Military Acad. Bessell, Jones, Nicholas, Yates.
WHITE PLAINS. Mary Benedicta.
WYOMING. Hartnell.

NORTH CAROLINA

CHAPEL HILL.
Univ. of North Carolina. Brauer, Browne, Cameron, Carpenter, Garner, Henderson, Hickerson, Hill, Hoke, Hoyle, Lasley, Mackie, Macon, Minton, Payne, Vause, Wahab, Whyburn.
CHARLOTTE. Hoyle.
DAVIDSON.
Davidson Coll. Martin, McGavock, Mebane, Peyton.
DURHAM. Pegram, Williams.
Duke Univ. Dressel, Elliott, Garrett, Gergen, Hickson, Patterson, Rankin, Roberts, Thomas.
GREENSBORO.
Woman's Coll. of the Univ. of North Carolina. Barton, Lewis, Strong, Walker.
GREENVILLE. Reynolds, Scott.
HICKORY. Dodson.
HIGH POINT. Adams.
MARS HILL. Howell.
OTEEN. Ripandelli.
RALEIGH. Downing.
North Carolina State Coll. Baker, Bullock, Carroll, Cell, Clayton, Levine, Lewis, Nahikian, Nolstad, Strobel, Watson.
SALISBURY. Ackerson, Dearborn.
TRYON. Dimick.
WARRENTON. Graham.
WILSON. Stark.

NORTH DAKOTA

FARGO.
North Dakota Agric. Coll. Arena, Grimes, Hill, Smith, Stennes.
GRAFTON. Westgate.
GRAND FORKS.
Univ. of North Dakota. Hankerson, Mason, McBride, Peterson, Rognlie, Staley.
JAMESTOWN. Jackson.
MINOT. Beckstrom.

OHIO

ADA. Ross, Whitted.
AKRON.
Univ. of Akron. Davis, Lowe, Mauch, Robbins, Ross, Selby.
ALLIANCE. Clark, Freese.
AMSTERDAM. Derflinger.
ATHENS.
Ohio Univ. Marquis, Reed, Starcher.
BARBERTON. Anderson.
BEREA. Annear.
BOWLING GREEN.
Bowling Green State Univ. Cornell, Krabill, Mathias, Moore, Overman, Tinnappel, Wohler.
CHILLICOTHE. Clinton.
CINCINNATI. Hobensack, Pinzka, Reilly, Rice, Stechschulte.
Univ. of Cincinnati. Barnett, Berman, Brand, Feige, Herwitz, Justice, Lipsich, Lubin, Merriman, Moore, Smith, Szasz, Wang, Yowell.
CLEVELAND. Brown, Burwell, Dustheimer, Garvin, Joliat, Johnson, Musselman, Simon, Voronovich.
Case Inst. of Tech. Brown, Green, Guenther, Leone, McCuskey, Morris, Nassau, Saltzer, Thomas.
Fenn Coll. Gutzman, Haskins, Higgins, Kelly, McGar, Topp, Van Voorhis.
COLUMBUS. Schultz.
Capital Univ. Barnhart, Heinke, Wildermuth.
Ohio State Univ. Adney, Alden, Bamforth, Bareis, Beatty, Blumberg, Caris, Davis, Fawcett, Hall, Helsel, Jones, Kuhn, Mickle, Miller, Morris, Myers, Naiditch, Pepper, Rado, Rankin, Rasor, Rechard, Reichelderfer, Rickard, Ryser, Sealander, Toops, Whitney, Swain.
DAYTON. Schawalder.
Wright Field. Fettis, Gephart, Loch, Millsaps, Price, Toney, Westbrook.
A.A.F. Inst. of Tech. Carson, Downing, Gatewood, Holtom.
Univ. of Dayton. Bellmer, Cassel, Hafner, Peckham, Schraut.
DEFIANCE. Godfrey, MacCullough.
DELAWARE. Crane, Rowland.
GAMBIER. Berg, Vandort.
GRANVILLE.
Denison Univ. Glabe, Kato, Ladner, Wiley.
HIRAM. Clarke, Olney.
KENT.
Kent State Univ. Boblett, Brooks, Brumfield, Dressler, Evans, Harshbarger, Iwanchuk, Jasper, Jenkins, Johnson, Lowenstein, Lowry, Manchester, E. J. Olson, F. R. Olson, Riggs, Wade, Warren.
LAKESIDE. Wolfe.
LANCASTER. Smart.
LISBON. Martin.
MARIETTA. Bennett. Sandt.
MIDDLETOWN. Orr.
MT. ST. JOSEPH. Corona, Dimond.
NAPOLEON. Yeager.
NEW CONCORD. Knight.
NEW LEXINGTON. Hoops.

OBERLIN. Yeaton.
Oberlin Coll. Graff, Sinclair, Tuckerman, Vance, Wagner, Yeaton.
OSBORN. Wells.
OXFORD. Tappan.
Miami Univ. Anderson, Miltenberger Pollard, Spenceley, Wolfe.
SPRINGFIELD.
Wittenberg Coll. Hahn, Krueger, Tripp.
STEUBENVILLE. Burke, Emmert.
SYLVANIA. Agneta.
TIFFIN. Menke.
TOLEDO. Ginther, Koley, Mercedes.
Univ. of Toledo. Blackall, Brandeberry, Calhoon, Cutler, Dancer, Davis, Shoemaker, Suprock.
WESTERVILLE. Glover.
WILBERFORCE. Curry, Woodson.
WILMINGTON. Spinks.
WOOSTER.
Coll. of Wooster. Fobes, Knight, Smyth, Williamson, Yanney.
YELLOW SPRINGS.
Antioch Coll. Astrachan, Hamilton, Luippold, Myatt.
YOUNGSTOWN. Yozwiak.

OKLAHOMA

ADA. Canada, Heimann, Winn.
ALVA. Huneke.
BARTLESVILLE. Rice.
CLAREMORE. Nemecek.
DURANT. Dwight, Krattiger.
ENID. Smith.
GOODWELL. Murphy.
NORMAN.
Univ. of Oklahoma. Andree, Bernhart, Boyd, Brixey, Brown, Court, Deal, Dragoo, Gorsline, Grau, Gregory, Hassler, Huff, LaFon, Lawson, Levy, Lewis, McFarland, McKnelly, Palmer, Pipes, Reaves, Ross, Spears, Springer, Yeilding.
OKLAHOMA CITY.
Oklahoma City Univ. Diamond, Meador, Pirrong.
PONCA CITY. Kaufman.
SHAWNEE. Doerfler.
STILLWATER.
Oklahoma A & M Coll. Allen, Barnett, Burns, Caskey, Diamond, Flanders, Hamilton, Krentel, McDole, McKinsey, Morrison, Pedrick, Scholz, Smith, Starch, Stavinoha, Zant.
TULSA. Doll, Duncan, Eisen, Shreve.
Univ. of Tulsa. Carter, Roth, Veatch, Whitt.
WARNER. Harrison.
WEATHERFORD. Linscheid.
WILBURTON. Holland.

OREGON

CORVALLIS.
Oregon State Coll. Beaty, Clark, Eves, Hoggatt, Hostetter, Kirkham, Li, Lonseth, Milne, Nickel, Pearson, Poole, Saunders, Smith, Stone, Williams, Wirshup.
CULVER. Bunch.
EUGENE.
Univ. of Oregon. Civin, Ghent, Moursund, Niven, Peterson, Shepherd, Sorenson, Wood, Young.
FOREST GROVE. Price.
LA GRANDE. Oesterle.
MCMINNVILLE. Dolan. Ramsey.
PORTLAND. Keeler, Merriss.
Reed Coll. Buschman, Griffin, L. J. Rosenbaum, R. A. Rosenbaum, Williams.
SALEM.
Willamette Univ. Clemans, Laidlaw, Luther.

PENNSYLVANIA

ALLENTOWN. Billig, Kunkel.
Muhlenberg Coll. Deck, Holt, Koehler, Nelson.
ALTOONA. Herpel.
ANNVILLE. Erickson.
BEAVER FALLS. Cleland, Justis.
BETHLEHEM. Rader.
Lehigh Univ. Ashbaugh, Beer, Chellevold, Cutler, Hailperin, Kenny, Latshaw, Mettler, Petrie, Pitcher, Raynor, Reynolds, Seebald, Shook, Smail, Spohn, Stoll, Tikson, Van Arnam, Wilansky.
BRYN MAWR. Atkinson.
Bryn Mawr Coll. Burton, Lehr, Oxtoby, Wheeler.
CARLISLE.
Dickinson Coll., Ayres, Kuebler, Nelson.
CHAMBERSBURG. Johnson, Loh.
CHESTER. Helms.
COLLEGEVILLE.
Ursinus Coll. Clawson, Dennis, Manning.
CORAOPOLIS. Zora.
DENVER. Marburger.
DUBOIS. Kocher.
EASTON. Sandwick.
Lafayette Coll. Benner, Cawley, Hatch, Keck, J. C. Smith, W. M. Smith, Tartler.
ERIE.
Gannon Coll. Kraus, Myers, Russell.
GREENSBURG. McNeil.
GROVE CITY. Carpenter.
HAVERFORD.
Haverford Coll. Allendoerfer, Oakley, Thomsen, Wilson.
HAZLETON. Liechty, Zerbe.
HERSHEY. Lanz.
HUNTINGDON. Stayer.
IMMACULATA. Hafner.
INDIANA. Stright.
JENKINTOWN. Van Sant.
KUTZTOWN. Knedler.
LANCASTER. Haag.
Franklin & Marshall Coll. Holzinger, Murray, Western.
LATROBE. Heid, Seubert.
LEBANON. Heilman.
LEWISBURG.
Bucknell Univ. Gold, Miller, Richardson.

LOCK HAVEN. Smith.
LORETTO. Mino.
MCKEES ROCKS. Arnold.
MEADVILLE. Steen.
MERION STATION. Evans.
MILLERSVILLE. Boyer.
MONTOURSVILLE. Price.
PHILADELPHIA. Carr, Cavalli, Connelly, Constable, Durand, Eggert, Eisenhart, Fudge, Hearn, Hilferty, Keralla, Koch, Latshaw, Levy, McDonough, Moliver, Neale, O'Conor, Russ, Slepin.
Drexel Inst. of Tech. Cherbas, Davis, McNeary.
Temple Univ. Hostinsky, Wilson, Wurster.
Univ. of Pennsylvania. Aissen, Anderson, Caris, Fine, Gottschalk, Kamel, Kline, Lehner, Patterson, Safford, Schafer, Schoenberg, Wilson.
PICTURE ROCKS. Price.
PITTSBURGH. Buker, Calkins, Frankel, Gray, Hallett, Harmon, Leifer, Michael, Mullan, Sommers, Taylor.
Carnegie Inst. of Tech. Dines, Hoover, Johnson, Lahti, Lemke, Moskovitz, Neelley, Nodvik, Olds, Peach, Rosenbach, Sacks, Saibel, Tyler, Whitman.
Duquesne Univ. Dunkelberger, Goodman, Hardy, Hnath, Ostrofsky, Smith.
Univ. of Pittsburgh. Blumberg, Bryson, Christiano, Gettig, Hovey, Knipp, Laush, Montroll, Mount, Sebesta, Taylor, Wells.
PLEASANTVILLE. Kerr.
POTTSVILLE. Cogan, Townsend.
READING. Speicher.
RIDGEWAY. Bauser.
SCRANTON. Bartley, Bertrand, Bohan, Rounds.
SHARON. Greene.
SLIPPERY ROCK. Lady.
SPRING GROVE. Martin.
STATE COLLEGE. Hutchison.
Pennsylvania State Coll. Black, Cohen, Curry, Dunlap, Erskine, Frink, Gordon, Gravatt, Hagen, Harrington, Johnson, Krall, Ormsby, F. W. Owens, H. B. Owens, Rupp, Sheffer, Story.
SWARTHMORE. Haseltine, Sevier.
Swarthmore Coll. Brinkmann, Dresden, Mariott, Wasow.
WASHINGTON.
Washington & Jefferson Coll. Bert, Shaub, Thomas
WAYNESBURG. Moston.
WILKES-BARRE. Richards.

PUERTO RICO

MAYAGUEZ. Garcia.
PONCE. Rathbun.
SAN JUAN. Piza.

RHODE ISLAND

KINGSTON.
Rhode Island State Coll. Bender, Haggerty, Stauffer.
NEWPORT. Cavanaugh, Chase.
PORTSMOUTH. Taliaferro.
PROVIDENCE. McKenney, McMurtrie.
Brown Univ. Ablow, Adams, Albert, Archibald, Bennett, Carlen, Gilman, Heins, Kennedy, Levy, Manning, Smiley.

SOUTH CAROLINA

CHARLESTON.
The Citadel. Conner, Dye, Folsom, Hair, Hutchison, Reves, Sutton, Zader.
CLEMSON.
Clemson Agric. Coll. Brewster, Brown, Bryan, Hind, Kirkwood, Lagrone, Park, Sheldon, Stanley, Sullivan.
COLUMBIA. Coleman.
Univ. of South Carolina. Buffkin, Croxton, Dinkines, E. G. Douglas, N. C. Douglas, Durst, E. A. Hedberg, M. Z. Hedberg, Jackson, Jones, Lee, Lytle, Martin, Novak, Perkins, Rabon, Rasor, Robinson, Shuler, B. R. Weber, W. W. Weber, Williams.
GREENVILLE. Blackwell, Mays, Pitts.
HARTSVILLE. Reaves, Saunders.
NEWBURY. Gaver.
ROCK HILL. Pepper.
SPARTANBURG. Patten.

SOUTH DAKOTA

BROOKINGS.
South Dakota State Coll. Engebretson, MacDougal, Walder, Wente.
HURON. Gantvoort.
MITCHELL. Knox.
RAPID CITY.
South Dakota School of Mines and Tech. Harbison, Swanson, Wilson.
SIOUX FALLS. Lyche.
VERMILLION. Alkire, Ekman.
WATERTOWN. Witwam.
WOLSEY. Meyer.
YANKTON. Howell.

TENNESSEE

ALCOA. Harris.
CHATTANOOGA. Massey.
CLARKSVILLE. Bright.
COOKEVILLE. Hutchinson, Moorman.
FOUNTAIN CITY. Keller.
GREENEVILLE. Johnston.
HARROGATE. Bowling.
JEFFERSON CITY. Sloan.
JOHNSON CITY. Carson.
KNOXVILLE.
Univ. of Tennessee. Blackstock, Bradley, Brown, Cooley, Eagle, Eaves, Givens, Hudson, Lee, Marr, Miller, Ritchie, Snyder, Wilson.
MARTIN. Taylor.
MARYVILLE. Sisk.
MEMPHIS. Coker, Kaltenborn, McBride, Thomas.
NASHVILLE. Boswell, Byrd, Clement, Dark, Gasaway, Van Horn, Wren.

Vanderbilt Univ. Blair, Boyce, Bryant, Clark, Graham, Hyden, Lundberg, Martin, N. P. Miser, W. L. Miser, Ratner, Shanks, Stiles, Thurman, Voorhees, Wesson.
OAK RIDGE. Chelius, Coveyou, Householder, Pollard, Rowe.
SEWANEE.
Univ. of the South. Dickerson, Hooke, Shotwell.

TEXAS

ABILENE. Burnam, Mullings, Tate.
AMARILLO, Davis, McCuan.
ARLINGTON. Howard.
AUSTIN.
Univ. of Texas. Batchelder, Bright, Craig, Decherd, Donaldson, Ettlinger, Greenwood, Hurt, Lubben, Osborn, Vandiver, Wall.
BROWNSVILLE. de la Garza.
BROWNWOOD. Johnson.
CANYON. Murray.
COLLEGE STATION.
A. &. M. Coll. of Texas. Basye, Beeman, Daum, Gandy, Klipple, Luther, McCulley, Moore, Tittle, Wapple.
COMMERCE. Wright.
DAINGERFIELD. Kennedy.
DALLAS. Aronofsky, Harris, Jonah, Mason, McNabb, Mouzon, Sorrells, Starr, Stulken, Thomas, Wood.
DENTON.
North Texas State Coll. Barksdale, Brown, Cooke, Ellis, Hanson, Parrish.
Texas State Coll. for Women. Ashburn, Miller, Willey.
EL PASO. Smith.
FORT WORTH.
Texas Christian Univ. Bramblett, Colquitt, Morgan, Neely, Ramsey, Sherer, Shore.
GEORGETOWN. Whitmore, Wilcox.
HILLSBORO. Hardy.
HOUSTON. Blau, Hardin, Howe, Pennington, Rainbow, Slotnick.
Univ. of Houston. Gibney, Gray, Grover, Newhouse, Rader, Rogers.
Rice Inst. Bray, Brunk, Calkin, Dean, Lovett, Taylor, Ulrich, White.
HUNTSVILLE.
Sam Houston State T. C. Lane, Querry, Vick, Wall, Wells.
KINGSVILLE. Dorroh.
LUBBOCK.
Texas Tech. Coll. Hazlewood, Heineman May, Michie, Parker, Rowland, Sparks, Underwood, Woodward.
PLAINVIEW. McCoy.
SAN ANTONIO. Dobbins, Jenke, Mary of Mercy, McNelly, Oesch, Schnepp.
Trinity Univ. Cullwell, Newton, Rees.
SAN MARCOS.
Southwest Texas S. T. C. Bernard, Cude, Porter.
STEPHENVILLE.
Tarleton State Coll. McSweeney, Redden, Worthington.
TEAGUE. Notley.
TYLER. Williams.
WACO. McLachlan.
WICHITA FALLS. Adams.

UTAH

LOGAN. Nelson, Hunsaker, Everett.
SALT LAKE CITY.
Univ. of Utah. Biesele, Hayes, Henriques, Horsfall, Mayer, Pehrson, Thorne, Wylie.

VERMONT

BURLINGTON.
Univ. of Vermont. Bullard. Butterfield, Larrivee, Millington, Swift.
MIDDLEBURY.
Middlebury Coll. Ballou, Bowker, Hazeltine.
NORTHFIELD. Dix.
SWANTON. Alliot.
WINOOSKI. Albiser.

VIRGINIA

ARLINGTON. McLynn, Thompkins.
ASHLAND. Blincoe, Simpson.
BLACKSBURG.
Virginia Poly. Inst. Hatcher, Horne, McFadden, O'Shaughnessy, Spencer.
CHARLOTTESVILLE.
Univ. of Virginia. Aylor, Bing, Botts, DeFrancesco, Dresser, Floyd, Gould, Johnson, Klee, Linfield, McShane, Oglesby, Rubenstein, Schlauch, Walker, Whyburn.
DAHLGREN. Bramble.
EMORY. Biggers.
FARMVILLE. Sutherland, Taliaferro.
FREDERICKSBURG. Burns, Frick.
HAMPTON. Hobbs.
HARRISONBURG. Ikenberry.
LANGLEY FIELD. Moore, Pinkerton.
LEXINGTON. Smith.
Virginia Military Inst. Byrne, Knox, Paxton, Purdie, Steward.
LYNCHBURG. Harris.
Randolph-Macon Woman's Coll. Humphreys, Larew, Wiggin.
MIDDLEBURG. Keppler.
NEWPORT NEWS. Raine.
NORFOLK. Norris, Snider.
PETERSBURG. Hunter.
RICHMOND. Drew.
Univ. of Richmond. Gaines, Grable, Pettus, Sleight, Wheeler.
SALEM. Carpenter, Suter.
STAUNTON. Taylor.
SWEET BRIAR. Lee.
WILLIAMSBURG.
Coll. of William and Mary. Barnes, Calkins, Phalen, Smith, Stetson, Wood.

WASHINGTON

BELLINGHAM. Gelder, Johnston.
CHENEY. Bell.
EVERETT. Van Arkel.
LACEY. Cebula.
MT. VERNON. Good, McKeehan.
PULLMAN.
State Coll. of Washington. J. V. Allhands, T. Allhands, Butler, Caton, Clement, Hacker, Irwin, Klotz, Knebelman, Salisbury, Tysver, Vatnsdal.
RICHLAND. Muller.
SEATTLE. Beegle, Bond, Boselly, Innis, Rowland, Vopni.
Univ. of Washington. Ball, Ballantine, Beaumont, Chapman, Cramlet, Dekker, Haller, Jerbert, Kingston, McFarlan, Street, Tang, Winger, Zuckerman.
SPOKANE. Barnard, Carlson.
Gonzaga Univ. Berard, Murray, Ryan.
TACOMA.
Coll. of Puget Sound. Anselone, Goman, Hoggatt, Rall.
VANCOUVER. Stair.
YAKIMA. Seamons.

WEST VIRGINIA

CHARLESTON. La Rue.
COWEN. Haller.
HUNTINGTON. Barron, Goins.
INSTITUTE. Ponds.
MORGANTOWN. Blum.
West Virginia Univ. Bauserman, Brown, Crisler, Cunningham, Davis, Eiesland, Heater, Peters, Reynolds, Sellers, Stewart, Turner, Vehse, Vest.

WISCONSIN

APPLETON.
Lawrence Coll. Berry, McGaughy, Stewart.
BELOIT.
Beloit Coll. Conwell, Emerson, Hood, Huffer.
EAU CLAIRE. Otteson.
LA CROSSE. Adkins, Malin.
MADISON. Erickson.
Univ. of Wisconsin. Allen, Arnold, Bruck, Buck, Chessin, Colvin, Eberlein, Evans, Finch, Fuller, Hart, Higgins, Hsiung, Ingraham, Kleene, Langer, MacDuffee, March, Mark, Mayor, Renno, Rose, Schurrer, Smith, Sokolnikoff, Trump, Zarling, Zemmer.
MARINETTE. Wagner.
MILWAUKEE. Baumann, Bigelow, Boehmer, Clark, Felice, Jautz, Mary Petronia, Norris, Overn, Roswitha.
Marquette Univ. Moeller, Pettit, Talacko, Wilczewski.
Univ. of Wisconsin. Bardell, Bartz, Battig, Kenney, Larson, Marden, Meyer, Parkinson, Spitzbart, Thompson, Vass, Wolf.
OSHKOSH. Bristow.
PLATTEVILLE. Harrell.
PLYMOUTH. Rusch.
RIVER FALLS. McLaughlin
SUPERIOR. Flogstad, Smith.
WAUKESHA. Dancey, Hopkins, Meadows.
WHITEFISH BAY. Anderson.
WEST DE PERE.
St. Norbert Coll. Bahng, Berner, Busch, DeCleene, Finkbeiner, Vande Castle, Watermolen.

WYOMING

LAMONT. Bellamy.
LARAMIE.
Univ. of Wyoming. Barr, Brady, Calvert, Goldbeck, Neubauer, Rechard, Schwid, S. R. Smith, W. N. Smith, Varineau.

CANADA

CALGARY, ALTA. Millar.
CHARLOTTETOWN, P.E.I. Roche.
EDMONTON, ALTA.
Univ. of Alberta. Campbell, Cook.
FREDERICTON. N. B. Edwards.
HAMILTON, ONT.
McMaster Univ. Bankier, Beesack, Findlay, McCallion, Terry.
KINGSTON, ONT.
Queens Univ. Halperin, Jeffery, Miller.
LONDON, ONT. Cole, Kingston.
MONTREAL, P. Q. Ayoub, Frechette, Gauthier, Gough, Lalonde, Pelletier.
McGill Univ. Bradley, Rosenthall, Williams.
OTTAWA, ONT. Dube, Duffie, Keyfitz, MacPhail.
QUEBEC, P. Q. Pouliot, Roland.
SACKVILLE, N. B. Crawford.
SASKATOON, SASK.
Univ. of Saskatchewan. Ferns, Gale, Miller.
TORONTO, ONT. Dobson, Grant, Pounder.
Univ. of Toronto. Beatty, Burk, Coxeter, Krieger, Pounder, Robinson, Solomon.
VANCOUVER, B. C.
Univ. of British Columbia. Buchanan, Derry, Gage, James, Jennings, Leimanis, Moyls, Murdoch, Simons, Yih.
VICTORIA, B.C. Wallace.
WINNIPEG, MAN.
Univ. of Manitoba. Mendelsohn, McEwen, Moser.
WOLFVILLE, N. S. Lane, Sheldon.

OTHER FOREIGN COUNTRIES

Argentina
Buenos Aires. Baidaff, Barral-Souto.

Belgium
Bruxelles. Errera.
Mons. Deaux.
St. Andre-lez-Bruges. Goormaghtigh.
St. Niklaas. Van Bergen.

Brazil
Rio de Janeiro. Murnaghan.

British Honduras
Cayo. Zimmerman.

Ceylon
Vaddukoddai. Lockwood.

Chile
Santiago. Moreno.

China
Amoy. Tan.

Cuba
Havana. González, Novoa, Rodríguez.
Santiago. Muguercia.

Egypt
Cairo. Dana.

England
Chessington. O'Beirne.
Chipping Norton. O'Hara.
Englefield. McCrea.
London. Dalal.
Sheffield. Mirsky.

France
Bourg-la-Reine. Minois.
Clermont-Ferrand. Droussent.
Paris. Belgodère.
Tennie. Thébault.

Greece
Athens. Lanckton.

Guatemala
Guatemala. Engel.

India
Aska. Das.
Bangalore. Madhava Rao.
Belgaum. Sharma.
Hyderabad. Ghani.

Ireland
Dublin. Broderick, Sandham, Synge.

Lebanon
Beirut. Jurdak.

Luxemburg
Differdange. Zahlen.

Mexico
Mexico. Napoles.
Monterrey. Lifshitz.

New Zealand
Auckland. Kania.
Dunedin. Martyn.

Nicaragua
Bluefield. Walbert.

Panama
Panama City. Linares, Thullen.

Peru
Lima. de Losada y Puga, Secada.

Philippines
Manila. Fitzgerald, Perez.

Portugal
Lisbon. Caraca.

South Africa
Bloemfontein. Arndt.

Spain
Madrid. Bachiller, Diaz.

Sweden
Stockholm. Blom.

Switzerland
Basle. Ostrowski.
Fribourg. Bays.
Neuchatel. DuPasquier.
Zurich. Burckhardt.

Uruguay
Montevideo. Calcagno.

Venezuela
Caracas. Colmenares-Carrillo, Goa, Michalup.

BY-LAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA (INC.)

(As amended to December 1, 1949)

ARTICLE I—NAME, PURPOSE AND CORPORATE SEAL

1. This organization shall be known as

THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED)

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field, by holding meetings in any part of the United States or Canada for the presentation and discussion of mathematical papers, by the publication of mathematical papers, journals, books, monographs and reports, by conducting investigations for the purpose of improving the teaching of mathematics, by accumulating a mathematical library and by coöperating with other organizations whenever this may be desirable for attaining these or similar objects.

3. The Corporate Seal of the Association shall have inscribed thereon the name of the Association and the words "Corporate Seal—Illinois."

ARTICLE II—MEMBERSHIP

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.

2. Election to membership shall be by vote of the Board upon written application from the individual seeking admission, endorsed by two members of the Association.

3. Those who were admitted to membership in The Mathematical Association of America (unincorporated) prior to October 1, 1920, and were in good standing as such on that date, were thereby admitted to membership in this Association (Incorporated).

ARTICLE III—BOARD OF GOVERNORS AND OFFICERS

1. The Officers of the Association shall be a President, a First Vice-President, a Second Vice-President, an Editor-in-Chief of the Official Journal (hereinafter called the "Editor"), a Secretary-Treasurer, and an Associate Secretary.

2. There shall be a Board of Governors (hereinafter called the "Board"), to consist of the Officers, the Ex-Presidents for terms of six years after the expiration of their respective presidential terms, and of additional elected members (hereinafter called "Governors"). It shall be the function of the Board to supervise all scholarly and scientific activities of the Association, to administer and control these activities, and to authorize expenditures of funds of the Association, except that at the demand of ten or more members of the Board, or at the demand of forty or more members of the Association, any proposal to alter or initiate a matter of policy shall be referred to the general membership of the Association for its decision. All members of the Board shall hold over until their respective successors are selected or appointed and qualify.

3. There shall be an Executive Committee advisory to the Board, and consisting of the President, the two Vice-Presidents, the Editor and the Secretary-Treasurer. It shall be the function of this Committee to review continually the policies and activities of the Association, to plan and organize new activities, to formulate in broad outline the programs of meetings and of publications, and in general to consider all matters of importance or of interest to the Association. This Committee shall prepare the agenda for meetings of the Board, and shall analyze the implications and aspects of all matters which are to come before the Board for decision. It shall present to the Board the viewpoints suggested by such analyses, as well as all such facts as may seem pertinent, or as may in any way facilitate the Board's work.

4. A statement regarding any proposed action of the Board which makes or alters a question of policy shall be published in the official journal, or notice of such proposed action shall be mailed to each member, before final action has been taken, so that members of the Association may make known to the Board their individual views.

5. The Board shall have authority to fill vacancies *ad interim* in any office, including vacancies in the Board, and to make any other appointments necessary for the transaction of the business of the Association.

6. At all meetings of the Board of Governors a quorum shall consist of not less than five (5) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Board, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than announcement at such meeting. Informal action based on a mail ballot by the members of the Board, if ratified at a properly convened meeting of the Board, shall be as valid and effective as if originally authorized at such meeting.

7. There shall be a Finance Committee responsible to the Board; at the direction of the Board it shall receive and administer the funds of the Association, control its properties and investments, make its contracts, and exercise such powers as may be delegated to it by the

Board. This committee shall consist of three members, of whom the Secretary-Treasurer shall be one.

8. (a) The Officers and Governors of the Association shall be elected in part by the Board, in part by the general membership, and in part by the membership in the Sections of the Association or by the membership in constituencies authorized by the Board for territory where Sections do not exist.

(b) The membership at large shall elect in alternate years respectively a President and a First Vice-President, each for a term of two years, and shall elect each year two Governors, for terms of three years.

(c) The membership in each Section shall elect triennially a Governor for a term of three years. For these elections, at least two nominations shall be submitted to the members by a committee appointed for that purpose by the Chairman of the Section.

(d) The Board shall elect at appropriate times by ballot and for the terms stated: a Second Vice-President for two years; an Editor, a Secretary-Treasurer, and an Associate Secretary, each for five years; and members of the Finance Committee (other than the Secretary-Treasurer) for four years.

(e) The President shall be ineligible for reëlection. The Vice-Presidents, the Editor, and the Governors shall be eligible for reëlection only after an interim equal to their respective terms of office.

(f) Elections by the Board shall be made from nomination by the Executive Committee. At least two nominations shall be made for each office to be filled in the case of the Second Vice-President and the members of the Finance Committee, and the Board may in any case reject all nominations made and call for a new list.

(g) The names of members to be printed upon the ballots, together with blank spaces in the case of elections by the general membership, shall be determined by a Nominating Committee to be appointed annually for that purpose by the President with the approval of the Board. Approximately six months before the date of the annual meeting all members shall be given an opportunity to nominate by mail a candidate for each office to be filled by the members for the ensuing year. Approximately one month before the annual meeting the Nominating Committee shall select a nominee for President out of the three persons who received the most votes for this office in the nominations; the Nominating Committee shall furthermore select two candidates for each other office to be filled by the members, one being the person who received the highest vote in the nominations and the other being selected from among the several nominees next in order. The election shall be by mail or in person and shall close on the day of the annual meeting.

9. The President shall be the Executive Officer of the Association, shall preside at all meetings of the Board of Governors and at the annual meeting of the Association. He shall have the usual duties pertaining to his office and such other duties as may from time to time be assigned him by the Board of Governors.

10. The Vice-Presidents shall, in the absence of the President, have and exercise the powers of the President, their order being determined alphabetically. The Board of Governors may assign to the Vice-Presidents such duties as may from time to time be determined.

11. The Secretary-Treasurer shall have the usual duties pertaining to the office of Secretary and of Treasurer, including the custody of the records of the Association and of its Corporate Seal, the keeping of minutes of the meetings of the Board of Governors and of the annual meeting and special meetings of members, and giving of due notice of all regular and special meetings of the Association and of the Board of Governors, and the supervision and safekeeping of the funds of the Association. The Secretary-Treasurer shall also have the duty of seeing that whenever Governors are elected, including the election of Governors to fill vacancies, a Certificate, under the Seal of the Association, giving the names of those elected and the term of their office, shall be recorded in the Office of the Recorder of Deeds for Cook County, Illinois. Such Certificates shall be signed by the Secretary-Treasurer and verified by oath of the President.

Article IV—Meetings

1. A meeting of the Association shall be held annually, at such time and place as the Board may direct. Special meetings of the Association may be called from time to time by the Board, or while the Board is not in session by the President of the Association, to be held at such time and place as may appear from the call.

2. The Board shall hold a meeting each year immediately preceding the annual meeting of the Association. Further meetings of the Board may be held from time to time at the call of the President or of any three (3) members of the Board.

3. Notice of any meeting of members of the Association shall be given by the Secretary-Treasurer at least thirty (30) days prior to the date set for each meeting. Notice of all meetings of the Board other than the regular meetings provided in Section 2 shall be given to each member of the Board at least fifteen (15) days prior to the date set therefor.

4. Any member of the Association or of the Board may waive notice with the same effect as if due notice had been given him.

5. At all meetings of the Association a quorum shall consist of not less than twenty-five (25) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Association, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than the announcement at such meeting.

6. Members may take part and vote in person or by proxy at all meetings of the Association.

ARTICLE V—SECTIONS

1. Any group of not less than ten (10) members of this Association may petition the Board for authority to organize a Section of the Association for the purpose of holding local meetings. The Board shall have power to specify the conditions under which such authority shall be granted.

2. The Association shall not be obligated to pay from its treasury any of the expenses of such Sections except as the Board may provide.

ARTICLE VI—OFFICIAL PUBLICATIONS

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.

2. The Board shall have full control of the publication and sale of the official journal and of all other official publications.

3. There shall be appointed by the Board a body of Associate Editors who shall give assistance in connection with the official journal.

4. The Board shall from time to time, as the need arises, make special provision for the management of any other official publications.

5. The Board shall fix the price of the official journal and of any other official publications of the Association, but in no case shall the journal be sold to non-members for less than the annual dues of individual members.

ARTICLE VII—DUES

1. Members of the Association shall pay an initiation fee of Two Dollars ($2) at the time of election.

2. The annual dues of each member shall be Four Dollars ($4), including a subscription to the official journal.

3. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list after due notice.

4. New members entering the Association after April 1 of any year shall have their dues pro-rated for the balance of the year, except when they desire to receive the full current volume of the official journal.

5. Any member who because of age is no longer in active service, who is in good standing at the time of his retirement and who has been a member of the Association for twenty years, may, upon notifying the Secretary of said retirement, be exempt from the payment of dues, with the privilege of obtaining the official journal at an annual cost of one dollar.

ARTICLE VIII—AMENDMENTS TO THE ARTICLES OF ASSOCIATION AND BY-LAWS

1. Changes in the Articles of Association or amendments to the By-Laws may be made at any annual meeting of the Association, or at any adjourned session, thereof, or at any special meeting of the Association called for such purpose, by a two-thirds ($\frac{2}{3}$) vote of those present and entitled to vote; *provided* that due notice concerning such amendment shall have been printed in the official journal, or mailed to each member, at least one (1) month before the date of such meeting.

2. No changes in the Articles of Association shall have legal effect until a certificate thereof, verified by oath of the President and under Seal of the Association, attested by the Secretary-Treasurer, shall be filed in the office of the Secretary of State of the State of Illinos and recorded in the office of the Recorder of Deeds for Cook County, Illinois.

PUBLICATIONS OF THE MATHEMATICAL ASSOCIATION OF AMERICA

THE CARUS MATHEMATICAL MONOGRAPHS

Number 1. *Calculus of Variations* by G. A. Bliss, xiii+189 pages. 1925.
Number 2. *Analytic Functions of a Complex Variable* by D. R. Curtiss, ix+173 pages. 1926.
Number 3. *Mathematical Statistics* by H. L. Rietz, ix+181 pages. 1927.
Number 4. *Projective Geometry* by J. W. Young, ix+185 pages. 1930.
Number 5. *History of Mathematics in America before 1900* by D. E. Smith and Jekuthiel Ginsburg, viii+210 pages. 1934.
Number 6. *Fourier Series and Orthogonal Polynomials* by Dunham Jackson, xiv+234 pages. 1941.
Number 7. *Vectors and Matrices* by C. C. MacDuffee, xi+192 pages. 1943.
Number 8. *Rings and Ideals* by N. H. McCoy, xii+216 pages. 1948.

Price $1.75 per copy to members of the Mathematical Association, one copy to each member, when ordered directly through the office of the Secretary-Treasurer of the Mathematical Association of America at the University of Buffalo, Buffalo 14, N. Y.

Additional copies for members, and copies for non-members, must be purchased from the Open Court Publishing Company, La Salle, Illinois, at the regular price of $3.00 per copy.

THE RHIND MATHEMATICAL PAPYRUS

Volume I, $11\frac{1}{4}$ by 8 inches, 8+210 pages, contains the Free Translation, Commentary, and Bibliography of Egyptian Mathematics, Volume II, $11\frac{1}{4}$ by $14\frac{1}{2}$ inches, contains 140 photographic plates in original colors, black and red, with Text and Introductions, and Literal Translation. Price to members of the Association $20.00 for the set; to non-members $25.00 for the set. Members should order copies through the office of the Secretary-Treasurer of the Mathematical Association of America at the University of Buffalo, Buffalo 14, N. Y. Non-members must purchase copies from the Open Court Publishing Company, La Salle, Illinois.

THE SLAUGHT MEMORIAL PAPERS

Number 1. *Fourier's Series, The Genesis and Evolution of a Theory* by R. E. Langer. v+86 pages. Paper.

Number 2. *Outline of the History of Mathematics* (6th edition) by R. C. Archibald. iv+114 pages. Paper.

Price $1.00 each. Copies should be ordered through the office of the Secretary-Treasurer of the Mathematical Association of America, University of Buffalo, Buffalo 14, N.Y.

The Mathematical Association of America

The Mathematical Association of America is a national organization of persons interested in mathematics at the collegiate level. It was organized at Columbus, Ohio in December 1915 and was incorporated in the State of Illinois on September 8, 1920.

Any person who is interested in the field of mathematics is eligible for election to membership in the Association. Election is by vote of the Board of Governors upon written application. Membership blanks may be secured from the office of the Secretary-Treasurer.

Each member pays an initiation fee of $2.00 and annual dues of $4.00. He receives a copy of the official journal, the American Mathematical Monthly, and is entitled to purchase the other publications of the Association at reduced prices.

The Association holds its Annual Meeting during the last week in December and its Summer Meeting during the first week in September. These meetings are held in conjunction with meetings of the American Mathematical Society.

The Association has established twenty-five Sections each comprising the members of the Association living in a certain geographical area. Each Section holds one-day or two-day meetings annually, usually in the spring. Some Sections meet twice a year, in the fall and in the spring. The programs presented at these meetings are similar to those of the national meetings.

The Association seeks the cooperation and support of all who desire to promote the interest of mathematics, particularly in the collegiate field. It invites into its membership all who are teaching mathematics in universities, colleges and junior colleges, all who use mathematics as a tool in their business or profession, and all who recognize the great service of mathematics in the development of civilization.

Further information about the Association, its publications and its activities may be obtained by writing to:

H. M. Gehman, *Secretary-Treasurer*
Mathematical Association of America
University of Buffalo
Buffalo 14, New York